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On the anomalous phase-relation between first and second harmonic in the radial velocity of ζ Geminorum and related stars, by *J. Woltjer Jr.*

The radial velocity of an adiabatically pulsating star generally is an uneven periodic function of a linear function of the time with period 2π . However, if the deviations from adiabatic conditions are taken into account, the symmetric periodic solution of the equations of motion does not longer hold: the periodic solution is asymmetric. This asymmetry is very small in a star like δ Cephei, but strongly developed in ζ Geminorum and in some stars of period intermediate between the periods of these two stars. The theoretical interpretation of this fact is the subject of this paper.

1. Consider a spherically symmetric star performing a radial pulsation. Refer the state of this star to a "normal" state: a state of hydrostatic equilibrium corresponding to given values of the entropy η of the unit of mass; the quantities belonging to this state are to be denoted by the subscript n . Introduce the functions $s_i(r_n)$ ($i = 1, 2, \dots$) of the radius-vector r_n , proportional to that value of $\log \frac{r}{r_n}$ which corresponds to an infinitesimal excitation of only the fundamental mode of vibration with number i , the factor of proportionality to be determined by the requirement:

$$\int s_i^2 \rho_n r_n^4 dr_n = 1,$$

ρ denoting the density and the integral extending from centre to surface. The quantity r may be expanded in the series

$$\frac{r - r_n}{r_n} = \sum_i C_i s_i(r_n),$$

the coefficients C_i being functions of the time t . Denote the total energy of the star (the sum of kinetic energy, internal energy and gravitational energy) by $4\pi H$; H is a function of the variables C_i , the derivatives of these variables with regard to t : $\dot{C}_i$ and of the time t introduced by the oscillation of the entropy η . Then, the hydrodynamical equation

may be resolved into the system of ordinary differential equations ¹⁾:

$$\frac{dC_i}{dt} = \frac{\partial H}{\partial \dot{C}_i}, \quad \frac{d\dot{C}_i}{dt} = -\frac{\partial H}{\partial C_i} \quad (i = 1, 2, \dots).$$

If only adiabatic pulsations are considered, η is constant with regard to the variable t ; the function η_n may be chosen equal to η .

Introduction of the variables J and w by the equations:

$$C_i = \sqrt{\frac{2J_i}{n_i}} \cos w_i, \quad \dot{C}_i = -\sqrt{2J_i n_i} \sin w$$

transforms the system of differential equations into the set:

$$\frac{dJ_i}{dt} = -\frac{\partial H}{\partial w_i}, \quad \frac{dw_i}{dt} = \frac{\partial H}{\partial J_i}.$$

These equations may be written, with a sufficient degree of approximation, in a form that separates the adiabatic and non-adiabatic parts:

$$\frac{dJ_i}{dt} = -2\alpha_i J_i - \frac{\partial H(\eta = \eta_n)}{\partial w_i}, \quad \frac{dw_i}{dt} = \frac{\partial H(\eta = \eta_n)}{\partial J_i};$$

α_i denotes the damping-constant belonging to the fundamental mode of vibration with number i .

2. If only the interaction of the first two degrees of freedom is considered and the development of the function H is restricted to those interaction terms of lowest order which are predominant if n_1 and n_2 are nearly commensurable in the ratio of 2:1, this function may be restricted to the terms:

$$H = n_1 J_1 + n_2 J_2 + k_{12} J_1 \sqrt{J_2} \cos(2w_1 - w_2).$$

Then a periodic solution of the differential equations exists in which w_1 and w_2 are linear functions of the time and J_1 , J_2 and the argument $\varphi \equiv 2w_1 - w_2$

¹⁾ A more extensive analysis may be found in a former paper: *B.A.N.* No. 303.

are constant. These constant values result from the equations:

$$-2\alpha_1 J_1 + 2k_{12} J_1 \sqrt{J_2} \sin \vartheta = 0$$

$$-2\alpha_2 J_2 - k_{12} J_1 \sqrt{J_2} \sin \vartheta = 0$$

$$2n_1 - n_2 + k_{12} J_1 \sqrt{J_2} \left(\frac{2}{J_1} - \frac{1}{2J_2} \right) \cos \vartheta = 0.$$

Hence:

$$\alpha_1 J_1 + 2\alpha_2 J_2 = 0$$

$$\alpha_1 = k_{12} \sqrt{J_2} \sin \vartheta$$

$$2n_1 - n_2 = \left(\frac{1}{2} \frac{J_1}{J_2} - 2 \right) k_{12} \sqrt{J_2} \cos \vartheta =$$

$$-\left(\frac{\alpha_2}{\alpha_1} + 2 \right) k_{12} \sqrt{J_2} \cos \vartheta.$$

These equations determine the pulsation: the amplitude of the first harmonic term and the relative amplitude and phase of the second harmonic term. As $w_2 = 2w_1 - \vartheta$, the value of ϑ measures the difference of phase between the two harmonic terms; this quantity results from the equation:

$$\tan \vartheta = -\frac{\alpha_2 + 2\alpha_1}{2n_1 - n_2}.$$

$$H = n_1 J_1 + n_2 J_2 + n_3 J_3 + k_{123} \sqrt{J_1 J_2 J_3} \cos(w_1 + w_2 - w_3) + k_{13} J_1^{3/2} \sqrt{J_3} \cos(3w_1 - w_3) + l_{11} J_1^2.$$

Then the values of the variables $J_1, J_2, J_3, \vartheta, \vartheta' \equiv 3w_1 - w_3$ in the periodic solution are determined by the equations:

$$-2\alpha_1 J_1 + k_{123} \sqrt{J_1 J_2 J_3} \sin(\vartheta' - \vartheta) + 3k_{13} J_1^{3/2} \sqrt{J_3} \sin \vartheta' = 0$$

$$k_{123} \sqrt{J_1 J_2 J_3} \sin(\vartheta' - \vartheta) = 0$$

$$-2\alpha_3 J_3 - k_{123} \sqrt{J_1 J_2 J_3} \sin(\vartheta' - \vartheta) - k_{13} J_1^{3/2} \sqrt{J_3} \sin \vartheta' = 0$$

$$2n_1 - n_2 + k_{123} \sqrt{J_1 J_2 J_3} \left(\frac{1}{J_1} - \frac{1}{2J_2} \right) \cos(\vartheta' - \vartheta) + 3k_{13} \sqrt{J_1 J_3} \cos \vartheta' + 4l_{11} J_1 = 0$$

$$3n_1 - n_3 + k_{123} \sqrt{J_1 J_2 J_3} \left(\frac{3}{2J_1} - \frac{1}{2J_3} \right) \cos(\vartheta' - \vartheta) + k_{13} J_1^{3/2} \sqrt{J_3} \left(\frac{9}{2} \frac{1}{J_1} - \frac{1}{2J_3} \right) \cos \vartheta' + 6l_{11} J_1 = 0.$$

$$\text{Hence: } \alpha_1 J_1 + 3\alpha_3 J_3 = 0, \sin(\vartheta' - \vartheta) = 0.$$

Thus the phase-relation between first and second harmonic in the radial velocity is determined by that

between first and third harmonic.

Suppose the solution of the equation $\sin(\vartheta' - \vartheta) = 0$ to be $\vartheta' - \vartheta = 0$. Then the values of $J_1, J_2, J_3, \vartheta'$ result from the equations:

$$\alpha_1 J_1 + 3\alpha_3 J_3 = 0$$

$$2\alpha_3 \sqrt{J_3} + k_{13} J_1^{3/2} \sin \vartheta' = 0$$

$$2n_1 - n_2 + k_{123} \sqrt{J_1 J_2 J_3} \left(\frac{1}{J_1} - \frac{1}{2J_2} \right) + 3k_{13} \sqrt{J_1 J_3} \cos \vartheta' + 4l_{11} J_1 = 0$$

$$3n_1 - n_3 + k_{123} \sqrt{J_1 J_2 J_3} \left(\frac{3}{2J_1} - \frac{1}{2J_3} \right) + k_{13} J_1^{3/2} \sqrt{J_3} \left(\frac{9}{2} \frac{1}{J_1} - \frac{1}{2J_3} \right) \cos \vartheta' + 6l_{11} J_1 = 0.$$

Denote the quantity $-\frac{1}{3} \frac{\alpha_1}{\alpha_3}$ by μ^4 ; it is determined by the internal constitution of the star, especially as

regards the functional relation between the generation of energy and the temperature. Then the values of J_1, J_2, J_3 and ϑ' are determined by the equations:

The radial velocity is with a sufficient degree of approximation equal to:

$$-n_1 r_n \left\{ \sqrt{\frac{2J_1}{n_1}} s_1 \sin w_1 + 2 \sqrt{\frac{2J_2}{n_2}} s_2 \sin(2w_1 - \vartheta) \right\}.$$

The anomalous phase-relation between the two harmonic terms which exists in the radial velocity of ζ Geminorum would involve a value of $2n_1 - n_2$ nearly equal to zero. However, as, to attain a sufficiently large amplitude of the first harmonic, it is necessary to suppose the coefficient k_{12} extremely small, this interpretation is somewhat artificial; moreover, a small value of k_{12} would exclude the possibility of restricting the consideration to two degrees of freedom only.

3. If the consideration includes three degrees of freedom, the equations which determine the elements of the periodic solution become less simple. However, to conform to actual conditions, the analysis is restricted to the case: $k_{12} = 0$ and $\alpha_2 = 0$. If only those terms of the function H that are essential, if n_1, n_2 and n_3 are nearly proportional to the values 1, 2 and 3, are taken into account, the development of this function is given by the relation:

$$\frac{J_3}{J_1} = \mu^4, \quad k_{13} J_1 \sin \vartheta' = -2\alpha_3 \mu^2$$

$$2n_1 - n_2 + k_{123} \mu^2 J_1 \sqrt{\frac{J_2}{J_1}} \left(\frac{1}{J_1} - \frac{1}{2J_2} \right) + 3k_{13} J_1 \mu^2 \cos \vartheta' + 4l_{11} J_1 = 0$$

$$3n_1 - n_3 - \frac{1}{2} \frac{k_{123}}{\mu^2} (1 - 3\mu^4) \sqrt{\frac{J_2}{J_1}} - \frac{1}{2} \frac{k_{13} J_1}{\mu^2} (1 - 9\mu^4) \cos \vartheta' + 6l_{11} J_1 = 0.$$

The solution of these equations may be effected by multiplication of the fourth equation with an appropriate factor and addition of the result to the third equation so as to eliminate the $\sqrt{\frac{J_2}{J_1}}$ -term. The

factor required is $\frac{2\mu^4}{1-3\mu^4}$ and the resulting sum-equation:

$$2n_1 - n_2 + \frac{2\mu^4}{1-3\mu^4} (3n_1 - n_3) - \frac{k_{123} \mu^2 J_1}{2 \sqrt{\frac{J_2}{J_1}}} + k_{13} J_1 \frac{2\mu^2}{1-3\mu^4} \cos \vartheta' + \left(4 + \frac{12\mu^4}{1-3\mu^4} l_{11} \right) J_1 = 0.$$

As μ^2 must be a small quantity the influence of $\cos \vartheta'$ on the resulting $\sqrt{\frac{J_2}{J_1}}$ value may be neglected.

Then, with a sufficient degree of approximation, $\sqrt{\frac{J_2}{J_1}}$ is given by the relation:

$$\frac{\sqrt{\frac{J_2}{J_1}}}{J_1} = \frac{1}{2} \frac{k_{123} \mu^2}{2n_1 - n_2 + 4l_{11} J_1}.$$

If the value of $2n_1 - n_2$ is nearly zero, the value of $\sqrt{\frac{J_2}{J_1}}$ is nearly equal to $\frac{1}{8} \frac{k_{123} \mu^2}{l_{11}}$. Then the fourth equation approximates to a determination of $\cos \vartheta'$ by the equation:

$$\frac{1}{2} \frac{k_{13}}{\mu^2} J_1 \cos \vartheta' = 3n_1 - n_3 - \frac{1}{16} \frac{k_{123}}{l_{11}} + 6l_{11} J_1.$$

4. A more detailed investigation of the preceding solution requires some knowledge of the coefficients k_{12} , k_{13} , k_{123} , l_{11} . A computation of these quantities involves the functional relation between the internal energy and the variables ρ and n , density and entropy. This computation may be considerably simplified by development in terms of a quantity that will be supposed small and be taken into account only as far as the first power: the ratio of gas-pressure to radiation-pressure in the "normal" state of the star.

The adiabatic relation connects the temperature T and density ρ by the equation:

$$\left(\frac{T}{T_n} \right)^3 \frac{\rho_n}{\rho} = 1 + \frac{1}{8} \frac{p_n}{\Pi_n} \log \frac{\rho}{\rho_n} - \frac{1}{8} \frac{p_n}{\Pi_n} \log \frac{T^3 \rho_n}{T_n^3};$$

p_n and Π_n denote gas-pressure and radiation-pressure in the "normal" state of the star. As only terms of the first order in $\frac{p_n}{\Pi_n}$ are to be retained, this equation

is equivalent to the relation:

$$\left(\frac{T}{T_n} \right)^3 = \frac{\rho}{\rho_n} \left\{ 1 + \frac{1}{8} \frac{p_n}{\Pi_n} \log \frac{\rho}{\rho_n} \right\} = \left(\frac{\rho}{\rho_n} \right)^{1 + \frac{3}{4} \delta \gamma}; \quad \delta \gamma = \frac{1}{6} \frac{p_n}{\Pi_n}.$$

The internal energy per unit mass, U , then is equal to:

$$3 \frac{\Pi_n}{\rho_n} (1 + 3 \delta \gamma) \left(\frac{\rho}{\rho_n} \right)^{\frac{1}{3}} + \delta \gamma.$$

As the total pressure P_n equals $\Pi_n (1 + 6 \delta \gamma)$, this equation is equivalent to the relation:

$$U = 3 \frac{P_n}{\rho_n} \left(\frac{\rho}{\rho_n} \right)^{\frac{1}{3}} \left(1 - 3 \delta \gamma + \delta \gamma \log \frac{\rho}{\rho_n} \right).$$

The factor $\frac{\rho}{\rho_n}$ is connected with the development of $\frac{r}{r_n}$ by the relation:

$$\rho r^2 \frac{dr}{dr_n} = \rho_n r_n^2.$$

Hence the function $\frac{1}{4\pi} \int \left(U - f \frac{M_r}{r} \right) dM_r$, f denoting the constant of gravitation and M_r the mass inside the sphere with radius r , may be developed in ascending powers of the functions $C_1, C_2, \dots$. It is to be taken into account that, if $\delta \gamma = 0$, the function s_1 is constant; hence the variation of s_1 from centre to surface and the differential coefficient $\frac{ds_1}{dr_n}$ are quantities of the order of $\delta \gamma$.

The functions $s_i(r_n)$ are solutions of the differential equation:

$$\frac{d}{dr_n} \left(\gamma P_n r_n^4 \frac{ds_i}{dr_n} \right) + s_i r_n^3 \left(3 \frac{d\gamma}{dr_n} \frac{P_n}{r_n} - 4 \frac{dP_n}{dr_n} + n_i^2 \rho_n r_n \right) = 0;$$

the quantity γ being equal to $\frac{4}{3} + \delta \gamma$. Hence, the possibility exists of transforming some integrals needed in the development of the energy-function

in powers of $C_1, C_2, \dots$. These transformations are first to be considered.

Multiplication of the s_2 -equation with s_3 and the s_3 -equation with s_2 , integration of the sum of the

products from centre to surface of the star and reduction of the result by partial integration leads to the equation:

$$\int \gamma P_n r_n^4 \frac{ds_2}{dr_n} \frac{ds_3}{dr_n} dr_n = 3 \int s_2 s_3 r_n^3 \frac{d(\partial\gamma P_n)}{dr_n} dr_n;$$

as $\gamma = \frac{4}{3} + \partial\gamma$, partial integration of the right-hand member reduces this relation to the equation:

$$\int P_n r_n^4 \frac{ds_2}{dr_n} \frac{ds_3}{dr_n} dr_n = - \int P_n r_n^2 \partial\gamma \left\{ \frac{3}{4} r_n^2 \frac{ds_2}{dr_n} \frac{ds_3}{dr_n} + \frac{9}{4} r_n \frac{d(s_2 s_3)}{dr_n} + \frac{27}{4} s_2 s_3 \right\} dr_n.$$

The integral $\int P_n r_n^4 \frac{ds_1}{dr_n} \frac{ds_2}{dr_n} dr_n$ may be shown by partial integration to be equal to:

$$- \int s_2 \frac{d}{dr_n} \left(P_n r_n^4 \frac{ds_1}{dr_n} \right) dr_n.$$

Hence, substitution of the value of the last factor of the integrand from the differential equation that determines s_1 leads to the equations:

$$\int P_n r_n^4 \frac{ds_1}{dr_n} \frac{ds_2}{dr_n} dr_n = \frac{9}{4} \int s_1 s_2 \frac{d(\partial\gamma P_n)}{dr_n} r_n^3 dr_n = - \frac{27}{4} \int s_1 s_2 P_n r_n^2 \partial\gamma \left(1 + \frac{1}{3} \frac{r_n}{s_2} \frac{ds_2}{dr_n} \right) dr_n.$$

The coefficients of the development of the energy-function $\int \left(U - f \frac{M_r}{r} \right) dM_r$ needed in the analysis of

the preceding sections are those of $C_1^2 C_2$, $C_1^3 C_3$, $C_1 C_2 C_3$ and C_1^4 ; the values to the required order of approximation in $\partial\gamma$ are:

$$\text{coefficient of } C_1^2 C_2: \quad 18\pi \int s_1^2 s_2 \left(1 + \frac{1}{3} \frac{r_n}{s_2} \frac{ds_2}{dr_n} \right) P_n r_n^2 \partial\gamma dr_n;$$

$$\text{coefficient of } C_1^3 C_3: \quad -60\pi \int s_1^3 s_3 \left(1 + \frac{1}{3} \frac{r_n}{s_3} \frac{ds_3}{dr_n} \right) P_n r_n^2 \partial\gamma dr_n;$$

$$\begin{aligned} \text{coefficient of } C_1 C_2 C_3: \quad & 4\pi \int \left(-\frac{28}{9} r_n^3 \frac{ds_1}{dr_n} \frac{ds_2}{dr_n} \frac{ds_3}{dr_n} + \partial\gamma \left\{ -4 r_n^2 \frac{\partial s_1}{\partial\gamma} \frac{ds_2}{dr_n} \frac{ds_3}{dr_n} + 9 s_1 s_2 s_3 + 3 s_1 r_n \frac{d(s_2 s_3)}{dr_n} - \right. \right. \\ & \left. \left. - 4 s_1 r_n^2 \frac{ds_2}{dr_n} \frac{ds_3}{dr_n} \right\} \right) P_n r_n^2 dr_n; \end{aligned}$$

$$\text{coefficient of } C_1^4: \quad 39\pi \int s_1^4 P_n r_n^2 \partial\gamma dr_n.$$

The value of n_1^2 may be derived to the same degree of approximation in $\partial\gamma$ by consideration of the differential equation which determines the function $s_1(r_n)$. Integration of this equation from centre to surface leads to the relation:

$$\int s_1 r_n^3 \left(3 \frac{d\partial\gamma P_n}{dr_n} + n_1^2 \rho_n r_n \right) dr_n = 0,$$

taking into account the boundary values required for the function $s_1(r_n)$; partial integration reduces this equation to the form:

$$n_1^2 = 9 \frac{\int \partial\gamma P_n r_n^2 dr_n}{\int \rho_n r_n^4 dr_n}.$$

This equation, together with the condition to be satisfied by the multiplicative constant of integration in the function s_1 : $\int \rho_n r_n^4 s_1^2 dr_n = 1$, allows a reduction of the computed coefficients in the development of the energy-function to a form in which a numerical estimate of their values may be performed more readily. Denote the central value of a function s by $s(c)$ and use the n_1^2 -equation in adding as denominator the integral $\int \partial\gamma P_n r_n^2 dr_n$; then, to the required order of approximation in $\partial\gamma$, the values of the coefficients are:

$$\begin{aligned} \text{coefficient of } C_1^2 C_2: \quad & 2\pi n_1^2 s_2(c) \frac{\int \frac{s_2(r_n)}{s_2(c)} \left(1 + \frac{1}{3} \frac{r_n}{s_2} \frac{ds_2}{dr_n} \right) P_n r_n^2 \partial\gamma dr_n}{\int P_n r_n^2 \partial\gamma dr_n}; \\ \text{coefficient of } C_1^3 C_3: \quad & -60\pi n_1^2 s_3(c) \frac{\int \frac{s_3(r_n)}{s_3(c)} \left(1 + \frac{1}{3} \frac{r_n}{s_3} \frac{ds_3}{dr_n} \right) P_n r_n^2 \partial\gamma dr_n}{\int P_n r_n^2 \partial\gamma dr_n}; \\ \text{coefficient of } C_1 C_2 C_3: \quad & 4\pi n_1^2 \frac{\int \left(-\frac{28}{9} r_n^3 \frac{ds_1}{dr_n} \frac{ds_2}{dr_n} \frac{ds_3}{dr_n} + \partial\gamma \left\{ -4 r_n^2 \frac{\partial s_1}{\partial\gamma} \frac{ds_2}{dr_n} \frac{ds_3}{dr_n} + 9 s_1 s_2 s_3 + 3 s_1 r_n \frac{d(s_2 s_3)}{dr_n} - \right. \right. \right. \\ & \left. \left. \left. - 4 s_1 r_n^2 \frac{ds_2}{dr_n} \frac{ds_3}{dr_n} \right\} \right) P_n r_n^2 dr_n}{\int P_n r_n^2 \partial\gamma dr_n}; \\ \text{coefficient of } C_1^4: \quad & 39\pi n_1^2 \frac{\int s_1^4 P_n r_n^2 \partial\gamma dr_n}{\int P_n r_n^2 \partial\gamma dr_n}. \end{aligned}$$

$$\text{coefficient of } C_1^3 C_3: -\frac{20}{3} \pi n_1^2 s_1(c) s_3(c) \frac{\int \frac{s_3(r_n)}{s_3(c)} \left(1 + \frac{1}{3} \frac{r_n}{s_3} \frac{ds_3}{dr_n}\right) P_n r_n^2 \delta\gamma dr_n}{\int P_n r_n^2 \delta\gamma dr_n};$$

$$\begin{aligned} \text{coefficient of } C_1 C_2 C_3: & 4 \pi n_1^2 \frac{s_2(c) s_3(c)}{s_1(c)} \left[\int \left(-\frac{28}{81} r_n^3 d \frac{s_1(r_n)}{s_1(c)} d \frac{s_2(r_n)}{s_2(c)} d \frac{s_3(r_n)}{s_3(c)} + \delta\gamma \left\{ -\frac{4}{9} \frac{s_1(r_n)}{s_1(c)} r_n^2 d \frac{s_2(r_n)}{s_2(c)} d \frac{s_3(r_n)}{s_3(c)} + \right. \right. \right. \\ & \left. \left. \left. + \frac{s_1(r_n) s_2(r_n) s_3(r_n)}{s_1(c) s_2(c) s_3(c)} + \frac{1}{3} r_n d \frac{s_2(r_n) s_3(r_n)}{s_2(c) s_3(c)} - \frac{4}{9} r_n^2 d \frac{s_2(r_n)}{s_2(c)} d \frac{s_3(r_n)}{s_3(c)} \right\} \right) P_n r_n^2 dr_n \right] : \int P_n r_n^2 \delta\gamma dr_n; \end{aligned}$$

$$\text{coefficient of } C_1^4: \frac{13}{3} \pi n_1^2 s_1^2(c).$$

Each integral-quotient is the mean value of the s -combination involved with weight-factor $P_n r_n^2 \delta\gamma dr_n$; the central value of each s -combination is equal to unity. Denote the mean values respectively by the symbols σ_{12} , σ_{13} , σ_{123} .

The determination of the coefficients k_{12} , k_{13} , k_{123} , l_{11} involves the transformation of the products $\cos^2 w_1 \cos w_2$, $\cos^3 w_1 \cos w_3$, $\cos w_1 \cos w_2 \cos w_3$, $\cos^4 w_1$ and retention only of the terms needed:

$$\cos(2w_1 - w_2), \cos(3w_1 - w_3), \cos(w_1 + w_2 - w_3), \cos w_1.$$

As each C_i moreover introduces a factor $\sqrt{\frac{2 J_i}{n_i}}$ and transition to the development of the function H requires a division by the factor 4π , the values of the coefficients to the required order of approximation in $\delta\gamma$ easily follow. These values are:

$$\begin{aligned} k_{12} &= \frac{1}{4} s_2(c) \sqrt{\frac{2 n_1}{n_2}} \sqrt{\frac{2 n_1}{n_2}} \sigma_{12}, \quad k_{13} = -5 \frac{\sqrt{3}}{18} s_1(c) s_3(c) \sqrt{\frac{3 n_1}{n_3}} \sigma_{13}, \\ k_{123} &= \frac{\sqrt{3}}{6} \frac{s_2(c) s_3(c)}{s_1(c)} \sqrt{\frac{2 n_1}{n_2}} \sqrt{\frac{3 n_1}{n_3}} \sigma_{123}, \quad l_{11} = \frac{13}{8} s_1^2(c). \end{aligned}$$

5. The values of the coefficients of development in the function H , derived in the preceding section, allow a more detailed consideration of the periodic solutions of the differential equations. The range of possibilities being large, it is superfluous to start from a supposed internal constitution in the "normal" state of the star. A more real elucidation of the problem is reached by assigning plausible values to the quantities involved, so as to produce a relation between the several harmonics in the radial velocity that closely resembles the actual condition of ζ Geminorum and related stars.

The coefficient k_{12} will be supposed to be zero; to this end an increase of $\delta\gamma$ from centre to boundary is necessary.

The factors $\sqrt{\frac{2 n_1}{n_2}}$ and $\sqrt{\frac{3 n_1}{n_3}}$ will be approximated to by the value 1.

Some knowledge of the increase of the functions $\frac{s_2(r_n)}{s_2(c)}$ and $\frac{s_3(r_n)}{s_3(c)}$ from centre to boundary being necessary, the surface values of these functions are taken to be 10 and 50.

The ratio of the amplitude of the third harmonic to that of the first harmonic is adopted to be $\frac{1}{6}$; then the ratio of $\frac{2 s_1^2 J_1}{n_1}$ to $\frac{18 s_3^2 J_3}{n_3}$ is equal to 36, the s -values being taken at the surface of the star. Hence, neglecting the difference between $s_1(r_n)$ and $s_3(r_n)$ in the determination of the multiplicative constant of integration involved, the ratio of J_3 to J_1 has the value

$$1 : 3 \times 36 \times 2500.$$

Then the value of μ^2 is equal to $1 : 300 \sqrt{3}$.

The value of $\frac{\alpha_3}{n_3}$ is estimated at $10^{-3} 1$.

The adopted values are plausible estimates from existing theoretical computations relating to some supposed internal constitution of the star in the "normal" state of equilibrium, supplemented as regards μ^2 by the observational value of the ratio of the amplitudes of first and third harmonic in the radial velocity. Starting from these values the periodic

¹⁾ Cf. Miss H. A. KLUYVER's computation of the damping-constant in B.A.N. No. 313, p. 293.

solution derived in section 3 may now be considered more closely.

Firstly, the value of the amplitude of the second harmonic can be determined if $2n_1 - n_2$ is sufficiently close to zero. As $\sqrt{J_2} = \frac{1}{8} \frac{k_{123}}{l_{11}} \mu^2$, the value of the amplitude of the second harmonic in $\frac{r-r_n}{r_n}$ is equal to:

$$\frac{\sqrt{3}}{78} \mu^2 \sigma_{123} \frac{s_2(r_n)}{s_1(c)} \frac{s_2(c)}{s_1^2(c)} \frac{s_3(c)}{s_1^2(c)}.$$

Hence, the value at the surface of the star is equal to

$$\frac{10\sqrt{3}}{78} \mu^2 \sigma_{123};$$

the deviations of $\frac{s_2(c)}{s_1(c)}$ and $\frac{s_3(c)}{s_1(c)}$ from 1 are neglected.

Substitution of the adopted μ^2 -value gives the value $\frac{\sigma_{123}}{2340}$. This value must be compared with an obser-

vational value of $\frac{1}{20} \times \frac{1}{2} \times \frac{1}{4} = \frac{1}{160}$. Hence, a value of $|\sigma_{123}|$ equal to about 15 is necessary to effect an agreement with the theoretical value in the considered circumstances.

As σ_{123} is the mean value of a form quadratic in the functions $\frac{s_2(r_n)}{s_2(c)}$, $\frac{s_3(r_n)}{s_3(c)}$ and their derivatives, which are large in the outer part of the star, no objection to this agreement exists on this account.

Secondly, the value of the amplitude of the first harmonic in $\frac{r-r_n}{r_n}$, determined by the equation

$$2\alpha_3 \mu^2 + k_{13} J_1 \sin \vartheta' = 0,$$

is equal to

$$s_1 \sqrt{\frac{2J_1}{n_1}} = \sqrt{\frac{-4\alpha_3 n_3}{n_3 n_1 k_{13} \sin \vartheta'}} \frac{\mu^2 s_1^2}{k_{13} \sin \vartheta'}.$$

As the anomalous phase-relation between the first and second harmonic is the object of the analysis, that value of $|\sin \vartheta'|$ is to be considered which is, as regards order of magnitude, 1. Substitution of the value of k_{13} and the adopted values of the remaining

constants reduces the right-hand member to the value $\frac{0.007}{\sqrt{|\sigma_{13}|}}$; the quantity $\sigma_{13} \sin \vartheta'$ has been supposed to

be positive. A value of $|\sigma_{13}|$ equal to about $\frac{1}{50}$ would suffice to reduce this amplitude to the value of about $\frac{1}{20}$, required by observation. As $|\sigma_{13}|$ is a mean value of a function of $\frac{s_3(r_n)}{s_3(c)}$ and its derivative analogous to

the quantity $|\sigma_{12}|$ as the mean value relating to $\frac{s_2(r_n)}{s_2(c)}$, which value has been supposed to be zero, a value equal to $\frac{1}{50}$, though rather small, may be admitted.

However, it is to be remembered that the adopted values of the constants involved in the theoretical analysis are rather uncertain.

Thirdly, the damping-constant α_1 , is determined by the adopted values of α_3 and μ^2 . The alternative procedure to derive μ^2 from theoretical values of α_3 and α_1 would have been preferable from a strictly theoretical point of view; however, as the theory of the energy-generation in the star is involved in the determination of α_1 , the procedure adopted is more convincing; moreover the procedures are aequivalent. As $\alpha_1 + 3\alpha_3 \mu^4 = 0$, the adopted values of α_3 and μ^4 require a value of the ratio $\frac{\alpha_3}{\alpha_1}$ equal to -0.9×10^5 . A rather uncertain theoretical estimate of this ratio, derived from Miss H. A. KLUYVER's¹⁾ computation of the damping-constants leads to a value of about -10^5 . As a change in the value of μ^2 involved is quite possible, this agreement must not be considered to have large weight.

6. The periodic solution considered in section 3 has been developed supposing the quantity $2n_1 - n_2$ sufficiently small. Hence, the admissible values of $3n_1 - n_3$ are restricted to a small range, as thrice the first quantity diminished by twice the second quantity is nearly constant. The quantity $3n_1 - n_3$ is related to $\cos \vartheta'$ by the equation:

$$3n_1 - n_3 + k_{123} \mu^2 \sqrt{J_2} \left(\frac{3}{2} - \frac{1}{2\mu^4} \right) + k_{13} J_1 \mu^2 \left(\frac{9}{2} - \frac{1}{2\mu^4} \right) \cos \vartheta' + 6l_{11} J_1 = 0.$$

Hence, the relation between the values of $3n_1 - n_3$ and $\cos \vartheta'$ is determined by the value of $\frac{1}{2} k_{13} J_1 \mu^{-2}$.

This quantity, being equal to $\frac{n_1 k_{13} \mu^{-2}}{4 s_1^2}$ multiplied by the square of the amplitude of the first harmonic in $\frac{r-r_n}{r_n}$, approximately equals $-\frac{5}{32} n_1 \sigma_{13}$, the amplitude re-

ferred to being taken equal to $\frac{1}{20}$.

Hence, the small range of values of $\frac{3n_1 - n_3}{n_1}$ considered in the preceding sections admits a considerable range in the value of $\cos \vartheta'$, whence in the value of ϑ' .

¹⁾ l.c. p. 297.