# Production and detection of three-qubit entanglement in the Fermi sea

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# Production and detection of three-qubit entanglement in the Fermi sea

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Building on a previous proposal for the entanglement of electron-hole pairs in the Fermi sea, we show how three qubits can be entangled without using electron-electron interactions. As in the two-qubit case, this electronic scheme works even if the sources are in (local) thermal equilibrium—in contrast to the photonic analog. The three qubits are represented by four edge-channel excitations in the quantum Hall effect (two hole excitations plus two electron excitations with identical channel index). The entangler consists of an adiabatic point contact flanked by a pair of tunneling point contacts. The irreducible three-qubit entanglement is characterized by the tangle, which is expressed in terms of the transmission matrices of the tunneling point contacts. The maximally entangled Greenberger-Horne-Zeilinger (GHZ) state is obtained for channel-independent tunnel probabilities. We show how low-frequency noise measurements can be used to determine an upper and lower bound to the tangle. The bounds become tighter the closer the electron-hole state is to the GHZ state.

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## I. INTRODUCTION

This paper continues the research program of Ref. 1: To develop methods for quantum entanglement and spatial separation of quasiparticle excitations in the Fermi sea, with the special property that they do not require electron-electron interactions. Interaction-free entanglement schemes provide an altogether different alternative to proposals based on the Coulomb[2–5] or superconductive pairing[6–10] interaction. Which method will first be realized experimentally remains to be seen. Theoretically, there is much to explore in parallel to the experimental developments.

Photons can be entangled without interactions, but not if the sources are in thermal equilibrium.[11–13] What was shown in Ref. 1 is that this optical "no-go theorem" does not apply to the Fermi sea. Entangled electron-hole excitations can be extracted from a degenerate electron gas at a tunnel barrier and then spatially separated by an electric field—even under conditions of (local) thermal equilibrium. Since this entanglement mechanism relies on single-particle elastic scattering, no control over electron-electron interactions is required.

Interaction-free entanglement in the Fermi sea has now been studied in connection with counting statistics,[14] teleportation,[15] the Hanbury-Brown–Twiss effect,[16] and chaotic scattering.[17] All these works deal with the bipartite entanglement of a pair of qubits. In the present paper we set the first step towards general multipartite entanglement, by studying the interaction-free entanglement of three qubits.

The proposed three-qubit entangler is sketched schematically in Fig. 1. As in the original three-photon entangler of Zeilinger *et al.*,[18] we propose to create three-qubit entanglement out of two entangled electron-hole pairs. The key distinction between the two schemes is that the sources in the electronic case are reservoirs in thermal equilibrium, in contrast to the single-photon sources of Ref. 18. In the next section we propose a physical realization of Fig. 1, using edge channels in the quantum Hall effect. A pair of edge channels represents a qubit, either in the spin degree of freedom (if the edge channels lie in the same Landau level), or in the orbital degree of freedom (if the spin degeneracy is not resolved and the edge channels lie in two different Landau levels).

The irreducible tripartite entanglement is quantified by the tangle $\tau$ of Coffman, Kundu, and Wootters,[19] which is the three-qubit analog of the concurrence.[20] The tangle is unity for the maximally entangled Greenberger-Horne-Zeilinger (GHZ) state and vanishes if one qubit is disentangled from the other two.[21] We would like to measure $\tau$ by correlating current fluctuations, following the same route as in the bipar-

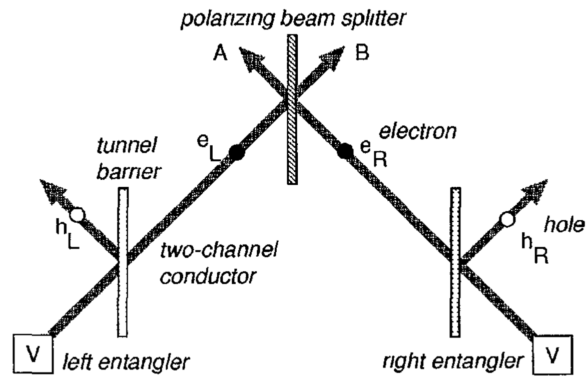


FIG. 1. Schematic description of the creation of three-qubit entanglement out of two entangled electron-hole pairs in the Fermi sea. The left and right entangler consist of a tunnel barrier over which a voltage $V$ is applied. For a simplified description we assume spin entanglement in the state $(|\uparrow_h\uparrow_e\rangle+|\downarrow_h\downarrow_e\rangle)/\sqrt{2}$, where the subscripts $e, h$ refer to electron and hole spin. (The more general situation is analyzed in Sec. II.) The two electrons meet at a polarizing beam splitter, which fully transmits the up spin and fully reflects the down spin. If the outgoing ports A, B contain one electron each, then they must both have the same spin. The corresponding outgoing state has the form $(|\uparrow_h\uparrow_h\uparrow_e\uparrow_e\rangle+|\downarrow_h\downarrow_h\downarrow_e\downarrow_e\rangle)/\sqrt{2}$. Since the two electrons at A, B are constrained to have the same spin, this four-particle GHZ state represents three independent logical qubits.

 

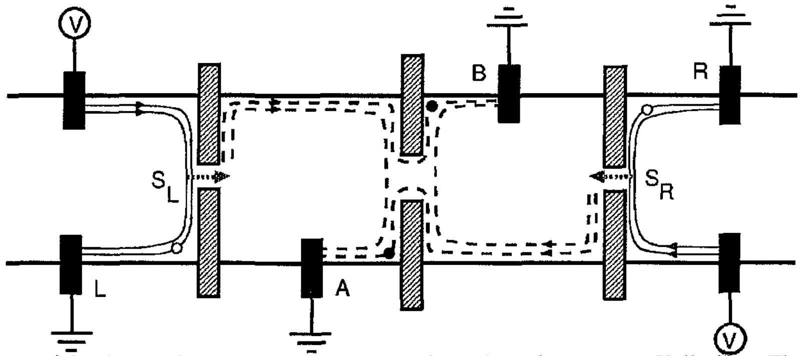


FIG. 2. Proposed realization of the three-qubit entangler, using edge channels in the quantum Hall effect. The left and right point contacts (scattering matrices $S_L$, $S_R$) each produce entangled electron-hole pairs in the Fermi sea. They partially transmit and reflect both edge channels, analogously to beam splitters in optics. The central point contact is the analog of a polarizing beam splitter. It fully transmits the inner edge channel and fully reflects the outer one. Three-qubit entanglement results if there is one excitation at each of the four edges $L,R,A,B$. The two electron excitations at $A$ and $B$ then have the same channel index, so they constitute a *single* qubit. This qubit forms a three qubit entangled state with the two hole excitations at $L$ and $R$.

tite case.[10,22,23] There the concurrence of the electron-hole pair could be related directly to second-order current correlators through the maximal violation of a Bell inequality[1,16,17]—at least in the absence of decoherence.[24]

While there exists a one-to-one relation between concurrence and Bell inequality for any pure state of two qubits,[25] no such relation is known for $\tau$. A recent numerical investigation[26] has found a simple set of upper and lower bounds for $\tau$. Since these bounds become tighter and tighter as the state approaches the GHZ state, they should be of practical use.

The outline of this paper is as follows. In Secs. II and III we construct the three-qubit state and calculate its tangle. Unlike the concurrence, the tangle depends not only on the transmission eigenvalues of the point contact entanglers, but also on the eigenvectors. In Sec. IV we give the bounds on $\tau$ determined by the maximal violation of a Bell inequality. Two tripartite inequalities are compared, one due to Mermin[27] and the other due to Svetlichny.[28]

The maximization in these inequalities is over local unitary transformations of the three qubits, represented by rotated Pauli matrices $\mathbf{c}\cdot\boldsymbol{\sigma}$ (with $\mathbf{c}$ a unit vector). In our case the third qubit is special, because it is composed of a pair of electrons with the same channel index. This defines a preferential basis for the third qubit. In Sec. V we derive that fourth-order irreducible current correlators give a *constrained* maximization of the Bell inequalities. The constraint is that the rotation vector $\mathbf{c}$ of the third qubit lies in the $x$-$y$ plane. The first and second qubits (each consisting of a single hole) can be rotated freely in all three directions. Since the bounds on $\tau$ are unaffected by this constraint, it is not a problem. For generality, we show in the Appendix how the constraint on the axis of rotation of the third qubit can be removed by including also information from second-order correlators.

We conclude in Sec. VI.

## II. PRODUCTION OF THE ENTANGLED STATE

Figure 2 shows our proposal for a physical realization of the schematic diagram in Fig. 1. A three-qubit entangler of edge channels in the quantum Hall effect is constructed by combining a pair of tunneling point contact entanglers from Ref. 1 with an adiabatic point contact (which acts as a polarizing beam splitter). Two voltage sources each excite two edge channels in a narrow energy range eV above the Fermi level. (We will disregard the energy as a separate degree of freedom in what follows.)

After scattering by the three point contacts, the four excitations are distributed in different ways over the four edges $L,R,A,B$. We consider only the terms with one excitation at each edge. This means one excitation (with creation operator $a_{L,i}$) of edge channel $i=1,2$ at the far left, another excitation $a^\dagger_{R,j}$ of edge channel $j$ at the far right, and two more excitations $a^\dagger_{A,k}$, $a^\dagger_{B,l}$ of edge channels $k,l$ at opposite sides of the central point contact. The polarizing beam splitter ensures that $k=l$, meaning that the two excitations at $A$ and $B$ have the same channel index. They constitute a single qubit, which is entangled with the two excitations at $L$ and $R$.

To extract the terms with one excitation at each edge from the full wave function $|\Psi\rangle$, we project out doubly occupied edges. (Note that if no edge is doubly occupied, then the four excitations must be distributed evenly over the four edges.) The projection operator is

$$\mathcal{P}=(1-n_{L,1}n_{L,2})(1-n_{R,1}n_{R,2})(1-n_{A,1}n_{A,2})(1-n_{B,1}n_{B,2}), \tag{2.1}$$

with number operator $n_{X,i}=a^\dagger_{X,i}a_{X,i}$. The projected wave function takes the form

$$\mathcal{P}|\Psi\rangle=\sum_{i,j,k}(r_L\sigma_y t_L^T)_{ik}(r_R\sigma_y t_R^T)_{jk}a^\dagger_{L,i}a^\dagger_{R,j}a^\dagger_{A,k}a_{B,k}|0\rangle, \tag{2.2}$$

which we normalize to unity,

$$|\Phi\rangle=w^{-1/2}\mathcal{P}|\Psi\rangle, \tag{2.3}$$

$$w=\sum_{k}(t_L\sigma_y r_L^\dagger r_L\sigma_y t_L^T)_{kk}(t_R\sigma_y r_R^\dagger r_R\sigma_y t_R^T)_{kk}. \tag{2.4}$$

Here $r_L, r_R, t_L, t_R$ are the $2\times 2$ reflection and transmission matrices of the left and right point contact, and $\sigma_y$ is a Pauli matrix.

We transform from electron to hole operators ($b_{L,i}^\dagger = a_{L,i}$, $b_{R,i}^\dagger = a_{R,i}$) at the left and right ends, and redefine the vacuum accordingly $|0'\rangle = a_{L,1}^\dagger a_{L,2}^\dagger a_{R,1}^\dagger a_{R,2}^\dagger|0\rangle$. The wave function $|\Phi\rangle$ transforms into

$$|\Phi'\rangle = \sum_{i,j,k} m_{ijk} b_{L,i}^\dagger b_{R,j}^\dagger a_{A,k}^\dagger a_{B,k}^\dagger |0'\rangle, \tag{2.5}$$

$$m_{ijk} = w^{-1/2}(\sigma_y r_L^T \sigma_y t_L^T)_{ik}(\sigma_y r_R^T \sigma_y t_R^T)_{jk}. \tag{2.6}$$

The wave function (2.5) describes an entangled state of a pair of holes at the left and right ends (creation operators $b_{L,i}^\dagger$ and $b_{R,j}^\dagger$), with a single qubit at the center consisting of two electrons sharing the same channel index (creation operator $a_{A,k}^\dagger a_{B,k}^\dagger$). This three-qubit state corresponds to the maximally entangled GHZ state $(|\uparrow\uparrow\uparrow\rangle + |\downarrow\downarrow\downarrow\rangle)/\sqrt{2}$ if $m_{ijk} = 2^{-1/2}\delta_{ik}\delta_{jk}$ (or, more generally, if $m_{ijk} = 2^{-1/2}U_{ik}V_{jk}$ with $U,V$ unitary matrices). The degree of entanglement in the general case is calculated in the next section.

## III. CALCULATION OF THE DEGREE OF ENTANGLEMENT

To quantify the irreducible three-qubit entanglement contained in the wave function (2.5), we use the tangle[19]

$$\tau = 2\left|\sum m_{ijk} m_{i'j'l} m_{npk'} m_{n'p'l'} \varepsilon_{ii'} \varepsilon_{jj'} \varepsilon_{kk'} \varepsilon_{ll'} \varepsilon_{nn'} \varepsilon_{pp'}\right|. \tag{3.1}$$

Here $\epsilon = i\sigma_y$ and the sum is over all indices. The expression between the modulus signs is the hyperdeterminant of a rank-three matrix.[29] Substituting Eq. (2.6), we find that in our case this hyperdeterminant factorizes into the product of two determinants of rank-two matrices,

$$\tau = 4w^{-2}|\mathrm{Det}(r_L t_L t_R r_R^T)|^2$$

$$= 4w^{-2}\prod_i T_{L,i}(1-T_{L,i})T_{R,i}(1-T_{R,i}). \tag{3.2}$$

Here $T_{L,1}, T_{L,2}$ are the two transmission eigenvalues of the left point contact (eigenvalues of $t_L t_L^\dagger$), and $T_{R,1}, T_{R,2}$ are the corresponding quantities for the right point contact.

The tangle reaches its maximal value of unity in the special case of channel-independent transmission eigenvalues $T_{L,1} = T_{L,2} \equiv T_L$ and $T_{R,1} = T_{R,2} \equiv T_R$. Then $w = 2T_L(1-T_L)T_R(1-T_R)$, hence $\tau = 1$—irrespective of the value of $T_L$ and $T_R$. In this special case the state $|\Phi'\rangle$ equals the GHZ state up to a local unitary transformation.

In the more general case of channel-dependent $T_{L,i}, T_{R,i}$ the tangle is less than unity. We are interested in particular in the limit that the left and right point contacts are weakly transmitting $T_{L,i} \ll 1$, $T_{R,i} \ll 1$. The reflection matrices $r_L$ and $r_R$ are then approximately unitary, which we may use to simplify the normalization constant (2.4). The result for the tangle in this tunneling limit is

$$\tau = \frac{4T_{L,1}T_{L,2}T_{R,1}T_{R,2}}{\left|\sum_k (t_L t_L)_{kk}(t_R t_R)_{kk}\right|^2}. \tag{3.3}$$

In contrast to the concurrence,[1] the tangle depends not only on the transmission eigenvalues but also on the eigenvectors [through the denominator in Eq. (3.3)].

## IV. THREE-QUBIT BELL INEQUALITIES

The tangle is not directly an observable quantity, so it is useful to consider also alternative measures of entanglement that are formulated entirely in terms of observables. These take the form of generalized Bell inequalities,[30,31] where the amount of violation of the inequality (the "Bell parameter") is the entanglement measure.

### A. Bell parameters

Bell inequalities for three qubits are constructed from the correlator

$$E(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}) = \langle\Phi|(\boldsymbol{a}\cdot\boldsymbol{\sigma})\otimes(\boldsymbol{b}\cdot\boldsymbol{\sigma})\otimes(\boldsymbol{c}\cdot\boldsymbol{\sigma})|\Phi\rangle$$

$$= \sum m_{ijk}^*(\boldsymbol{a}\cdot\boldsymbol{\sigma})_{ii'}(\boldsymbol{b}\cdot\boldsymbol{\sigma})_{jj'}(\boldsymbol{c}\cdot\boldsymbol{\sigma})_{kk'} m_{i'j'k'}. \tag{4.1}$$

Here $\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}$ are real three dimensional vectors of unit length that define a rotation of the Pauli matrices, for example, $\boldsymbol{a}\cdot\boldsymbol{\sigma} \equiv a_x\sigma_x + a_y\sigma_y + a_z\sigma_z$. We choose a pair of vectors $\boldsymbol{a},\boldsymbol{a}'$, $\boldsymbol{b},\boldsymbol{b}'$, and $\boldsymbol{c},\boldsymbol{c}'$ for each qubit and construct the linear combinations

$$\mathcal{E} = E(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}') + E(\boldsymbol{a},\boldsymbol{b}',\boldsymbol{c}) + E(\boldsymbol{a}',\boldsymbol{b},\boldsymbol{c}) - E(\boldsymbol{a}',\boldsymbol{b}',\boldsymbol{c}'), \tag{4.2}$$

$$\mathcal{E}' = E(\boldsymbol{a}',\boldsymbol{b}',\boldsymbol{c}) + E(\boldsymbol{a}',\boldsymbol{b},\boldsymbol{c}') + E(\boldsymbol{a},\boldsymbol{b}',\boldsymbol{c}') - E(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}). \tag{4.3}$$

Mermin's inequality[27] reads $|\mathcal{E}| \leqslant 2$, while Svetlichny's inequality[28] is $|\mathcal{E}-\mathcal{E}'| \leqslant 4$. The GHZ state violates these inequalities by the maximal amount ($|\mathcal{E}| = 4$ and $|\mathcal{E}-\mathcal{E}'| = 4\sqrt{2}$ for suitably chosen rotation vectors), while the violation is zero for a separable state. The maximal violation of Mermin or Svetlichny's inequality is a measure of the degree of entanglement of the state. These "Bell parameters" are defined by

$$\mathcal{M}_\mathrm{M} = \max|\mathcal{E}|, \quad \mathcal{M}_\mathrm{S} = \max|\mathcal{E}-\mathcal{E}'|. \tag{4.4}$$

The maximization is over the vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, $\boldsymbol{a}'$, $\boldsymbol{b}'$, $\boldsymbol{c}'$ for a given state $|\Phi'\rangle$.

For later use we also define a second set of Bell parameters,

$$\mathcal{M}_\mathrm{M}' = \max_{c_z=0=c_z'} |\mathcal{E}|, \quad \mathcal{M}_\mathrm{S}' = \max_{c_z=0=c_z'} |\mathcal{E}-\mathcal{E}'|, \tag{4.5}$$

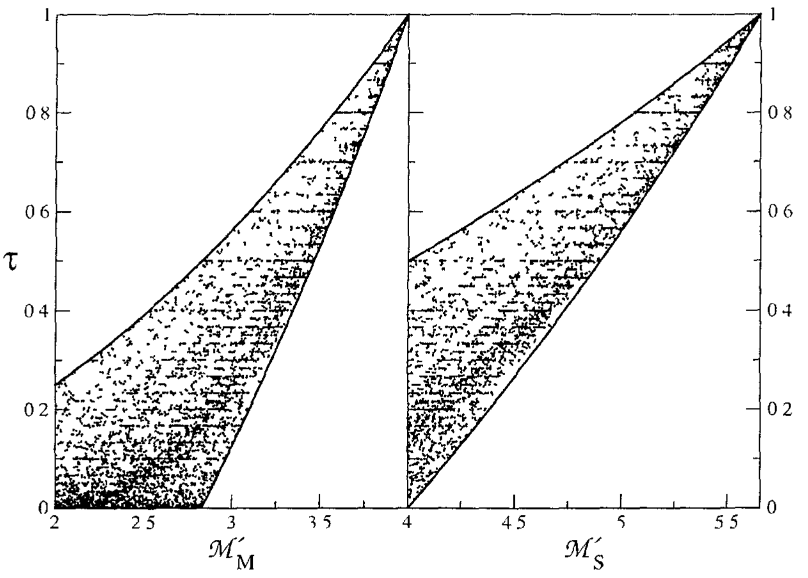


FIG. 3. Numerically determined maximal violation of the Mermin ($\mathcal{M}'_M$) and Svetlichny ($\mathcal{M}'_S$) inequalities for the three-parameter state (4.6). The primes refer to a maximization constrained by rotation vectors $c,c'$ in the x-y plane. A range of values for the tangle $\tau$ gives the same maximal violation. The solid curves are the upper and lower bounds (4.8) and (4.9). The same bounds apply also to the unconstrained Bell parameters $\mathcal{M}_M$ and $\mathcal{M}_S$.[26]

with $\hat{z}$ a unit vector in the z direction. The maximization is therefore constrained to rotation vectors $\boldsymbol{c},\boldsymbol{c}'$ in the $x$-$y$ plane. (The other rotation vectors $\boldsymbol{a},\boldsymbol{a}',\boldsymbol{b},\boldsymbol{b}'$ may vary in all three directions.)

### B. Relation between tangle and Bell parameters

We seek the relation between the tangle and these Bell parameters for states of the form (2.5). These states constitute a three-parameter family, with equivalence up to local unitary transformations. (The full set of three-qubit pure states form a five-parameter family.[29]) A convenient spinor representation is[32]

$$|\Phi\rangle=\cos\alpha\left|\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\right\rangle+\sin\alpha\left|\begin{pmatrix}\cos\beta\\\sin\alpha\end{pmatrix}\begin{pmatrix}\cos\gamma\\\sin\gamma\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}\right\rangle, \tag{4.6}$$

with angles $\alpha,\beta,\gamma\in(0,\pi/2)$. The tangle (3.1) is given in terms of these angles by

$$\tau=(\sin 2\alpha\sin\beta\sin\gamma)^2. \tag{4.7}$$

The special case $\beta=\pi/2=\gamma$ was studied by Scarani and Gisin.[33] Even in that one-parameter case no exact analytical formula could be derived for the maximal violation of the Bell inequality. The lower bound $\mathcal{M}_M>\max(4\sqrt{\tau},2\sqrt{1-\tau})$ was found numerically to be very close to the actual value.

In the more general three-parameter case (4.6) there is no one-to-one relation between tangle and maximal violation of a Bell inequality. Still, the Bell inequalities are useful because they give upper and lower bounds for the tangle, which become tighter the larger the violation. This was found in Ref. 26 for the unconstrained Bell parameters.

The bounds hold in the nonclassical interval $2<\mathcal{M}_M<4$, $4<\mathcal{M}_S<4\sqrt{2}$. For a given Bell parameter in this interval the tangle is bounded by

$$\max(0,\mathcal{M}_M^2/8-1)<\tau<\mathcal{M}_M^2/16, \tag{4.8}$$

$$\mathcal{M}_S^2/16-1<\tau<\mathcal{M}_S^2/32. \tag{4.9}$$

The numerical results shown in Fig. 3 demonstrate that the same bounds apply also to the constrained maximization. These bounds do not have the status of exact analytical results, but they are reliable representations of the numerical data. As expected,[34] the same violation of the Svetlichny inequality gives a tighter lower bound on the tangle than the Mermin inequality gives.

## V. DETECTION OF THE ENTANGLED STATE

For the entanglement detection each contact to ground $X=L,R,A,B$ is replaced by a channel mixer (represented by a unitary $2\times 2$ matrix $U_X$), followed by a channel selective current meter $I_{X,i}$ (see Fig. 4). Low-frequency current fluctuations $\delta I_{X,i}(\omega)$ are correlated for different choices of the $U_X$, and the outcome is used to determine the Bell parameters. These correlators can be calculated using the general theory of Levitov and Lesovik.[35]

All second- and third-order correlators involving both contacts $L$ and $R$ vanish. The first nonvanishing correlator involving both $L$ and $R$ is of fourth order,

$$\langle\langle \delta I_{L,i}(\omega_1)\,\delta I_{R,j}(\omega_2)\,\delta I_{A,k}(\omega_3)\,\delta I_{B,l}(\omega_4)\rangle\rangle$$
$$=(e^5V/h)2\pi\delta\left(\sum_{n=1}^{4}\omega_n\right)C_{ij,kl}. \tag{5.1}$$

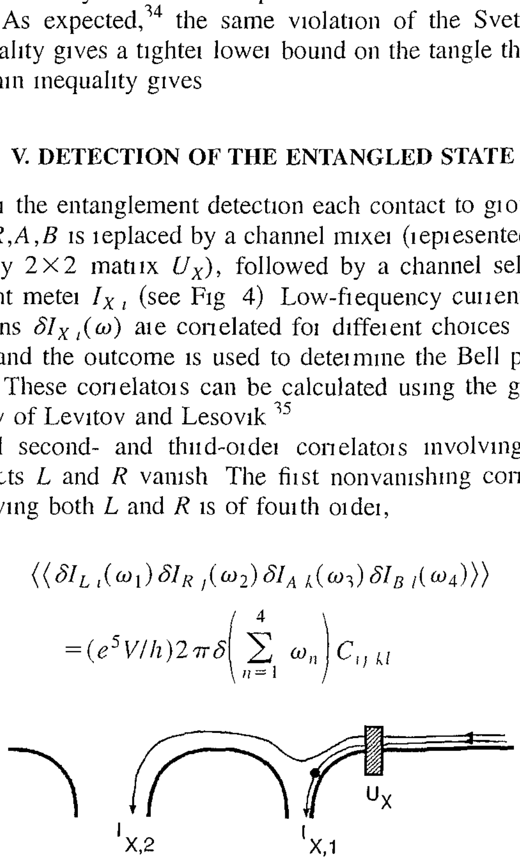


FIG. 4. Schematic diagram of a channel mixer $U_X$ followed by a channel-resolved current detector, needed to measure the Bell parameters. Each contact to ground in Fig. 2 is replaced by such a device (with $X=L,R,A,B$).

Here $\langle\langle\ \ \rangle\rangle$ denotes the irreducible part of the correlator, defined generally by

$$\langle\langle \delta x_1 \delta x_2 \delta x_3 \delta x_4 \rangle\rangle = \langle \delta x_1 \delta x_2 \delta x_3 \delta x_4\rangle - \langle \delta x_1 \delta x_2\rangle\langle \delta x_3 \delta x_4\rangle - \langle \delta x_1 \delta x_3\rangle\langle \delta x_2 \delta x_4\rangle - \langle \delta x_1 \delta x_4\rangle \times \langle \delta x_2 \delta x_3\rangle \tag{5.2}$$

The polarizing beam splitter ensures that there is only a single independent irreducible correlator with respect to variation of the indices $k$ and $l$:

$$C_{ij,11} = C_{ij,22} = -C_{ij,12} = -C_{ij,21} \equiv \mathcal{C}_{ij}. \tag{5.3}$$

We obtain the following expression for $\mathcal{C}_{ij}$ in terms of the transmission and reflection matrices:

$$\mathcal{C}_{ij} = 2\,\mathrm{Re}\{\alpha\beta(t_L r_L^\dagger U_L^\dagger)_{1i}(t_R r_R^\dagger U_R^\dagger)_{1j}(t_L r_L^\dagger U_L^\dagger)^*_{2i}(t_R r_R^\dagger U_R^\dagger)^*_{2j}\}, \tag{5.4}$$

$$\alpha = U^\dagger_{A\,11} U_{A\,12}, \quad \beta = U^\dagger_{B\,11} U^\dagger_{B\,12}. \tag{5.5}$$

We write $\alpha\beta \equiv \zeta$.

We wish to relate the current correlator to the matrix of coefficients $m_{ijk}$ that characterizes the three-qubit state (2.5). This becomes possible in the tunneling limit, when $r_L$ and $r_R$ may be approximated by two unitary matrices. We apply to Eq. (2.6) the identity[15]

$$U\sigma_y = (\mathrm{Det}\, U)\sigma_y U^*, \tag{5.6}$$

valid for any $2\times 2$ unitary matrix $U$. Note that the determinant $\mathrm{Det}\,U$ is simply a phase factor $e^{i\phi}$. We find

$$\mathcal{C}_{ij} = 2w|\zeta|\mathrm{Re}\, e^{i\Omega}\tilde{m}_{ij1}\tilde{m}^*_{ij2}, \tag{5.7}$$

$$\tilde{m}_{ijk} = \sum_{i',j'} U^r_{L\,ii'} U^*_{R\,jj'} m_{i'j'k}. \tag{5.8}$$

The weight $w$ in the tunneling limit can be obtained by measuring separately the current into contacts $A$ and $B$ when either the left or the right voltage source is switched off. If the right voltage source is off, then we measure the mean currents $I_{L\to A} = (e^2V/h)(t_L^\dagger t_L)_{22}$ and $I_{L\to B} = (e^2V/h)(t_L^\dagger t_L^\dagger)_{11}$. Similarly, if the left voltage source is off, we measure $I_{R\to A} = (e^2V/h)(t_R t_R^\dagger)_{11}$ and $I_{R\to B} = (e^2V/h)(t_R t_R^\dagger)_{22}$. The weight factor is given by

$$w = (e^2V/h)^{-2}(I_{L\to B} I_{R\to A} + I_{L\to A} I_{R\to B}). \tag{5.9}$$

We are now ready to express the Bell parameters of Sec. IV A in terms of current correlators. We define the linear combination

$$F(U_L, U_R, \zeta) = w^{-1}(\mathcal{C}_{11} + \mathcal{C}_{22} - \mathcal{C}_{12} - \mathcal{C}_{21}). \tag{5.10}$$

Using Eq. (5.7) we arrive at

$$F(U_L, U_R, \zeta) = |\zeta| \sum m_{ijk}(U_L^\dagger \sigma_z U_L)_{ii'}(U_R \sigma_z U_R)_{jj'} \times (\sigma_x \cos\Omega + \sigma_y \sin\Omega)_{kk'} m^*_{i'j'k'}, \tag{5.11}$$

where $\zeta = |\zeta| e^{i\Omega}$. Note that $(U^\dagger \sigma_z U)$ with unitary $U$ and $\boldsymbol{a}\cdot\boldsymbol{\sigma}$ with unit vector $\boldsymbol{a}$ are equivalent representations of rotated Pauli matrices. We indicate this equivalence relation by writing $(U^\dagger\sigma_z U)^{eq} = \boldsymbol{a}_U\cdot\boldsymbol{\sigma}$.

Comparing Eqs. (4.1) and (5.11) we thus conclude that

$$F(U_L, U_R, \zeta) = |\zeta| E(\boldsymbol{a}_{U_L}, \boldsymbol{b}_{U_R}, \boldsymbol{c}), \quad \boldsymbol{c} = (\cos\Omega, \sin\Omega, 0). \tag{5.12}$$

The two correlators $F$ and $E$ are equivalent provided that the unit vector $\boldsymbol{c}$ lies in the $x$-$y$ plane. The unit vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are not so constrained.

The Bell parameters $\mathcal{M}'_{\mathrm{M}}$ and $\mathcal{M}'_{\mathrm{S}}$ follow from

$$\mathcal{M}'_{\mathrm{M}} = 4\max|F(U_L, U_R, \zeta') + F(U_L, U'_R, \zeta) + F(U'_L, U_R, \zeta) - F(U'_L, U'_R, \zeta')|, \tag{5.13}$$

$$\begin{aligned}\mathcal{M}'_{\mathrm{S}} = 4\max|&F(U_L, U_R, \zeta') + F(U_L, U'_R, \zeta) + F(U'_L, U_R, \zeta) \\ &- F(U'_L, U'_R, \zeta') - F(U'_L, U'_R, \zeta) - F(U'_L, U_R, \zeta') \\ &- F(U_L, U'_R, \zeta') + F(U_L, U_R, \zeta)|.\end{aligned} \tag{5.14}$$

The maximization is over the $2\times 2$ unitary matrices $U_L, U_R, U_A, U_B, U'_L, U'_R, U'_A, U'_B$. (We have used that the maximum is reached for $|\zeta|, |\zeta'| = 1/4$.)

Equations (5.13) and (5.14) demonstrate that the irreducible fourth-order current correlators measure the constrained Bell parameters $\mathcal{M}'_{\mathrm{M,S}}$. The constraint is that the rotation vector of the third qubit lies in the $x$-$y$ plane. As discussed in Sec. IV B, these quantities contain essentially the same information about the tangle of our three-qubit state as the unconstrained Bell parameters $\mathcal{M}_{\mathrm{M,S}}$.

One might wonder whether it is possible at all to express the unconstrained Bell parameters in terms of low-frequency current correlators. The answer is Yes, as we show in the appendix. The constraint on the rotation of the third qubit can be removed by including also products of second-order correlators.

## VI. CONCLUSION

We conclude by listing similarities and differences between the scheme for three-qubit entanglement in the Fermi sea presented here and the two-qubit scheme of Ref. 1. This comparison will also point to some directions for future research.

(1) Both schemes require neither electron-electron interactions nor single-particle sources. Elastic scattering from a static potential and sources in thermal equilibrium suffice. This sets apart the present solid-state proposal from existing quantum optics proposals,[18] which require either nonlinear media or single-photon sources to produce a GHZ state.

(2) The scheme of Ref. 1 is capable of producing the most general two-qubit entangled pure state, by suitably choosing the scattering matrix of the tunnel barrier. The present scheme, in contrast, is limited to the production of the three-parameter subset (4.6) of the most general five-parameter family of three-qubit entangled pure states.[29] This subset is characterized by the property that tracing over the third qubit

results in a mixed two-qubit state which is not entangled. The origin of this restriction is that the three-qubit state is constructed out of two separate entangled electron-hole pairs.

(3) The two-qubit entangler can produce maximally entangled Bell pairs as well as partially entangled states, as quantified by the concurrence. Similarly, the three-qubit entangler can produce maximally entangled GHZ states as well as states that have a smaller degree of tripartite entanglement, as quantified by the tangle.[19] However, in the three-qubit case there is a second class of states that are irreducibly entangled and cannot be obtained from the GHZ state by any local operation.[21] These so-called $W$ states are not accessible by our scheme. It would be interesting to see if there exists an interaction-free method to extract the $W$ state out of the Fermi sea, or whether this is impossible as a matter of principle.

(4) The concurrence of the electron-hole pair can be measured using second-order low-frequency current correlators.[1,16] We have found that the tangle can be determined from fourth-order correlators, but the method presented here only gives upper and lower bounds. The bounds become tight if the state is close to the maximally entangled GHZ state,[26] so they are of practical use. Still, it would be of interest to see if there exists an alternative method to measure the actual value of the tangle, even if the state is far from the maximally entangled limit.

(5) Low-frequency noise measurements can determine the degree of entanglement within the context of a quantum mechanical description, but they cannot be used to rule out a description in terms of local hidden variables. That requires time resolved detection.[23] For the tunnel barrier entangler the detection time should be less than the inverse $e/\bar{I}$ of the mean current, corresponding to the mean time between subsequent current pulses. For our three-qubit entangler the requirement is more stringent. The detection time should be less than the coherence time $h/eV$, corresponding to the width of a current pulse. This is the same condition of "ultracoincident detection" as in the quantum optical analog.[18]

(6) We have restricted ourselves to entanglers in the tunneling regime. In the two-qubit case, it is possible to measure the concurrence even if the transmission probabilities of the entangler are not small compared to unity.[17] A similar generalization is possible in the three-qubit case (cf. the Appendix).

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## APPENDIX: RELATION BETWEEN UNCONSTRAINED BELL PARAMETERS AND CURRENT CORRELATORS

To relate the unconstrained Bell parameters $\mathcal{M}_M$ and $\mathcal{M}_S$ to low-frequency current fluctuations we need to consider also second-order correlators. These have the general form

$$\langle \delta I_{X,i}(\omega_1)\,\delta I_{Y,j}(\omega_2)\rangle=(e^3V/h)2\pi\delta(\omega_1+\omega_2)K_{ij}^{XY}, \tag{A1}$$

with $X,Y\in\{L,R,A,B\}$ and $i,j\in\{1,2\}$. We seek the combination

$$K_{ik}^{LA}K_{jl}^{RB}+K_{il}^{LB}K_{jk}^{RA}\equiv K_{ij,kl} \tag{A2}$$

involving all four contacts. It is determined by the transmission and reflection matrices of the left and right point contact,

$$K_{ij,kl}=\sum_{p=1,2}|(U_A)_{kp}(U_B)_{lp}(t_L r_L^{\dagger}U_L^{\dagger})_{pi}(t_R r_R^{\dagger}U_R^{\dagger})_{pj}|^2 \tag{A3}$$

We now take the tunneling limit to relate the current correlators to the matrix of coefficients (2.6). Using the identity (5.6) we find

$$K_{ij,kl}=w\sum_{p}|(U_A)_{kp}(U_B)_{lp}\tilde{m}_{ijp}|^2 \tag{A4}$$

The weight $w$ can be determined from the mean currents, as explained in Sec. V, or alternatively from $w=\Sigma_{i,j,k,l}K_{ij,kl}$.

The two real numbers $|\alpha|^2=|U_{A,11}|^2(1-|U_{A,11}|^2)$ and $|\beta|^2=|U_{B,11}|^2(1-|U_{B,11}|^2)$ in Eq. (5.4) can be determined separately by measuring what fraction of the mean current in contacts $A$ or $B$ ends up in channel 1. We use this to construct the function

$$\begin{aligned}\tilde{F}(U_L,U_R,\Omega,|\alpha|)&=2w^{-1}|\beta|^{-1}(C_{11}+C_{22}-C_{12}-C_{21})\\&=2|\alpha|\sum m_{ijk}^{*}(U_L\sigma_z U_L^{\dagger})_{ii'}^{*}(U_R^{*}\sigma_z U_R^{T})_{jj'}^{*}\\&\quad\times(\sigma_x\cos\Omega+\sigma_y\sin\Omega)_{kk'}m_{i'j'k'},\end{aligned} \tag{A5}$$

with $\alpha\beta=|\alpha||\beta|e^{i\Omega}$. Comparing Eqs. (4.1) and (A5) we see that

$$\tilde{F}(U_L,U_R,\Omega,|\alpha|)=2|\alpha|E(\boldsymbol{a}_{U_i},\boldsymbol{b}_{U_R},\boldsymbol{c}),$$
$$\boldsymbol{c}=(\cos\Omega,\sin\Omega,0) \tag{A6}$$

Equation (A6) has the constraint that $\boldsymbol{c}$ is in the $x$-$y$ plane. In order to access also components of $\boldsymbol{c}$ in the $z$ direction we include the product of second-order correlators

$$\begin{aligned}G(U_L,U_R,\xi)&=w^{-1}\sum_{i,j,k,l=1,2}(-1)^{i+1}(-1)^{j+1}\\&\quad(-1)^{k+1}K_{ij,kl}\\&=\sum m_{ijk}^{*}(U_L\sigma_z U_L^{\dagger})_{ii'}(U_R\sigma_z U_R^{\dagger})_{jj'}\\&\quad\times(\xi\sigma_z)_{kk'}m_{i'j'k'},\end{aligned} \tag{A7}$$

with $\xi=2|U_{A,11}|^2-1$. Adding $\tilde{F}$ and $G$ we arrive at

$$\bar{F}(U_L,U_R,\Omega,|\alpha|)+G(U_L,U_R,\xi)=E(\boldsymbol{a}_{U_L},\boldsymbol{b}_{U_R},\boldsymbol{c}),$$

$$\boldsymbol{c}=(2|\alpha|\cos\Omega,2|\alpha|\sin\Omega,\xi). \quad \text{(A8)}$$

Note that $\xi^2+4|\alpha|^2=1$, so $\boldsymbol{c}$ is a unit vector—as it should be.

By varying over the unitary matrices $U_L$, $U_R$, and $U_A$ one can now determine the unconstrained Mermin and Svetlichny parameters (4.4), using only low-frequency noise measurements.

Equation (A8) still requires the tunneling regime ($T_{L,i},T_{R,i}\ll 1$). It is possible to relax this condition, by adding products of mean currents to the second- and fourth-order irreducible correlators. The entire expression then takes the form of a fourth-order *reducible* correlator, which is directly related to a Bell inequality formulated in terms of equal-time correlators of the currents at contacts $L,R,A,B$. This is analogous to the calculation of the concurrence in Ref. 17.

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