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Anomalous Temperature Dependence of the Supercurrent Through a Chaotic Josephson Junction

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Abstract—We calculate the supercurrent through a Josephson junction consisting of a phase-coherent metal particle (quantum dot), weakly coupled to two superconductors. The classical motion in the quantum dot is assumed to be chaotic on time scales greater than the ergodic time τ_{erg}, which itself is much smaller than the mean dwell time τ_{dwell}. The excitation spectrum of the Josephson junction has a gap E_{gap}, which can be less than the gap Δ in the bulk superconductors. The average supercurrent is computed in the ergodic regime $\tau_{erg} \ll \hbar/\Delta$, using random-matrix theory, and in the non-ergodic regime $\tau_{erg} \gg \hbar/\Delta$, using a semiclassical relation between the supercurrent and dwell-time distribution. In contrast to conventional Josephson junctions, raising the temperature above the excitation gap does not necessarily lead to an exponential suppression of the supercurrent. Instead, we find a temperature regime between E_{gap} and Δ where the supercurrent decreases logarithmically with temperature. This anomalously weak temperature dependence is caused by long-range correlations in the excitation spectrum, which extend over an energy range $\hbar/\tau_{erg}$ greater than $E_{gap} \simeq \hbar/\tau_{dwell}$. A similar logarithmic temperature dependence of the supercurrent was discovered by Aslamazov, Larkin and Ovchinnikov in a Josephson junction consisting of a disordered metal between two tunnel barriers. © 1997 Elsevier Science Ltd

1. INTRODUCTION

The dissipationless flow of a current through a superconductor–normal-metal–superconductor (SNS) junction is a fundamental demonstration of the 'proximity effect': a normal metal borrows characteristic properties from a nearby superconductor. The energy gap Δ in the bulk induces a suppression of the density of states inside the normal metal near the Fermi level, depending on the phase difference ϕ between the superconductors. The resulting ϕ-dependence of the free energy F implies the flow of a current $I = (2e/\hbar)dF/d\phi$ in equilibrium. In contrast to the original Josephson effect in tunnel junctions, the separation of the superconductors in an SNS junction can be much greater than the superconducting coherence length. Recent experiments on mesoscopic Josephson junctions [1–6] have revived theoretical interest in this subject [7–9], which goes back to work by Kulik [10] and Aslamasov, Larkin and Ovchinnikov [11]. (For more references, see the review [12].)

In this paper, we consider the case when the normal region consists of a chaotic quantum dot. A quantum dot is a small metal particle within which the motion is phase coherent, weakly coupled to the superconductors by means of point contacts. We assume that the classical dynamics in the quantum dot are chaotic on time scales longer than the time τ_{erg} needed for ergodic exploration of the phase space of the quantum dot. (In order of magnitude, $\tau_{erg} \simeq L/v_F$ for a quantum dot of size L without impurities, where v_F is the Fermi velocity.) On energy scales smaller than $\hbar/\tau_{erg}$, the spectral statistics of a chaotic quantum dot is described by random-matrix theory [13, 14]. On larger energy scales, the non-ergodic dynamics on time scales below τ_{erg} become dominant [15]. The condition of weak coupling

We now perform a partial integration and close the integration contour in the upper half of the complex plane. The integrand has poles at the Matsubara frequencies $i\omega_n = (2n + 1)i\pi kT$. Summing over the residues, one finds that

$$I = -\frac{e}{\hbar} 2kT \frac{d}{d\phi} \sum_{n=0}^{\infty} \ln \det[1 - \mathscr{S}_A(i\omega_n)\mathscr{S}_N(i\omega_n)]. \tag{8}$$

This equation is the starting point for our evaluation of the average supercurrent through a chaotic Josephson junction.

3. SUPERCURRENT THROUGH A CHAOTIC JOSEPHSON JUNCTION

We consider the case when the normal region has a chaotic classical dynamics on time scales greater than the ergodic time τ_{erg}. In this section, we assume that $\tau_{\text{erg}} \ll \hbar/\Delta$, so that we may use random-matrix theory to evaluate the ensemble average of the supercurrent. We postpone to Section 5, a discussion of the regime $\tau_{\text{erg}} \gtrsim \hbar/\Delta$, in which the non-ergodic dynamics on time scales shorter than τ_{erg} starts to play a role. We assume that the normal metal is weakly coupled to the superconductors, so that the mean dwell time $\tau_{\text{dwell}} \gg \tau_{\text{erg}}$. No assumption is made regarding the relative magnitudes of τ_{dwell} and $\hbar/\Delta$.

We use a relationship [24, 25] between the scattering matrix S of the normal metal and its Hamiltonian H

$$S(\varepsilon) = 1 - 2\pi i W^{\dagger}(\varepsilon - H + i\pi WW^{\dagger})^{-1}W. \tag{9}$$

The Hamiltonian H (representing the isolated normal metal region) is taken from the Gaussian ensemble of random-matrix theory [26],

$$P(H) \propto \exp\left(-\frac{1}{4}M\lambda^{-2} \operatorname{tr} H^2\right), \tag{10}$$

where M is the dimension of H (taken to infinity at the end) and λ is a parameter that determines the average level spacing $\delta = \lambda\pi/2M$ of the excitation spectrum in the normal region. (This spacing δ is half the level spacing of H, because it combines electron and hole levels together.) The matrix H is real and symmetric. The coupling matrix W is an $M \times N$ matrix [27, 28] with elements

$$W_{mn} = \frac{1}{\pi}\delta_{mn}(2M\delta)^{1/2}(2\Gamma_n^{-1} - 1 - 2\Gamma_n^{-1}\sqrt{1 - \Gamma_n})^{1/2}. \tag{11}$$

Here Γ_n is the transmission probability of mode n in the contacts to the superconductor. For ballistic contacts $\Gamma_n = 1$, while $\Gamma_n \ll 1$ for tunneling contacts.

We now substitute from equation (9) into equation (1) and then substitute $\mathscr{S}_N$ into equation (8) for the supercurrent. Using also equations (2a) and (2b), we find after some straightforward matrix algebra that

$$I = -\frac{2e}{\hbar} 2kT \frac{d}{d\phi} \sum_{n=0}^{\infty} \ln \det[i\omega_n - \mathscr{H} + \mathscr{W}(i\omega_n)], \tag{12}$$

where we have introduced the $2M \times 2M$ matrices

$$\mathscr{H} = \begin{pmatrix} H & 0 \\ 0 & -H \end{pmatrix}, \tag{13a}$$

$$\mathscr{W}(\varepsilon) = \frac{\pi\Delta}{\sqrt{\Delta^2 - \varepsilon^2}} \begin{pmatrix} (\varepsilon/\Delta)WW^{\mathrm{T}} & We^{\imath\Phi}W^{\mathrm{T}} \\ We^{-\imath\Phi}W^{\mathrm{T}} & (\varepsilon/\Delta)WW^{\mathrm{T}} \end{pmatrix}. \tag{13b}$$

The matrix $\mathscr{H} - \mathscr{W}(\varepsilon)$ is the effective Hamiltonian of Refs [18, 29] (where the regime $\varepsilon \ll \Delta$ was considered, in which the ε-dependence of $\mathscr{W}(\varepsilon)$ can be neglected).

We define the $2M \times 2M$ Green function

$$\mathscr{G}(\varepsilon) = [\varepsilon - \mathscr{H} + \mathscr{W}(\varepsilon)]^{-1}, \tag{14}$$

which determines the density of states according to

$$\rho(\varepsilon) = -\pi^{-1} \operatorname{Im} \operatorname{tr} \mathscr{G}(\varepsilon + i0). \tag{15}$$

Equation (15) is equivalent to equation (4). The expression for the supercurrent in terms of $G(\varepsilon)$ is

$$I = \frac{2e}{\hbar} 2kT \frac{\mathrm{d}}{\mathrm{d}\phi} \sum_{n=0}^{\infty} \ln \det \mathscr{G}(\imath\omega_n)$$

$$= -\frac{2e}{\hbar} 2kT \sum_{n=0}^{\infty} \operatorname{tr} \mathscr{G}(\imath\omega_n) \frac{\mathrm{d}}{\mathrm{d}\phi} \mathscr{W}(\imath\omega_n). \tag{16}$$

The average supercurrent follows from the average Green function $\langle \mathscr{G}(\varepsilon) \rangle$, since $\mathscr{W}$ is a fixed matrix. The average over the random Hamiltonian H (determining $\mathscr{G}$) is done with the help of the diagrammatic technique of Refs [30, 31]. We consider the regime $M, N, |\varepsilon|/\delta \gg 1$, in which only planar diagrams need to be considered. Resummation of these diagrams leads to a self-consistency equation which is similar to Pastur's equation [32]

$$\langle \mathscr{G}(\varepsilon) \rangle = [\varepsilon + \mathscr{W}(\varepsilon) - (\lambda^2/M)\mathscr{P}(\varepsilon) \otimes \mathbb{1}_M]^{-1}. \tag{17}$$

The symbol $\otimes$ indicates the direct product between the $M \times M$ unit matrix $\mathbb{1}_M$ and the 2×2 matrix

$$\mathscr{P} = \begin{pmatrix} \langle \operatorname{tr} \mathscr{G}^{ee} \rangle & -\langle \operatorname{tr} \mathscr{G}^{eh} \rangle \\ -\langle \operatorname{tr} \mathscr{G}^{he} \rangle & \langle \operatorname{tr} \mathscr{G}^{hh} \rangle \end{pmatrix}. \tag{18}$$

We seek the solution of equations (17) and (18) which satisfies

$$\varepsilon\mathscr{G}(\varepsilon) \to \mathbb{1}_{2M} \text{ if } |\varepsilon| \gg \lambda. \tag{19}$$

It is convenient to define a self energy

$$\Sigma = \begin{pmatrix} \Sigma^{ee} & \Sigma^{eh} \\ \Sigma^{he} & \Sigma^{hh} \end{pmatrix} = \frac{\lambda}{M} \begin{pmatrix} \operatorname{tr} \mathscr{G}^{ee} & \operatorname{tr} \mathscr{G}^{eh} \\ \operatorname{tr} \mathscr{G}^{he} & \operatorname{tr} \mathscr{G}^{hh} \end{pmatrix}. \tag{20}$$

Equations (17)–(20) contain a closed set of equations from which $\langle \Sigma \rangle$ can be determined.

We are interested in the limit $M \to \infty$, $\lambda \to \infty$, keeping N and $\varepsilon/\delta = 2\varepsilon M/\lambda\pi$ fixed. In this limit, the equations for $\langle\Sigma\rangle$ become

$$\langle\Sigma^{ee}\rangle = \langle\Sigma^{hh}\rangle, \quad \langle\Sigma^{eh}\rangle\langle\Sigma^{he}\rangle - \langle\Sigma^{ee}\rangle^2 = 1, \tag{21a}$$

$$\frac{\pi\varepsilon}{2\delta}\langle\Sigma^{eh}\rangle + \sum_{j=1}^{N} K_j(\varepsilon\langle\Sigma^{eh}\rangle + \Delta e^{i\Phi_{jj}}\langle\Sigma^{ee}\rangle) = 0, \tag{21b}$$

$$\frac{\pi\varepsilon}{2\delta}\langle\Sigma^{he}\rangle + \sum_{j=1}^{N} K_j(\varepsilon\langle\Sigma^{he}\rangle + \Delta e^{-i\Phi_{jj}}\langle\Sigma^{ee}\rangle) = 0. \tag{21c}$$

The function $K_j(\varepsilon)$ is defined through

$$\Gamma_j/K_j = (4 - 2\Gamma_j)\sqrt{\Delta^2 - \varepsilon^2} + \Gamma_j(\Delta e^{i\Phi_{jj}}\langle\Sigma^{eh}\rangle + \Delta e^{-i\Phi_{jj}}\langle\Sigma^{he}\rangle + 2\varepsilon\langle\Sigma^{ee}\rangle). \tag{21d}$$

(We have substituted from equation (11) for the matrix W.) The boundary condition in equation (19) becomes ineffective in the limit $\lambda \to \infty$. Instead, we seek the solution of equations (21) with $\langle\Sigma^{ee}\rangle = \langle\Sigma^{hh}\rangle \to -i$ for $\varepsilon \to i\infty$, corresponding to a constant density of states $\rho(\varepsilon) = 1/\delta$ for $|\varepsilon| \gg \Delta$. From $\langle\Sigma\rangle$, we find $\langle\mathcal{G}\rangle$ and hence the ensemble averaged supercurrent $\langle I\rangle$ is found to be

$$\langle I\rangle = \frac{2e}{i\hbar} kT\Delta \sum_{n=0}^{\infty} \sum_{j=1}^{N} \mathrm{sign}(\Phi_{jj})K_j[e^{i\Phi_{jj}}\langle\Sigma^{he}(i\omega_n)\rangle - e^{-i\Phi_{jj}}\langle\Sigma^{eh}(i\omega_n)\rangle]. \tag{22}$$

Equations (21) and (22) contain all the information needed to determine the average supercurrent through a chaotic Josephson junction.

An analytic solution of equations (21) is possible in certain limiting cases. Here we discuss the case of high tunnel barriers, $\Gamma_j \ll 1$, for all j. Then we may approximate $K_j = (1/4)\Gamma_j(\Delta^2 - \varepsilon^2)^{-1/2}$ and find

$$\langle\Sigma^{ee}\rangle = \langle\Sigma^{hh}\rangle = -\varepsilon(\sqrt{\Delta^2 - \varepsilon^2} + E_T)[|\Omega|^2 \Delta^2 - \varepsilon^2(\sqrt{\Delta^2 - \varepsilon^2} + E_T)^2]^{-1/2}, \tag{23a}$$

$$\langle\Sigma^{eh}\rangle = \Omega\Delta[|\Omega|^2 \Delta^2 - \varepsilon^2(\sqrt{\Delta^2 - \varepsilon^2} + E_T)^2]^{-1/2}, \tag{23b}$$

$$\langle\Sigma^{he}\rangle = \Omega^*\Delta[|\Omega|^2 \Delta^2 - \varepsilon^2(\sqrt{\Delta^2 - \varepsilon^2} + E_T)^2]^{-1/2}, \tag{23c}$$

$$\Omega(\phi) = \frac{\delta}{2\pi}\sum_{j=1}^{N}\Gamma_j e^{i\Phi_{jj}}, \quad E_T = \frac{\delta}{2\pi}\sum_{j=1}^{N}\Gamma_j = \Omega(0). \tag{23d}$$

The energy E_T is related to the mean dwell-time through $E_T = \hbar/2\tau_{\mathrm{dwell}}$. (The dwell-time is defined as $\tau_{\mathrm{dwell}} = (\hbar/iN)(\partial/\partial\varepsilon)(\ln\det S(\varepsilon))$, see Ref. [33].) The excitation gap in the spectrum of the Josephson junction is of order $|\Omega(\phi)|$ when $\tau_{\mathrm{dwell}} \gg \hbar/\Delta$ [18]. Substitution from the above equations into equation (22) yields the supercurrent

$$\langle I\rangle = \frac{2\pi}{e}kTGE_T \sum_{n=0}^{\infty} \frac{\Delta^2 \sin\phi}{\sqrt{\Delta^2 + \omega_n^2}}[|\Omega|^2 \Delta^2 + \omega_n^2(\sqrt{\Delta^2 + \omega_n^2} + E_T)^2]^{-1/2}, \tag{24}$$

where

$$G = \frac{2e^2}{h}\frac{\sum_{i=1}^{N_1}\sum_{j=N_1+1}^{N}\Gamma_i\Gamma_j}{\sum_{k=1}^{N}\Gamma_k} \tag{25}$$

is the conductance of the Josephson junction when the superconductors are in the normal state.

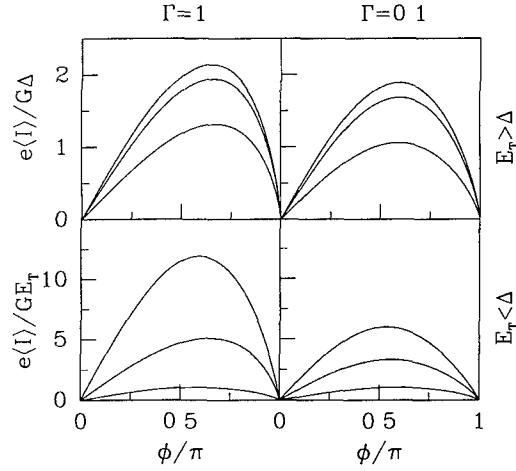For arbitrary transmission probabilities Γ_j, it is necessary to solve equations (21)

Fig. 2. Average supercurrent at zero temperature, computed from equations (21) and (22) for the case $N_1 = N_2 = N/2$, $\Gamma_j = \Gamma$, for all j. Left panels: $\Gamma = 1$; right panels $\Gamma = 0.1$. The upper panels show $\langle I \rangle$ in the short dwell-time regime for $E_T/\Delta = 1$ (bottom curve), 10 and 100 (top curve). The bottom panels show $\langle I \rangle$ in the long dwell-time regime for $E_T/\Delta = 0.01$ (top curve), 0.1 and 1 (bottom curve). The conductance $G = (2e^2/h)N\Gamma/4$ and the Thouless energy $E_1 = N\Gamma\delta/2\pi$. Notice that $\langle I \rangle$ is in units of $G\Delta/e$ in the top panels and in units of GE_T/e in the bottom panels.

numerically. We have studied the case when both point contacts have an equal number of modes ($N_1 = N_2 = N/2$), and that all transmission probabilities are equal ($\Gamma_j = \Gamma$, for all j). The average supercurrent at zero temperature for $\Gamma = 0.1$ and $\Gamma = 1$ is shown in Fig. 2.

4. ERGODIC REGIME

The general result in equation (22) describes the supercurrent in the ergodic regime $\tau_{erg} \ll \hbar/\Delta$. Within this regime, we can distinguish two further regimes, depending on whether the dwell time $\tau_{dwell} = \hbar/2E_T$ is short or long compared with $\hbar/\Delta$. We discuss these two regimes in two separate subsections.

4.1. Short dwell-time regime

In the short dwell-time regime (when $\tau_{dwell} \ll \hbar/\Delta$ or, equivalently, $E_T \gg \Delta$), the magnitude of the critical current $I_c = \max_\phi I(\phi)$ is set by the energy gap Δ in the bulk superconductor: $I_c \simeq G\Delta/e$ at zero temperature. The temperature dependence of I_c can be neglected as long as $kT \ll \Delta$, i.e. for temperatures T much less than the critical temperature T_c of the bulk superconductor. In the case of tunneling contacts, evaluation of equation (24) with $E_T \gg \Delta \gg kT$ yields

$$\langle I \rangle = \frac{G\Delta}{e} \frac{K(\gamma \sin(\phi/2))\sin\phi}{\sqrt{1 - \gamma^2 \sin^2(\phi/2)}}. \tag{26}$$

The conductance G was defined in equation (25), the function K is the complete elliptic integral of the first kind, and we abbreviated

$$\gamma = 2\left(\sum_{i=1}^{N_1} \sum_{j=N_1+1}^{N} \Gamma_i\Gamma_j\right)^{1/2} \left(\sum_{k=1}^{N} \Gamma_k\right)^{-1}. \tag{27}$$

The parameter γ equals 1 for two identical point contacts with mode-independent tunnel probabilities.

The result in equation (26) could also have been obtained directly from the general formula for the zero-temperature supercurrent in the short dwell-time regime [19]

$$\langle I \rangle = \frac{e\Delta}{2\hbar} \int_0^1 dt\, \rho(t) \frac{t \sin\phi}{\sqrt{1 - t \sin^2(\phi/2)}}, \tag{28}$$

which relates $\langle I \rangle$ to an integral over the transmission eigenvalues t of the junction in the normal state, with density $\rho(t)$. The transmission eigenvalues density for a chaotic cavity with two identical tunneling contacts ($N_1 = N_2 = N/2$, $\Gamma_j = \Gamma_{j+N/2}$, for $j = 1,2,...,N/2$) is given [34] by

$$\rho(t) = \sum_{j=1}^{N/2} \frac{\Gamma_j(2 - \Gamma_j)}{\pi(\Gamma_j^2 - 4\Gamma_j t + 4t)\sqrt{t(1 - t)}}. \tag{29}$$

One can check that equation (28) agrees with equation (26) with $\gamma = 1$ if $\Gamma_j \ll 1$, for all j. For two identical ballistic point contacts ($N_1 = N_2 = N/2$, $\Gamma_j = 1$, for all j), the density is $\rho(t) = N(2\pi)^{-1}[t(1 - t)]^{-1/2}$ [35, 36], which yields

$$\langle I \rangle = \frac{2e\Delta}{i\pi\hbar \sin(\phi/2)} GE(i\,\text{arsinh}[\tan(\phi/2)], i\cot\text{an}(\phi/2)). \tag{30}$$

Here $G = N/4$ and E is the elliptic integral of the second kind.

4.2. Long dwell-time regime

In the long dwell-time regime (when $\tau_{\text{dwell}} \gg \hbar/\Delta$ or, equivalently, $E_T \ll \Delta$), the magnitude of the critical current is set by the Thouless energy, but retains a logarithmic dependence on Δ, so that $I_c \simeq (GE_T/e)\ln(\Delta/E_T)$. The temperature dependence of I_c can be neglected so long as $kT \ll E_T$. If $kT \gg E_T$ (but still $T \ll T_c$) the critical current decreases, though only logarithmically, so that $I_c \simeq (GE_T/e)\ln(\Delta/kT)$. For the case of tunneling contacts, we find from equation (24) the expressions

$$\langle I \rangle = \frac{GE_T}{e} \sin\phi \ln\left(\frac{2\Delta/E_T}{\sqrt{1 - \gamma^2 \sin^2(\phi/2)}}\right), \quad kT \ll E_T, \tag{31a}$$

$$\langle I \rangle = \frac{GE_T}{e} \sin\phi \left[\ln\left(\frac{2\Delta}{\pi kT}\right) + c_{\text{Euler}}\right], \quad kT \gg E_T,\, T \ll T_c, \tag{31b}$$

where $c_{\text{Euler}} \approx 0.58$ is Euler's constant. For ballistic contacts, we do not have such simple expressions as these, but the parametric dependence of I on Δ, E_T and kT is the same as for tunneling contacts (see Fig. 2).

The logarithmic dependence on Δ of the supercurrent when $E_T \ll \Delta$ arises because the Thouless energy E_T is not an effective cutoff for the Matsubara sum in equation (8) or, equivalently, for the energy integration of equation (7). Spectral correlations exist up to energies of order $\hbar/\tau_{\text{erg}} \gg E_T$. These long-range spectral correlations are responsible for the weak decay $\Sigma^{eh} \propto 1/\omega$ of the self-energy and $\rho - \delta^{-1} \propto 1/\varepsilon^2$ of the density of states. The superconducting energy gap Δ has to serve as a cutoff energy for the otherwise logarithmically divergent equations (7) and (8), which explains the logarithm $\ln\Delta$ in equations (31a) and (31b).

5. NON-ERGODIC REGIME

When $\tau_{\text{dwell}} \gtrsim \hbar/\Delta$, a random-matrix theory of the Josephson effect is no longer possible, because the non-ergodic dynamics on time scales shorter than τ_{erg} start to play a role. To

study the average supercurrent in this non-ergodic regime, we return to equation (8). On substitution of equations (1) and (2), we obtain an expression for I in terms of the scattering matrix S of the normal region

$$I = \frac{2\pi}{e}kT\sum_{n=0}^{\infty}F(\omega_n),$$

(32a)

$$F(\omega) = -\frac{4e^2}{h}\frac{\mathrm{d}}{\mathrm{d}\phi}\,\mathrm{tr}\,\ln[1-\alpha(\mathrm{i}\omega)^2 S(\mathrm{i}\omega)\mathrm{e}^{\mathrm{i}\Phi}S^*(-\mathrm{i}\omega)\mathrm{e}^{-\mathrm{i}\Phi}].$$

(32b)

The evaluation of the scattering matrix at the imaginary energy $\mathrm{i}\omega_n$ is equivalent to the evaluation of the scattering matrix at the Fermi level in the presence of absorption, with rate $1/\tau_{\mathrm{abs}} = 2\omega_n/\hbar = (2n+1)2\pi kT/\hbar$. We first consider temperatures $kT \gg E_{\mathrm{T}}$. Since $\omega_n \gg E_{\mathrm{T}} = \hbar/2\tau_{\mathrm{dwell}}$ for all n in this high temperature regime, absorption is strong, $\tau_{\mathrm{abs}} \ll \tau_{\mathrm{dwell}}$. The formal correspondence between Matsubara frequency and absorption rate helps to understand that, to lowest order in $\tau_{\mathrm{abs}}/\tau_{\mathrm{dwell}} = E_{\mathrm{T}}/\omega$, the diagonal elements of $S(\mathrm{i}\omega)$ are given by the reflection amplitudes of the tunnel barriers in the contacts, $S_{jj} = (1-\Gamma_j)^{1/2}$, while the off-diagonal elements satisfy

$$\langle|S_{ij}(\mathrm{i}\omega)|^2\rangle = \frac{\Gamma_i\Gamma_j}{\sum_{k=1}^{N}\Gamma_k}\int_0^{\infty}\mathrm{d}\tau P_{ij}(\tau)\exp(-2\omega\tau/\hbar),\quad i\neq j.$$

(33)

The function P_{ij} is the classical distribution of dwell times for particles that enter the quantum dot through mode j and exit through mode i. Because of the smallness of $\langle|S_{ij}(\mathrm{i}\omega)|^2\rangle = O(E_{\mathrm{T}}/\omega)$, it is sufficient to keep only the lowest order term in an expansion of $\langle F(\omega)\rangle$ in the off-diagonal scattering matrix elements,

$$\langle F(\omega)\rangle = -\frac{4e^2}{h}\sum_{i=1}^{N_1}\sum_{j=N_1+1}^{N}\frac{2\alpha(\mathrm{i}\omega)^2\langle|S(\mathrm{i}\omega)_{ij}|^2\rangle\sin\phi}{[1-\alpha(\mathrm{i}\omega)^2(1-\Gamma_i)][1-\alpha(\mathrm{i}\omega)^2(1-\Gamma_j)]}.$$

(34)

Equations (32a), (32b), (33) and (34) permit a semiclassical calculation of the average supercurrent in the non-ergodic regime for temperatures $kT \gg E_{\mathrm{T}}$, where random-matrix theory fails. The only input required is the classical distribution of dwell times.

On time scales greater than τ_{erg}, the distribution P_{ij} is exponential with the same mean dwell time $\tau_{\mathrm{dwell}} = \hbar/2E_{\mathrm{T}}$, for all i,j:

$$P_{ij}(\tau) = \frac{2E_{\mathrm{T}}}{\hbar}\exp(-2E_{\mathrm{T}}\tau/\hbar).$$

(35)

The non-chaotic dynamics on time scales shorter than τ_{erg} enter through a non-universal form of P_{ij} for $\tau \lesssim \tau_{\mathrm{erg}}$. We consider the case of a ballistic dynamics (size L of the normal region much less than the mean free path ℓ). The ergodic time $\tau_{\mathrm{erg}} \simeq L/v_{\mathrm{F}}$ is then a lower cutoff on P_{ij}, since the minimum dwell time L/v_{F} is the time needed to cross the system ballistically. A qualitative estimate of $\langle I\rangle$ is obtained if we set $P_{ij}(\tau) = 0$ for $\tau < L/v_{\mathrm{F}}$ and approximate it by equation (35) for larger times. Substitution of this dwell-time distribution into equation (33) gives

$$\langle|S_{ij}(\mathrm{i}\omega)|^2\rangle = \frac{\Gamma_i\Gamma_j}{\sum_{k=1}^{N}\Gamma_k}\frac{E_{\mathrm{T}}}{\omega}\exp(-2\omega L/\hbar v_{\mathrm{F}}),\quad \omega \gg E_{\mathrm{T}},\quad i\neq j.$$

(36)

We next compute $\langle F(\omega)\rangle$ from equation (34), replacing $\alpha(i\omega)$ by its value $-i$ for $\omega \ll \Delta$. The result is

$$\langle F(\omega)\rangle = \frac{\tilde{G}E_T}{\omega}\exp(-2\omega L/\hbar v_F)\sin\phi, \tag{37a}$$

$$\tilde{G} = \frac{2e^2}{h}\frac{\sum_{i=1}^{N_1}\sum_{j=N_1+1}^{N}\Gamma_i\Gamma_j(2-\Gamma_i)^{-1}(2-\Gamma_j)^{-1}}{\sum_{k=1}^{N}\Gamma_k}. \tag{37b}$$

Notice that $\tilde{G} = G$ for the case of high tunnel barriers ($\Gamma_j \ll 1$, for all j). We can now calculate the average supercurrent from equations (32a) and (32b). Equation (37a) is valid for $E_T \ll \omega \ll \Delta$, $E_T \ll \hbar v_F/L \ll \Delta$, and is sufficient to determine the supercurrent in the temperature range $E_T \ll kT \ll \Delta$. Substitution from equation (37) into equation (32) gives

$$\langle I\rangle = \frac{\tilde{G}E_T}{e}\sin\phi\,\ln(\hbar v_F/\pi kTL), \quad E_T \ll kT \ll \hbar v_F/L \ll \Delta, \tag{38a}$$

$$\langle I\rangle = \frac{\tilde{G}E_T}{e}2\sin\phi\,\exp(-2\pi kTL/\hbar v_F), \quad E_T \ll \hbar v_F/L \ll kT \ll \Delta. \tag{38b}$$

Equations (38a) and (38b) have the same temperature dependence as the result of Ref. [11] for the double-barrier SNS junction.

We now turn to low temperatures $kT \lesssim E_T$. In this temperature regime, the Matsubara sum in equation (32) contains terms with $\omega_n \lesssim E_T$, for which the off-diagonal scattering matrix elements $S_{ij}(i\omega_n)$ are not small and the approximation of equation (34) is no longer valid. However, since $E_T \ll \hbar/\tau_{\text{erg}}$, these Matsubara frequencies are well within the validity range of random-matrix theory. Therefore, we can use the results of Section 3 to compute $\langle F(\omega)\rangle$ for $\omega \lesssim E_T$ and the semiclassical formula in equation (34) for $\omega \gtrsim E_T$. These two results match at $\omega \simeq E_T$, because the validity range $\omega \ll \hbar/\tau_{\text{erg}}$ of random-matrix theory and the validity range $\omega \gg E_T$ of the semiclassical theory overlap (assuming $\tau_{\text{erg}} \ll \tau_{\text{dwell}} = \hbar/2E_T$).

For the case of high tunnel barriers, random-matrix theory gives [cf. equation (24)]

$$\langle F(\omega)\rangle = \frac{GE_T\sin\phi}{\sqrt{|\Omega(\phi)|^2 + \omega^2}}, \quad \omega \ll \hbar v_F/L \ll \Delta, \tag{39}$$

while the semiclassical formula of equation (34) gives

$$\langle F(\omega)\rangle = \frac{GE_T\sin\phi}{\omega}\exp(-2\omega L/\hbar v_F), \quad E_T \ll \omega \ll \Delta. \tag{40}$$

(The function $\Omega(\phi)$ was defined in equation (23d).) The two results in equations (39) and (40) have a common range of validity $E_T \ll \omega \ll \hbar v_F/L$ within which they can be matched. The result is a formula valid for all $\omega \ll \Delta$, for a ballistic quantum dot with high tunnel barriers:

$$\langle F(\omega)\rangle = \frac{GE_T\sin\phi}{\sqrt{|\Omega(\phi)|^2 + \omega^2}}\exp(-2\omega L/\hbar v_F), \quad \omega \ll \Delta. \tag{41}$$

After substitution from equation (41) into equation (32) we obtain the average supercurrent in the low-temperature regime

$$\langle I\rangle = \frac{GE_T}{e}\sin\phi\left[\ln\left(\frac{\hbar v_F}{LE_T\sqrt{1-\gamma^2\sin^2(\phi/2)}}\right) - c_{\text{Euler}}\right], \quad kT \ll E_T \ll \hbar v_F/L \ll \Delta. \tag{42}$$

Table 1 Parametric dependence of the zero-temperature critical current I_c on the three time scales τ_{dwell}, τ_{erg} and $\hbar/\Delta$ The Thouless energy $E_T = \hbar/2\tau_{dwell}$

$I_c \times e/G$	Weak coupling ($\tau_{dwell} \gg \tau_{erg}$)		Strong coupling ($\tau_{dwell} \simeq \tau_{erg}$)
	Ergodic ($\tau_{erg} \ll \hbar/\Delta$)	Non-ergodic ($\tau_{erg} \gg \hbar/\Delta$)	
Short dwell time ($\tau_{dwell} \ll \hbar/\Delta$)	Δ	—	Δ
Long dwell time ($\tau_{dwell} \gg \hbar/\Delta$)	$E_T \ln(\Delta/E_T)$	$E_1 \ln(\hbar/E_T \tau_{erg})$	E_T

(The parameter γ was defined in equation (27).) The results in equations (38) and (42) cover the entire temperature range below T_c.

6. CONCLUSIONS

In Table 1, we summarize the parametric dependence of the critical current at zero temperature on the three time scales τ_{dwell}, τ_{erg} and $\hbar/\Delta$. We show the three new regimes for a weakly coupled normal region ($\tau_{dwell} \gg \tau_{erg}$), and have included for comparison also the two old regimes for a strongly coupled normal region ($\tau_{dwell} \simeq \tau_{erg}$). Apart from a logarithmic factor, the critical current is given by $I_c \simeq (G/e)\min(\hbar/\tau_{dwell}, \Delta)$ in each of the five regimes. There is an additional logarithmic dependence on $\min(\tau_{dwell}/\tau_{erg}, \tau_{dwell}\Delta/\hbar)$ in two of the three new regimes. Upon raising the temperature, the critical current is suppressed at a characteristic temperature given by $\min(\hbar/\tau_{erg}, \Delta)$. At lower temperatures, I_c has a logarithmic T-dependence so long as $T \gtrsim \hbar/\tau_{dwell}$ and becomes T-independent at still lower T.

In this work, we did not address the sample-to-sample fluctuations of the supercurrent, but calculated only the ensemble average. For strongly coupled diffusive Josephson junctions ($\tau_{dwell} \simeq \tau_{erg}$, $L \gg \ell$), the root-mean-squared of the fluctuations is a factor e^2/hG smaller than the average critical current [19, 37]. Preliminary calculations in the ergodic regime indicate that the same is true for weakly coupled Josephson junctions ($\tau_{dwell} \gg \tau_{erg}$), i.e. the r.m.s. fluctuations of I_c are given by the entries in Table 1 multiplied by e/h.

We close with a remark on quantum dots with an integrable classical dynamics, such as rectangular or circular ballistic cavities. For energies $\varepsilon \lesssim E_T$, the excitation spectrum of an integrable Josephson junction is quite different from its chaotic counterpart [18]. The density of states $\rho(\varepsilon)$ of a chaotic cavity in contact with a superconductor shows a gap of size E_T around the Fermi level $\varepsilon = 0$, while $\rho(\varepsilon)$ vanishes linearly when $\varepsilon \to 0$ for a rectangular or circular cavity It is an interesting open problem to compute the supercurrent through an integrable cavity and compare with the results for the chaotic case obtained in this paper.

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