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Diffuse Reflection and Transmission by a Very Thick Plane-Parallel Atmosphere with Isotropic Scattering

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Radiation from arbitrary directions incident on a thick, homogeneous, conservative atmosphere with isotropic scattering is mostly diffusely reflected. A smaller part diffuses through the atmosphere and emerges at the other side. Sobolev's equations, which give the asymptotic forms of the reflection and transmission functions for large thickness, b, in terms of the function $H(\mu)$ known from the theory of semi-infinite atmospheres, are newly derived. It is shown by two examples that knowledge of these asymptotic forms makes it possible accurately to interpolate any quantity dependent on b between $b = 1$ and $b = \infty$.

The Problem

By well-known theory (Chandrasekhar, 1950) the diffuse reflection and transmission of an arbitrary incident radiation field by a homogeneous plane-parallel atmosphere with isotropic scattering can be expressed in terms of two functions $X(b,a,\mu)$ and $Y(b,a,\mu)$. Here b = total optical depth of the atmosphere, a = albedo of each volume element ($a = 1$ is conservative scattering), and μ is the cosine of the angle with the outside normal at the top or bottom surface ($0 \leqslant \mu \leqslant 1$).

We discuss in this paper the manner in which these functions approach the well-known limits valid for a semi-infinite atmosphere

$$\lim_{b\to\infty} X(b,a,\mu) = H(a,\mu)$$

$$\lim_{b\to\infty} Y(b,a,\mu) = 0.$$

This problem has been independently solved by Sobolev and by Mullikin. The present paper may nevertheless have some interest to research workers because:

(a) Sobolev's solution is not illustrated by numerical examples. The title of the relevant Section 7 of Chap. 3 of his book (Sobolev, 1956) is wrongly translated into English (Sobolev, 1963) so that it seems to refer to infinitely large optical thickness. (The original says large optical thickness.) The most systematic account of the asymptotic formulas is not given in this book but in a separate paper (Sobolev, 1957).[1]

(b) The work by Mullikin (1963, 1964), based on the use of singular integral equations, is more general but at the same time more abstract. The nonmathematical reader may find the access to these formulations difficult. Moreover, the asymptotic formulas have not explicitly been given but may easily be derived (Mullikin, private communication).

(c) The present method, as illustrated by Fig. 1, makes it possible to visualize exactly how the radiation diffuses through the layer.

(d) A surprisingly accurate interpolation method between $b = 1$ and $b = \infty$ is suggested here.

(e) Less accurate solutions of the same problem have been derived independently in the literature on heat transfer (Viskanta and Grosh, 1962; Probstein, 1963; Eckert, 1963). I am grateful to Prof. Goulard for pointing this out to me.

[1] Reference to this paper should have been made in the footnote on p. 155 of the English translation of the book. The publisher's omission of the date of the original book combined with the literal translation "in press" makes this footnote very puzzling.

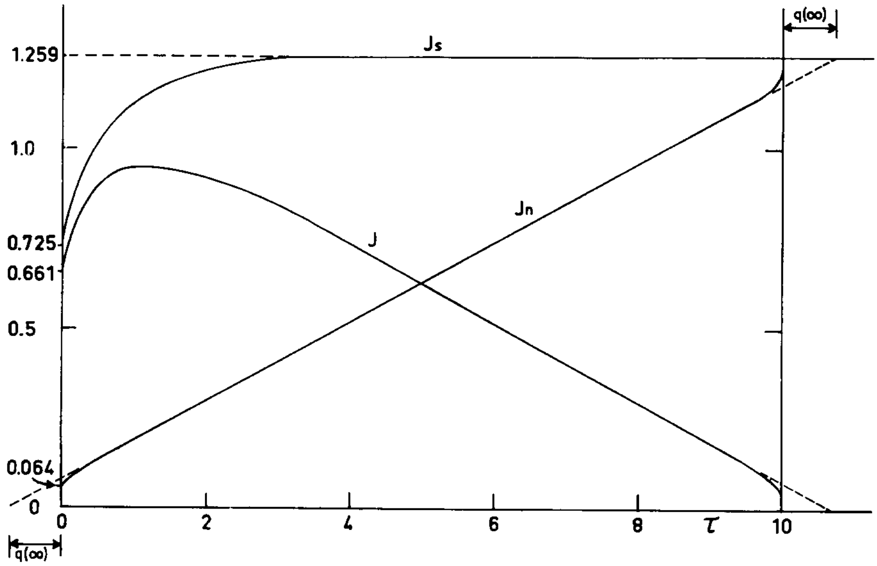

FIG. 1. Source function J_s in a semi-infinite atmosphere and J in an atmosphere with optical thickness 10 when perpendicular radiation is incident from the left.

We measure optical depth τ from the top $\tau = 0$ to the bottom $\tau = b$ and consider arbitrary radiation with intensity $I(0, -\mu)$ incident on the top. We wish to find not only the emergent radiation fields at top and bottom, but also the fluxes reflected and transmitted and the average intensity (or source function) at any depth inside.

The role of the albedo a may be summarized as follows: For $a < 1$ absorption losses occur with every scattering. The radiation field deep inside the atmosphere decreases with depth as $e^{-p\tau}$, where p is a well-known function of a (Case, De Hofmann, and Placzek, 1953, Table 9; Sobolev, 1963, Table 3.11) and $1/p$ is called the "diffusion length." Beyond a certain value of b the actual depth of the atmosphere is irrelevant for practical purposes. The error in the reflected intensity made by taking $b = \infty$ instead of finite remains below an approximate 1 or 2% limit if

$$b > 2/(1 - a)^{1/2}.$$

For $a > 1$ gains occur with every scattering. When b increases, a critical depth is reached beyond which the radiation in the layer is self-sustained and can have nonzero values even in the absence of incident radiation. The critical relation between b and the first eigenvalue a is known (Mullikin, 1962; van de Hulst, 1963) and has the asymptotic form, for a near 1, b large:

$$1 - (1/a) = \pi^2/3(b + 1.42)^2.$$

The eigenvalues of the homogeneous Milne equation for the semi-infinite atmosphere form a continuous set, $a \geqslant 1$. Physically, this means that the diffuse reflection from an infinitely thick layer with any $a > 1$ is undetermined because the appropriate eigenfunction may be added with an arbitrary factor. These functions have been discussed by Case, De Hofmann, and Placzek (1953, Section 18.1).

The remaining case $a = 1$ is the only one in which a finite nonzero fraction of the incident radiation reaches the bottom for very large b. The further sections treat only this conservative case, $a = 1$. We shall now omit the argument a from the functions X, Y, H, etc.

INTERENCE FROM INTEGRAL EQUATIONS

The functions $X(b,\mu)$ and $Y(b,\mu)$ satisfy the integral equations

$$X(b,\mu) = 1 + 2\mu \int_0^1 R(b,\mu,\mu_0)d\mu_0$$

$$Y(b,\mu) = e^{-b/\mu} + 2\mu \int_0^1 T_{\text{diff}}(b,\mu,\mu_0)d\mu_0,$$

where

$$R(b,\mu,\mu_0) = \frac{1}{4(\mu + \mu_0)} [X(b,\mu)X(b,\mu_0) - Y(b,\mu)Y(b,\mu_0)]$$

$$T_{\text{diff}}(b,\mu,\mu_0) = \frac{1}{4(\mu_0 - \mu)} [X(b,\mu)Y(b,\mu_0) - Y(b,\mu)X(b,\mu_0)].$$

We call R the (diffuse) reflection function and T_{diff} the diffuse (nondirect) part of the transmission function. They also satisfy the differential equations

$$\partial X(b,\mu)/\partial b = y_{-1}(b)Y(b,\mu)$$

$$\partial Y(b,\mu)/\partial b = -(1/\mu)Y(b,\mu) + y_{-1}(b)X(b,\mu),$$

where

$$y_{-1}(b) = \tfrac{1}{2} \int_0^1 Y(b,\mu)\, d\mu/\mu.$$

We may first observe that the term $e^{-b/\mu}$, which represents the directly transmitted radiation in the physical interpretation of this integral equation (van de Hulst, 1948), becomes numerically insignificant for large b. If this term is omitted, the integral equations are identically satisfied by

$$X(b,\mu) = [1 - \mu f(b)]H(\mu)$$

$$Y(b,\mu) = \mu f(b)H(\mu),$$

where $f(b)$ is an arbitrary function. For we find by direct substitution

$$R(b,\mu,\mu_0) = [1 - (\mu + \mu_0)f(b)]R(\infty,\mu,\mu_0)$$

$$T_{\text{diff}}(b,\mu,\mu_0) = (\mu + \mu_0)f(b)R(\infty,\mu,\mu_0),$$

with

$$R(\infty,\mu,\mu_0) = [1/4(\mu + \mu_0)]H(\mu)H(\mu_0)$$

and the integral equations for X and Y are satisfied, because of known integral properties of the H functions. Since direct transmission is supposed negligible, the total transmission function equals T_{diff}.

It is easily checked that the various relations between the moments of the X and Y- functions are also satisfied, again with neglect of the term representing direct transmission.

We now have $y_{-1}(b) = f(b)$. Hence both differential equations reduce to $df/db = -f^2$, so that

$$f(b) = 1/(b + \text{constant}).$$

The value of the constant will be found in the next section to be

$$2q(\infty) = 1.42089218.$$

The Source Function

The source function $J(\tau)$ in a plane-parallel, homogeneous atmosphere with isotropic scattering is a times the specific intensity at depth τ averaged over the full solid angle 4π, including the similarly defined average, $J_1(\tau)$ of the (weakened) direct radiation from external and/or internal sources. It satisfies Milne's equation

$$J(\tau) = J_1(\tau) + \frac{a}{2} \int_0^b J(x)E_1(|\tau - x|)dx.$$

We now specify radiation incident on the top with intensity $I(0,-\mu)$ normalized to a flux π falling on a unit area of the top surface, so that

$$\int_0^1 I(0,-\mu)2\mu d\mu = 1$$

and

$$J_1(\tau) = \tfrac{1}{2}a \int_0^1 I(0,-\mu)e^{-\tau/\mu}d\mu.$$

Again we restrict the discussion to conservative scattering ($a = 1$). We make use of two known results for semi-infinite atmospheres ($b = \infty$).

1. A constant outward net flux πF corresponds to the source function

$$J(\tau) = \tfrac{3}{4}F[\tau + q(\tau)]$$

and emerges from the atmosphere with the intensity

$$I(0,\mu) = (\sqrt{3}/4)FH(\mu).$$

2. Incident intensity as specified above gives rise to the source function (suffix s refers to semi-infinite atmosphere):

$$J_s(\tau) = \tfrac{1}{2} \int_0^1 I(0,-\mu)W(\tau,\mu)d\mu$$

and the radiation emerges again with the diffusely reflected intensity

$$I(0,\mu) = \int_0^1 R(\infty,\mu,\mu_0)I(0,-\mu_0)2\mu_0 d\mu_0$$
$$= \frac{H(\mu)}{2}\int_0^1 I(0,-\mu_0)H(\mu_0)\,\frac{\mu_0 d\mu_0}{\mu+\mu_0}.$$

The functions $q(\tau)$, "Hopf's function," and $H(\mu)$ are treated in all standard texts on radiative transfer. For an accurate table of $q(\tau)$ see Kourganoff (1952, p. 138). $W(\tau,\mu)$, which we call the "point-direction gain," is a less familiar function. It is in essence the same as Sobolev's "probability for quantum exit," $p(\tau,\mu)$, the relation being (for arbitrary albedo a)

$$p(a,\tau,\mu) = (a/4\pi)W(a,\tau,\mu).$$

We prefer the different name (and normalization) because it refers equally to the two reciprocal physical meanings which can be attached to the function and because it gives a slight simplification in the formulas. A three-figure table of $W(\tau,\mu)/H(\mu)$ is given by Sobolev (1963, p. 139). Some more accurate values and moments of $W(\tau,\mu)$ were derived by van de Hulst (1964). The necessary relations in the present context are those for $\tau = 0$ and $\tau \to \infty$:

$$W(0,\mu) = H(\mu)$$
$$W(\infty,\mu) = \mu\sqrt{3}H(\mu).$$

In order to employ these results we first observe that the Milne equation for finite b may be formally written as one for $b = \infty$ if we specify that $J(\tau) = 0$ for $\tau > b$ and require the equation to be valid only in the interval $0 < \tau < b$. We now propose a solution written in the entire range $(0,\infty)$ as

$$J(\tau) = J_s(\tau) - J_n(\tau),$$

where the first term is identical to the known solution for a semi-infinite atmosphere mentioned above and $J_n(\tau)$ is associated with the known net-flux solution as follows:

$$J_n(\tau) = \begin{cases} \frac{3}{4}F[\tau + q(\tau) + q(\infty) - q(b-\tau)] & 0 < \tau < b \\ \frac{3}{4}F[b + 2q(\infty)] & \tau > b. \end{cases}$$

We shall verify that this is an accurate solution if (and to the same accuracy as) it is possible to define a range of depths in the atmosphere which is far enough from the top *and* bottom surfaces to replace both $q(\tau)$ and $q(b-\tau)$ by $q(\infty)$, $J_1(\tau)$ by 0 and $J_s(\tau)$ by

$$J_s(\infty) = \frac{\sqrt{3}}{2}\int_0^1 I(0,-\mu)\mu H(\mu)d\mu.$$

If these conditions are satisfied, and if $J(\tau)$ is made 0 for $\tau > b$, which fixes the value of F at

$$F = \frac{(4/3)J_s(\infty)}{b + 2q(\infty)},$$

then we observe that:

(1) For small and intermediate τ, $J(\tau)$ satisfies the Milne equation because $J_s(\tau)$ is a solution of the nonhomogeneous equation and $J_n(\tau)$ is a solution of the homogeneous equation.

(2) For small and intermediate $b - \tau$, the equation is homogeneous and $J_n(\tau)$ is a solution because it has the form $\frac{3}{4}F[(b-\tau) + q(b-\tau)]$ for $\tau < b$ and 0 for $\tau > b$.

The physical significance of F is that πF is the net flux carried through the atmosphere from top to bottom: it is subtracted from the flux π which would be reflected from the top by a semi-infinite atmosphere and it emerges as a net flux at the bottom.

For unidirectional incidence

$$J_s(\infty) = (\sqrt{3}/4)H(\mu_0),$$
$$F = H(\mu_0)/\sqrt{3}[b + 2q(\infty)].$$

Figure 1 illustrates this solution by an actual example, namely perpendicular incidence on a layer with $b = 10$. The "intermediate range" is about from $\tau = 3$ to 7. We have $J_s(\infty) = (\sqrt{3}/4)H(1) = 1.2591$ and $F = \frac{4}{3}J_s(\infty)/11.4209 = 0.1470$, which means that nearly 15% of the incident flux emerges from the bottom.

Having thus found the source density we can immediately write the emergent intensities, for convenience referring to unidirectional incidence. These are:

Top:

$$I(0,\mu) = R(b,\mu,\mu_0) = (R(\infty,\mu,\mu_0) - \sqrt{3}/4)FH(\mu)$$
$$= \frac{H(\mu)H(\mu_0)}{4(\mu+\mu_0)} - \frac{H(\mu)H(\mu_0)}{4[b + 2q(\infty)]}$$

Bottom:

$$I(b,-\mu) = T(b,\mu,\mu_0) = \frac{\sqrt{3}}{4} FH(\mu)$$
$$= \frac{H(\mu)H(\mu_0)}{4[b + 2q(\infty)]}.$$

These are indeed the forms derived in the second section with

$$f(b) = 1/[b + 2q(\infty)]$$

Interpolation

Most numerical data about scattering by finite layers in the published literature are confined to $b \leqslant 1$. Extrapolation of such results to larger b is hazardous. However, the formulas for large b just derived make this into an interpolation which gives surprisingly accurate results in a simple manner.

This may be illustrated by two examples. In the conservative case, if the incident radiation follows Lambert's law, the transmitted fraction of the incident flux is

$$t_1 = UTU = t_c$$

and the reflected fraction is

$$r_1 = 1 - t_1 = URU = r_c = \bar{s}.$$

Here we have simply listed some notations under which the same function of b, which is important for calculating the influence of

TABLE I

Moments of $Y(b,\mu)$ Interpolated for Thick Layers between $b = 1$ and ∞[a]

b	t_1	$\frac{1}{t_1} - \frac{3}{4}b$	β_0	$\frac{1}{\beta_0} - \frac{1}{2}b\sqrt{3}$
0	1	1	1	1
0.05	0.95484	1.0098	0.9130	1.0520
0.10	0.91566	1.0171	0.8578	1.0792
0.15	0.88073	1.0229	0.8142	1.0983
0.20	0.84906	1.0278	0.7772	1.1135
0.25	0.82013	1.0318	0.7448	1.1261
0.50	0.7040	1.0455	0.6254	1.1660
1.00	0.5534	1.0570	0.48370	1.2014
2	(0.3900)	(1.064)	(0.3384)	(1.223)
4	(0.2459)	(1.066)	(0.2130)	(1.2302)
10	(0.1322)	(1.066)	0.1011042	1.23054
100	(0.013146)	(1.066)	0.01138523	1.23056
∞	0	1.06566914	0	1.23052876

[a] ( ) means interpolated values.

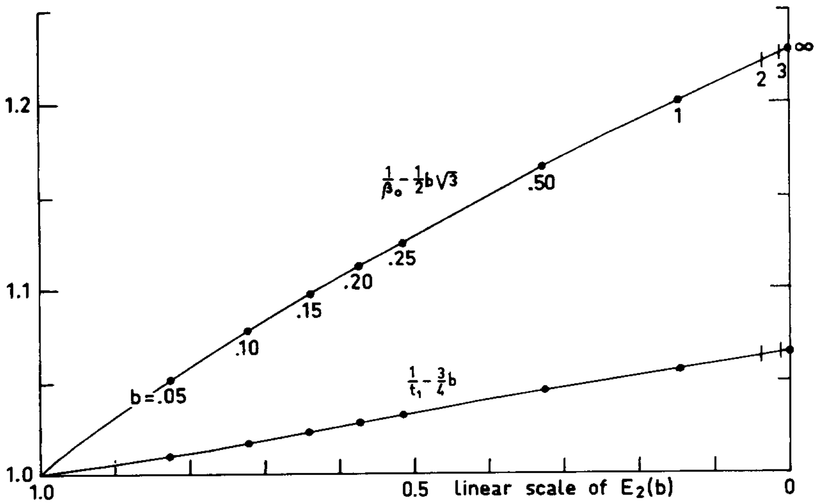

Fig. 2. Interpolation graphs for quantities depending on atmosphere thickness b.

a planetary solid surface below a scattering atmosphere, has appeared in different publications (Chandrasekhar and Elbert, 1952; Mayers, 1962; van de Hulst, 1948, 1963). We now find by simple integration that

$$t_1 = \tfrac{4}{3}f(b) = \frac{\tfrac{4}{3}}{(b + 1.42)}$$

for large b. This suggests that we use the function $1/(t_1 - \tfrac{3}{4}b)$ for interpolation. Table I and Fig. 2 show that this works quite well. The choice of $E_2(b)$ as abscissa is not based on theory, but proves to be convenient to make the graph about linear. This graph leaves no doubt that, e.g. at $b = 2$, the difference can be estimated within 0.2%, hence t_1 within 0.08% and r_1 within 0.06%. This accuracy rapidly increases with increasing b.

As a second example, Table I and Fig. 2 show values of

$$NTU = \beta_0 = 2y_0 = \int_0^1 Y(b,\mu)d\mu,$$

which physically means the fraction of the total flux transmitted through the atmosphere if the incident radiation comes from an isotropic source (or a layer of such sources) above the atmosphere. The asymptotic expression is

$$\beta_0 = 2f(b)/\sqrt{3} = 1/(0.8660b + 1.2305),$$

which suggests interpolation of the quantity $1/\beta_0 - 0.8660b$, which reaches the limit $\sqrt{3} \cdot q(\infty) = 1.2305$ for $b \to \infty$. For curiosity it may be noted that this limit also equals $3/2\alpha_2$, where α_2 is the second moment of the function $H(\mu)$. The table of Mullikin (private communication), from which the values for $b = 10$ and 100 are taken was computed with six-figure accuracy; probably it is more accurate to use simply the asymptotic formula for any $b \geqslant 10$.

Similar interpolation methods may be applied to the other moments, or to the functions $Y(b,\mu)$ or the differences $H(\mu) - X(b,\mu)$ for any fixed μ. An important conclusion is that there would be little point in calculating elaborate tables for the X and Y functions beyond $b = 3$, because four-figure accuracy and better can be reached by simple interpolation.

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