14

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LETTER TO THE EDITOR

COMMENT ON "A GENERALIZED LANGEVIN EQUATION FOR 1/*f* NOISE"

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We point out that in a recent generalized Langevin equation approach to 1/*f* noise, the frequency dependent transport coefficient violates the Kramers–Kronig causality relations. An investigation of the possibility of using a generalized Langevin equation that does not violate causality leads to the conclusion that this idea is not a viable approach to the problem of 1/*f* noise.

It is well known [1–3] that in certain resistors the power spectrum $S(\omega)$, defined as the Fourier transform of the voltage–voltage correlation function, contains besides a term due to the thermal noise (Johnson noise) a contribution proportional to I_0^2/f. Here I_0 is the average current through the resistor, and $f \equiv \omega/2\pi$ the frequency. This $1/f$ noise term is found over several frequency decades and is observed in many different materials (and even in other systems as well [4,5]). In spite of its generality, its nature is not yet well understood.

The fact that the strength of $1/f$ noise is proportional to I_0^2 has been interpreted as an indication that this effect is due to fluctuations in the resistance of the material. In particular, it has been suggested that these, in turn, could be induced by temperature fluctuations. [6]. For this reason, two of us [7] (Llebot and Rubí) recently postulated the following one-dimensional generalized Langevin equation for the temperature field $T(x, t)$:

$$\frac{\partial T(x,t)}{\partial t} = \int_{-\infty}^{\infty} \lambda(t-t') \frac{\partial^2 T(x,t')}{\partial x^2} \mathrm{d}t' + F(x,t), \quad (1)$$

where $F(x, t)$ is a random force and λ a generalized time dependent heat diffusivity. Llebot and Rubí tried to investigate under what conditions the above equations would be compatible with a $1/f$ power spectrum. However, they overlooked the fact that in the course of their analysis, they had made assumptions about the Fourier transform $\lambda(\omega)$ of $\lambda(\tau)$ which violate the Kramers–Kronig relations [8, 9] (and hence causality). In this letter, we correct this error and arrive at the opposite conclusions from the ones obtained on the basis of the faulty analysis: we argue that the generalized Langevin equation approach, though possible in principle, is not a promising route to an understanding of $1/f$ noise.

The basis of the Kramers–Kronig relations is the observation that $\lambda(\tau)$ in eq. (1) must satisfy

$$\lambda(\tau) = 0 \qquad \text{for } \tau < 0, \quad (2)$$

since the change of the temperature at time t cannot depend on its value at later times (causality principle). If one defines the Fourier transform with respect to time by

$$\lambda(\omega) \equiv \int_{-\infty}^{+\infty} \mathrm{d}\tau\, \lambda(\tau)\, \mathrm{e}^{\mathrm{i}\omega\tau}, \quad (3)$$

it follows from eq. 2 that $\lambda(\omega)$ has no poles in the upper half of the complex ω plane. By complex integration it can then be shown that [8, 9] (P denotes the principle part of the integral)

$$\mathrm{P}\int_{-\infty}^{+\infty} \mathrm{d}\omega \, \frac{\lambda(\omega)}{\omega-\omega'} - \mathrm{i}\pi\lambda(\omega') + \mathrm{i}\pi\lambda_\infty = 0\,. \tag{4}$$

Here, we have allowed for a nonzero but constant value λ_∞ of $\lambda(\omega)$ in the limit $|\omega| \to \infty$. The Kramers–Kronig relation (4) connects the real part $\lambda'(\omega)$ and the imaginary part $\lambda''(\omega)$ of $\lambda(\omega)$: once $\lambda'(\omega)$ is known $\lambda''(\omega)$ can be computed from (4), or vice versa.

In the analysis of ref. [7], it is assumed that $\lambda(\omega)$ obeys the symmetry relation $\lambda(\omega) = \lambda(-\omega)$. Since this implies that also $\lambda(\tau) = \lambda(-\tau)$, $\lambda(\tau)$ will then in view of eq. (2) be zero for $\tau > 0$ and $\tau < 0$, suggesting that the only $\lambda(\tau)$ obeying this symmetry and causality is a delta-function. Indeed the Kramers–Kronig relation confirms this idea: if $\lambda(\omega) = \lambda(-\omega)(=\lambda^*(\omega))$, $\lambda(\omega)$ is real so that $\lambda''(\omega) = 0$, and according to eq. (4) $\lambda'(\omega)$ is in that case independent of the frequency and equal to the constant λ_∞. For $\lambda(\tau)$, its inverse Fourier transform, one therefore obtains a delta-function.

Of course, if $\lambda(\tau)$ is a delta function, eq. (1) reduces to the usual Langevin equation without memory for the fluctuating temperature field, and no $1/f$ spectrum is found. If, on the other hand, one follows Llebot and Rubí in taking $\lambda(\omega)$ real but not a constant, one violates causality. Nevertheless, one could in principle extend their analysis to cases where $\lambda(\omega)$ is not real, since one can always write down a generalized Langevin equation that reproduces *any* given power spectrum. To see this, consider e.g. the generalized Langevin equation for a single variable $\alpha(t)$.

$$\frac{\mathrm{d}\alpha(t)}{\mathrm{d}t} = -\int_{-\infty}^{t} \lambda(t-t')\alpha(t')\,\mathrm{d}t' + F(t)\,, \tag{5}$$

where $F(t)$ is a 'random force'. In order that the fluctuation dissipation theorem is satisfied, the correlation of $F(t)$ should obey

$$\langle F(\omega)F^*(\omega')\rangle = 2k_\beta T_0 \lambda'(\omega)\delta(\omega-\omega')\,, \tag{6}$$

with k_β Boltzmann's constant and T_0 the equilibrium temperature. From eqs. (5) and (6), one obtains for the power spectrum $S_{\alpha\alpha}(\omega)$

$$\begin{aligned} S_{\alpha\alpha}(\omega)\delta(\omega-\omega') &\equiv \langle \alpha(\omega)\alpha^*(\omega')\rangle \\ &= 2k_\beta T_0 \frac{\lambda'(\omega)}{\lambda'(\omega)^2 + (\lambda''(\omega)-\omega)^2} \\ &\quad \times \delta(\omega-\omega')\,. \end{aligned} \tag{7}$$

For a *given* spectrum $S_{\alpha\alpha}(\omega)$, one can in principle solve this equation, together with (4), for $\lambda'(\omega)$ and $\lambda''(\omega)$. (In practice, this will be difficult since the Kramers–Kronig relation is an integral equation.) Of course, as such this result is rather useless since it is based on a mere mathematical manipulation that does not tell us whether the generalized Langevin equation physically makes any sense at all. If one obtains e.g. in this way from a simple $1/f$ spectrum two complicated functions $\lambda'(\omega)$ and $\lambda''(\omega)$, there is not much reason to expect the physical explanation of these functions to be simpler than the one of the $1/f$ spectrum itself. In fact, the recently advanced ideas about the nature of $1/f$ noise do not give much hope for the adequancy of a (generalized) Langevin equation either. The intuitive picture behind such an equation is that there should be a large separation of time scales that enables one to represent the fast changes due to molecular motion by a random force and the slow damping by a friction term, which contains at most only a few microscopic relaxation times. However, the presently prevailing idea [2–4, 10] is that $1/f$ noise is a manifestation of a process involving an ensemble of relaxation times with a distribution that is scale invariant over several orders of magnitude. If so, no clear separation of time scales exists and it appears doubtful that a convincing interpretation of a generalized Langevin equation can be given.

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