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ANALYTICAL EXAMINATION OF VIRIAL PROPERTIES OF GROUPS OF GALAXIES

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ABSTRACT

By extending work done long ago by Limber, and by Limber and Mathews, we derive approximate relations for the standard deviation of log M_{VT}/L_T and other virial parameters of groups of galaxies caused by (*a*) the projection of randomly oriented galactic velocity vectors on the line of sight and separation vectors on the celestial sphere and (*b*) the changes in the orbital radius vectors and velocities of galaxies as they move in a two-body or a homogeneous mass distribution of the group. Using these results and the estimated uncertainties in observational data, we find that the observed standard deviation of log M_{VT}/L_T for groups, and the observed correlation between M_{VT}/L_T and the virial velocity dispersion (*V*), are too large to be explained by the combined contributions of projection and quasi-equilibrium effects, plus uncertainties in galactic radial velocities, galactic masses, group luminosities, and distances. Either certain observational and/or theoretical errors are very much larger than is currently believed, or there exists a $\pm 2\,\sigma$ range of a factor of 30 in the cosmic M_{VT}/L_T.

The hypothesis that the large values of $\langle M_{VT}/L_T \rangle$ of groups and clusters are caused by the existence of a cosmological force with negative cosmological constant (Λ) is untenable because (*a*) it predicts a much more centrally concentrated density distribution in rich clusters than is observed, (*b*) it predicts a dependence of virial velocity dispersion on radius of gyration of groups which differs from that observed, and (*c*) the observed range in the calculated values of Λ for individual groups is larger than can be accounted for by observational and statistical uncertainties alone.

Subject headings: cosmology — galaxies: clusters of — galaxies: redshifts

I. INTRODUCTION

Observationally derived virial mass to light ratios of groups of galaxies (M_{VT}/L_T) cover a large range of values and are correlated with virial velocity dispersion (*V*) (e.g., Rood, Rothman, and Turnrose 1970). Can this range and correlation be explained entirely by observational, theoretical, and statistical uncertainties? Rood and Dickel (1978, herein called Paper I) concluded from a comprehensive but tedious analysis with the best available data that they probably could not—a cosmic range and correlation may exist. In Paper I, estimates from observational data were made of uncertainties in galactic radial velocities (*v*), galactic masses (*m*), group luminosities (L_T), and group distances (*d*). Results of computer *N*-body simulations of bound groups kindly provided by Dr. E. L. Turner were used to estimate the effect of the projection of galactic velocities on the line of sight and separations on the celestial sphere (called the projection effect, *p*), and the effect of changes in the orbital radius vectors and velocities of galaxies as they move in their orbits (quasi-equilibrium effect, *e*). While the *N*-body results probably yield realistic estimates of the *pe* effects, they should be complemented by necessarily more approximate but also more fundamental analytical estimates which provide a check. The present paper contains analytical derivations of standard deviations of virial parameters (§§ II and III) and contributions to correlation coefficients (§ IV) caused by *pe* effects. The results of Paper I are found to be confirmed (§ IV). The equations are also used to examine the hypothesis that the large $\langle M_{VT}/L_T \rangle$ values of groups and clusters are caused by the presence of a cosmological force with negative cosmological constant (Λ). This hypothesis is found to be untenable (§ V). Hypotheses to explain the observed virial properties of systems of galaxies are discussed in § VI.

Although we have considered numerous sources of uncertainty in our analysis, we have not dealt with the problem of ambiguous assignment of galaxies in groups. We have treated the STV groups (identified by Sandage and Tammann 1975 and de Vaucouleurs 1975) and TG groups (identified by Turner and Gott 1976) (further details in Paper I) as if (*a*) their membership assignment is unambiguous and (*b*) they are discrete groups. This is probably not the case, but the reliable evaluation of uncertainties caused by this problem is an extensive project in itself. Results of the comprehensive study by Soneira (1979) on the role of selection effects in group identification and M_{VT}/L_T estimates could possibly be applied for this purpose. The uncertainties caused by difficulties with

group membership could then be readily added to the analysis, because the theoretical framework and derivations presented in this paper remain valid.

II. PROJECTION EFFECTS

Consider a group of galaxies with randomly oriented velocity and separation vectors. An estimate of a virial parameter from line-of-sight velocities, projected separations, and the application of average projection factors (Limber and Mathews 1960) will typically be different for different directions of observation. The virial parameter as derived from randomly chosen directions will show a dispersion (standard deviation) about the mean value, which is the mean error of the estimated virial parameter caused by random projection effects, $\sigma_p(x)$.

In Appendix I, $\sigma_p(x)$ is derived for $x = \log_{10} V$ (the virial velocity dispersion defined by Rood, Rothman, and Turnrose 1970), $x = \log R$ (the virial radius), $x = \log M_{\mathrm{VT}}/L_T = \log V^2 R/GL_T$ (the virial mass-to-light ratio), and $x = \log R_I$ (the radius of gyration). It is assumed that the velocity and separation vectors are randomly oriented, independent of galactic mass, and uncorrelated with one another. The derivations of the mean errors are logical extensions of the derivations of average projection factors by Limber and Mathews (1960). The mean errors are derived from the straightforward application of probability functions. We find

$$\sigma_p(\log V) = 1/[C(5N)^{1/2}], \tag{1}$$

$$\sigma_p(\log R) = \frac{0.816}{C[\tfrac{1}{2}N(N-1)]^{1/2}}, \tag{2}$$

$$\sigma_p(\log M_{\mathrm{VT}}/L_T) = \{4[\sigma_p(\log V)]^2 + [\sigma_p(\log R)]^2\}^{1/2}, \tag{3}$$

$$\sigma_p(\log R_I) = 1/[2C(5N)^{1/2}], \tag{4}$$

where $C = 2.303$, the natural logarithm of 10, and N is the number of galaxies sampled in the group. N entered the equations through the application of the central limit theorem (see, e.g., Brunk 1960, p. 156). Although the central limit theorem applies rigorously only when N is large, we have followed Limber (1961) in assuming that it applies, on the average, to a large ensemble of groups, each with small N, when a characteristic or effective value of N is adopted.

In the derivation of equation (2), we have used the definition (applicable when galactic mass weighting can be neglected)

$$R = N^2 \Big/ \left(\sum_{\mathrm{pairs}} r^{-1}\right) = \frac{N^2}{\tfrac{1}{2}N(N-1)\langle r^{-1}\rangle} = \frac{2N}{(N-1)\langle r^{-1}\rangle}, \tag{5}$$

where r is the distance separating two galaxies in a group. The quantity $\langle 1/r\rangle^{-1}$ is the mean harmonic radius of the group. Note that as $N \to \infty$, $R \to 2\langle 1/r\rangle^{-1}$. We emphasize this definition because the literature contains numerous erroneous statements to the effect that the virial radius is numerically equivalent to the mean harmonic radius.

The number 0.816 in equation (2) applies rigorously only when the ratio of the mean separation of the galaxies in a group to the mean galactic diameter is $y_M = 25$. However, the medium y_M for the STV and TG groups is ~ 25 and the range about this value is small enough so that 0.816 is a good approximation for these samples. *Compact groups*, on the other hand, have such small values of y_M that 0.816 in equation (2) should be replaced by a much smaller value, and average projection factors differ significantly from the Limber and Mathews (1960) values. Formulae applicable to groups with *any* y_M are presented in Appendix I.

The quantity under the square-root in equation (2) is the number of galactic separations sampled. It has been assumed that these galactic separations are uncorrelated with one another whereas actually they *are* correlated. The value of $\sigma_p(\log R)$ obtained after allowance for correlated separations would be smaller than that derived from equation (2).

III. QUASI-EQUILIBRIUM EFFECTS

The first treatment of quasi-equilibrium effects was by Limber (1961), who considered the idealized model in which each group member in turn moves under the attraction of the other galaxies in the group approximated by a single point mass. Application of the two-body equations of motion then provided an estimate of the maximum range in estimated virial mass due to the motion of galaxies in orbits of characteristic eccentricity e. In Appendix II, we extend the work begun by Limber in two basic ways: (1) We derive the time-average mean error (instead of the maximum range) of virial parameters due to quasi-equilibrium effects, $\sigma_e(x)$. We let $x = \log R$, $\log R_I$, $\log V$, and $\log M_{\mathrm{VT}}/L_T$ in turn. (2) We derive $\sigma_e(x)$ for galaxies moving in a group approximated by (i) a point mass and (ii) a homogeneous spherical mass distribution. Results for model groups with $N = 3$ effective members and several values of e are tabulated in Table 1.

TABLE 1

THEORETICAL STANDARD DEVIATIONS OF VIRIAL PARAMETERS CAUSED BY
QUASI-EQUILIBRIUM EFFECTS ($N = 3$)

$\sin^{-1} e$ (degrees)	e	$\sigma_e(\log R)$	$\sigma_e(\log V)$	$\sigma_e[\log (M_{\mathrm{VT}}/L_T)]$
Two-Body Model				
0.............	0.00	0.000	0.000	0.000
18.............	0.31	0.053	0.069	0.059
37.............	0.60	0.097	0.253	0.145
53.............	0.80	0.117	0.744	0.271
64.............	0.90	0.123	1.738	0.438
76.............	0.97	0.125	5.87	0.897
Homogeneous Model				
0.............	0.00	0.000	0.000	0.000
18.............	0.31	0.004	0.004	0.004
37.............	0.60	0.020	0.019	0.020
53.............	0.80	0.044	0.039	0.041
64.............	0.90	0.071	0.056	0.054
76.............	0.97	0.117	0.081	0.052

IV. OBSERVATIONAL VERSUS THEORETICAL VIRIAL PROPERTIES

a) Standard Deviations

The observed standard deviations of the virial parameters, $\sigma(x)$, for the samples of groups from Paper I are presented in Table 2 along with correlation coefficients to be discussed in § IVb. The data comprise the 42 STV groups [deV 8 and 45 (designations from de Vaucouleurs 1975) are omitted because they may be spurious] and 19 TG groups with a fractional mean error in V^2 caused by radial velocity uncertainties, $\sigma_v(V^2)/V^2$, less than 0.5. We want to compare these observational results with the total contribution to the standard deviations expected from theoretical estimates of the p, e, v, m, L_T, and d effects.

TABLE 2

OBSERVED VERSUS THEORETICAL STANDARD DEVIATIONS, CORRELATION COEFFICIENTS, AND
REGRESSION-LINE SLOPES

Sample or Effect	$\sigma(\log R)$	$\sigma(\log V)$	$\sigma(\log M_{\mathrm{VT}}/L_T)$	$\sigma(\log R_1)$
STV ($n = 42$, observed)..............	0.41	0.34	0.72	0.44
TG ($n = 19$, observed)..............	0.42	0.31	0.69	0.56
$pevmL_Td$...........................	0.26	0.15	0.36	0.11
p...................................	0.20	0.11	0.30	0.06
e...................................	0.07	0.06	0.05	0.07
v...................................	0.00	0.05	0.10	0.00
m...................................	0.14	0.06	0.16	0.15
L_T.................................	0.00	0.00	0.04	0.00
d...................................	0.05	0.00	0.05	0.05
STV (cosmic)........................	0.32	0.31	0.62	0.43
TG (cosmic).........................	0.33	0.27	0.59	0.55

Sample or Effect	ρ_{VR} [a]	ρ_{MR} [a]	ρ_{MV} [a]	S_{VR} [b]	S_{MR} [b]	S_{MV} [b]
STV ($n = 42$, observed)..............	0.15	0.65	0.71	0.13	1.15	1.52
TG ($n = 19$, observed)..............	0.14	0.46	0.74	0.11	0.76	1.61
$pevmL_Td$...........................	-0.03	0.18	0.17	-0.02	0.25	0.41
p...................................	0	1	1	...	...	...
e...................................	-1	-1	1	...	...	...
v...................................	0	0	1	...	...	...
m...................................	0	1	1	...	...	...
L_T.................................	0	0	0	...	...	...
d...................................	0	-1	0	...	...	...
STV (cosmic)........................	0.18	0.47	0.54	0.17	0.91	1.08
TG (cosmic).........................	0.17	0.28	0.57	0.14	0.50	1.25

[a] $\rho_{VR} = \rho(\log V, \log R)$, $\rho_{MR} = \rho[\log (M_{VT}/L_T), \log R]$, $\rho_{MV} = \rho[\log (M_{VT}/L_T), \log V]$.
[b] $S_{VR} = S(\log V, \log R)$, $S_{MR} = S[\log (M_{VT}/L_T), \log R]$, $S_{MV} = S[\log (M_{VT}/L_T), \log V]$.

The individual contributions $\sigma_p(x)$, $\sigma_e(x)$, $\sigma_v(x)$, $\sigma_m(x)$, $\sigma_{L_T}(x)$, $\sigma_d(x)$, and the combined contribution, $\sigma_{pevmL_Td}(x)$, are evaluated in Appendix III from the formulae presented in §§ II and III and application of detailed observational data and its estimated uncertainties. Results are presented in Table 2.

From Table 2, we see that $\sigma_{pevmL_Td}(\log M_{VT}/L_T) = 0.36$, whereas the observed $\sigma(\log M_{VT}/L_T) = 0.70$. A standard f-test (see, e.g., Brunk 1960, p. 247) shows that these standard deviations differ significantly. Hence, we are forced to conclude that *a source of dispersion in M_{VT}/L_T exists in addition to that produced by pevmL_Td effects.* If this source is a cosmic dispersion in M_{VT}/L_T, then the $\pm 2 \sigma$ range in the values of the cosmic M_{VT}/L_T of groups is a factor of 30! This result agrees both qualitatively and quantitatively with that obtained in Paper I.

b) Correlations

The definition of the correlation coefficient of the virial parameters y and x of a sample of n groups is (see, e.g., Trumpler and Weaver 1953, p. 49)

$$\rho(y, x) = \left(\sum_{i=1}^{n} \Delta x_i \Delta y_i \right) \bigg/ n\sigma_x \sigma_y , \qquad (6)$$

where σ_x and σ_y are the standard deviations of x_i and y_i, Δx_i is the observed displacement of x_i, and Δy_i is the corresponding displacement of y_i from the respective averages for the n groups.

The corresponding regression-line slope, which represents the form of the dependence, is (Trumpler and Weaver 1953, p. 39)

$$S(y, x) = \rho(y, x)(\sigma_y/\sigma_x) . \qquad (7)$$

The observed correlation coefficients of the virial parameters, $\rho(y, x)$, evaluated for the STV and TG samples, are presented in Table 2. We want to compare these observed results with the total contribution to the correlation coefficients expected from theoretical estimates of the $pemvL_Td$ effects. To estimate this contribution, we first need to know the correlation coefficients which would be induced by each effect acting alone, $\rho_j(y, x)$, where $j = p, e, m, v, L_T$, and d in turn. The values of $\rho_j(y, x)$ are estimated in Appendix IV from basic properties of the various effects, and tabulated in Table 2. Then, in Appendix V, they are combined with equation (6) to derive the following useful relations for the theoretical correlation coefficients induced by $pemvL_Td$ effects and all additional effects with combined correlation coefficients $\rho_A(y, x)$ and standard deviations $\sigma_A(x)$ and $\sigma_A(y)$:

$$\rho(\log V, \log R) = \frac{-\sigma_e(x)\sigma_e(y) + \rho_A(y, x)\sigma_A(x)\sigma_A(y)}{\sigma_x \sigma_y} , \qquad (8)$$

where $y = \log V$, $x = \log R$; and

$$\rho\left(\log \frac{M_{VT}}{L_T}, \log R\right) = [\sigma_p^2(x) - \sigma_e(x)\sigma_e(y) + \sigma_m^2(x) - \sigma_d^2(x) + \rho_A(y, x)\sigma_A(x)\sigma_A(y)](\sigma_x \sigma_y)^{-1} , \qquad (9)$$

where now $y = \log M_{VT}/L_T$ and $x = \log R$; and

$$\rho\left(\log \frac{M_{VT}}{L_T}, \log V\right) = [2\sigma_p^2(x) + \sigma_e(x)\sigma_e(y) + 2\sigma_v^2(x) + 2\sigma_m^2(x) + \rho_A(y, x)\sigma_A(x)\sigma_A(y)](\sigma_x \sigma_y)^{-1} , \qquad (10)$$

where now $y = \log M_{VT}/L_T$ and $x = \log V$.

If we adopt $\rho_A(y, x) = 0$, i.e., if we assume that there are no correlation effects in addition to $pevmL_Td$ effects, then equations (8)–(10) reduce to the contribution to the observed correlation coefficients by $pevmL_Td$ effects, $\rho_{pevmL_Td}(y, x)$.

From Table 2, we see that the observed correlation coefficient $\rho(\log M_{VT}/L_T, \log V) = 0.72$, whereas the contribution by p, e, v, m, L_T, d effects is $\rho_{pevmL_Td}(\log M_{VT}/L_T, \log V) = 0.13$. A test with the z-statistic (Hewlett-Packard 1974) shows that these correlation coefficients differ significantly. Hence, our assumption that $\rho_A(x, y) = 0$ cannot be correct. We are forced to conclude that a *source of correlation relating* $\log M_{VT}/L_T$ *to* $\log V$ *exists in addition to that produced by pevmL_Td effects.* If this source is a dependence of cosmic M_{VT}/L_T on cosmic V, then the cosmic correlation coefficient is $\rho_A(\log M_{VT}/L_T, \log V) = 0.60$. These results agree both qualitatively and quantitatively with those obtained in Paper I, where $\rho_{pe}(y, x)$ were estimated from results of Turner's N-body experiments and correlation coefficients were evaluated directly from equations (6), (A30), and (A31) of this paper.

V. THE COSMOLOGICAL FORCE

The virial properties of groups derived in Paper I and the present work can also be used to test the hypothesis of Jackson (1970) and Forman (1970) that the large M_{VT}/L_T values of groups are caused by the existence of a dominant cosmological force with negative cosmological constant (Λ) and origin at the group barycenter. The

equation of motion of a galaxy in a group moving in a cosmological force field is $\ddot{r} = \frac{1}{3}\Lambda r \hat{u}_r$ and the orbit is an ellipse with center at the group barycenter. Hartwick (1978) finds that there is a statistically significant correlation between log V and log R_I for low-density groups, and points out that this is consistent with the cosmological force hypothesis.

There are at least three observational results in conflict with the universal (applicable to all groups) validity of the cosmological force hypothesis.

1. Zwicky (1957, pp. 28, 136, and references therein) first recognized and Ostriker (1977) recently emphasized that Newtonian theory predicts an Emden isothermal density distribution in a rich regular cluster having an approximately radially independent velocity dispersion, and this density distribution is actually observed (see Bahcall 1977a, and references therein). Following Zwicky (1957, p. 136), we find from the equation of dynamical hydrostatic equilibrium that if the cluster is bound by a negative Λ cosmological force instead of a Newtonian gravitational force, then the radial density distribution, $\rho(r)$, is given by

$$\frac{\rho(r)}{\rho_c} = \exp\left[-\frac{\Lambda m}{6kT}(r^2 - r_c^2)\right], \tag{11}$$

where ρ_c is the density at a lower cutoff (or core) radius r_c, m is a characteristic value of galactic mass, k is Boltzmann's constant, and T is the "temperature" corresponding to the galactic velocity dispersion of the cluster. Equation (11) is a much more centrally concentrated density distribution than the density distributions observed in rich clusters (Bahcall 1977a).

The space density distribution for an ideal cluster in a steady state set up by the combined gravitational force and cosmological force with various negative cosmological constants has been solved numerically by Chen and Sachdev (1975).

2. A group of galaxies moving under the combined action of a Newtonian gravitational force and a cosmological force satisfies the relation (Rood 1974b)

$$\log V = \log\left(\frac{\mu}{\mu - 1}\right)^{1/2} R_I + \frac{\log(-\Lambda)}{2} - \frac{\log 3}{2}, \tag{12}$$

where V and R_I were defined earlier, and μ is the ratio of the virial mass M_{VT} derived with $\Lambda = 0$ to the true mass, $M = L_T f$, where f is the actual mass-to-light ratio of a group. (In the calculations, we adopt $f = 10.2$ solar units, the average mass-to-light ratio of individual galaxies derived by Dickel and Rood 1978.) The ratio μ for individual groups can be derived empirically from their observed M_{VT}/L_T and $f = 10.2$. If the Newtonian force were entirely absent, then $\mu = \infty$.

Equation (12) says that the slope of the log V on the log $[\mu/(\mu - 1)]^{1/2}R_I$ regression line should be

$$S_{V,\ [\mu/(\mu - 1)]^{1/2}R_I} = 1.$$

The predicted slope is reduced to 0.91 when $pevmL_Td$ effects are taken into account ($\mu = \infty$ was assumed to evaluate the $pevmL_Td$ contribution). The observed slopes calculated for STV, TG, and de Vaucouleurs groups with $\mu > 1.25$ and $\mu > 5$, and all groups in the STV, TG, and deV samples when $\mu = \infty$ is adopted, are contained in Table 3. The observed slopes are significantly smaller than the predicted slope. Although V and R_I for the TG and deV samples are correlated, the observed *slopes* of the regression lines are in conflict with the theoretical prediction. We emphasize that this result applies to groups in general, and not necessarily to the low-density subset examined by Hartwick (1978).

TABLE 3

SLOPES OF OBSERVED AND THEORETICAL REGRESSION LINES OF LOG V ON LOG $[\mu/(\mu - 1)]^{1/2}R_I$, AND STANDARD DEVIATIONS OF LOG Λ

Sample or Effect	S_{V,R_I}	Mean Error	$S_{V,[\mu/(\mu-1)]^{1/2}R_I}$	Mean Error	$\log(-\Lambda)$ [a]	$\sigma_{(\log \Lambda)}$
de V ($n = 41$)	0.60	0.17	...	...	...	...
STV ($n = 53$)	0.13	0.12	...	...	...	...
TG ($n = 28$)	0.43	0.15	...	...	...	...
de V ($n = 23$, $\mu \geq 5$)	0.19	0.23	0.16	0.24	5.10	0.65
STV ($n = 31$, $\mu \geq 5$)	−0.08	0.14	−0.09	0.15	5.42	0.94
TG ($n = 11$, $\mu \geq 5$)	0.21	0.10	0.21	0.10	6.02	0.97
de V ($n = 37$, $\mu \geq 1.25$)	...	...	0.44	0.20	4.90	0.65
STV ($n = 48$, $\mu \geq 1.25$)	...	...	0.03	0.12	5.24	1.00
TG ($n = 20$, $\mu \geq 1.25$)	...	...	0.27	0.17	5.67	1.02
$pevmL_Td$	−0.09	...	(−0.09)	...	...	0.30

[a] Λ in units of km^2 s^{-2} Mpc^{-2}.

3. The average value of $\log(-\Lambda)$ and the observed standard deviation about the average, $\sigma(\log \Lambda)$, derived from the STV, TG, and deV samples, are also contained in Table 3. Because Λ is a constant, $\sigma(\log \Lambda)$ must be produced entirely by $pvmL_Td$ effects and possibly other uncertainties if the cosmological force hypothesis is valid. Quasi-equilibrium effects do not contribute to $\sigma(\log \Lambda)$ because Λ is a constant during the orbital motion of a galaxy. V and R are uncorrelated when the p, v, m, L_T, d effects act alone. If we adopt the approximation

$$\frac{R}{R_I} = \left(\frac{2N}{N-1}\right)\frac{\langle 1/r_{ij}\rangle^{-1}}{\langle r_i^2\rangle^{1/2}} = \frac{2N}{N-1} = 3 , \tag{13}$$

then by differentiating equation (12) and taking logarithms, we obtain

$$\sigma_{pvmL_Td}(\log \Lambda) \approx \left[\left\langle \frac{4\mu^2}{(\mu-1)^2} \right\rangle \sigma_{pvmd}{}^2(\log V) + \left\langle \left(\frac{2\mu-3}{\mu-1}\right)^2 \right\rangle \sigma_{pvmd}{}^2(\log R_I) \right.$$
$$\left. + \left\langle \frac{1}{(\mu-1)^2} \right\rangle \sigma_{L_T}{}^2(\log L_T) \right]^{1/2} . \tag{14}$$

From Table 3, we see that the observed $\sigma(\log \Lambda) = 0.65$–$0.97$ whereas $\sigma_{pvmL_Td}(\log \Lambda) = 0.30$. Hence, if the contribution to $\sigma(\log \Lambda)$ by effects other than $pvmL_Td$ effects are small, then the values of Λ as determined by data for different groups cover a larger range than that produced if Λ is a constant, again in conflict with the cosmological force hypothesis.

VI. DISCUSSION

The major conclusion of our analysis of groups is that the observed range in M_{VT}/L_T and correlation between M_{VT}/L_T and V are larger than the combined contributions of projection and quasi-equilibrium effects, uncertainties in galactic radial velocities, galactic masses, group luminosities, and distances. Several possible explanations of this result are discussed below.

1. *Observational and theoretical uncertainties.*—Despite our efforts to procure and use high-quality data, and to evaluate uncertainties realistically, a possibility still exists that our conclusion will be found to be spurious in the future. However, we are confident of the basic validity of our estimates of projection and quasi-equilibrium effects (analytical formulae and Turner's N-body experiments produce concordant results), and of uncertainties caused by errors in galactic radial velocities, group luminosities, and distances. Our estimates of uncertainties due to errors in galactic masses are reliable if the mass derived from the internal dynamical structure or Holmberg luminosity of a galaxy is approximately proportional to its total mass (whether most of this mass is in the disk or an extended halo).

For our analysis, we implicitly assumed that the STV and TG groups are real and uncontaminated groups. The mean mass-to-light ratio and crossing time of these groups are consistent with the values derived from the statistical virial theorem of Geller and Peebles (1973). The STV groups represent an improvement over the original de Vaucouleurs groups in that group members are selected from a more accurate and complete radial velocity sample, and the group identification expertise of Sandage and Tammann was taken into account. It could therefore be significant that the virial effects observed for the STV groups are at least as pronounced as those observed originally for the original de Vaucouleurs groups (Rood, Rothman, and Turnrose 1970). The STV groups are found primarily, but not exclusively, in the Local Supercluster where problems with isolating groups exist. It is therefore encouraging that well isolated groups in the foreground of the Coma cluster have values of M_{VT}/L_T similar to those derived for the STV groups (Gregory and Thompson 1978).

The STV groups were identified by the rough application of a "surface density contrast" criterion and a loose "radial velocity contrast" criterion. The TG groups represent a refinement because they were identified by the quantitative application of a strict density contrast criterion, but they too rely on a loose criterion of radial-velocity contrast. The further refinement—to quantitatively apply a strict criterion of radial-velocity contrast—will be incorporated in the several important projects to identify groups which are currently in progress. It should, however, be emphasized that many researchers believe that the problem of accurate and unbiased membership assignment is *the central difficulty* in understanding groups of galaxies. We hope to allow for this problem in a future analysis (see § I).

2. *Intrinsic redshifts.*—Radial velocities of galaxies are derived on the assumption that the displacements of the spectral lines of known elements from their rest wavelengths are determined by the Doppler effect. If a non-Doppler (intrinsic, origin unknown) effect primarily determined the line displacement, then the derived values of V and M_{VT}/L_T would be wrong and correlated. Evidence for and against the possibility of intrinsic redshifts in some peculiar objects is discussed by Field, Arp, and Bahcall (1973).

Tifft has found bands in the (galactic redshift, nuclear magnitude)-diagram for the core of the Coma cluster and possibly other clusters (Tifft 1974 and references therein), and he suggests an intrinsic redshift origin. The present authors do not know the cause(s) of the bands observed by Tifft, but we find it difficult to accept an intrinsic redshift origin for the following reasons: (a) The optical redshift of the stellar and ionized gaseous components of

the *nuclear* region of a galaxy agrees with the redshift obtained from the 21 cm line of neutral hydrogen in the galactic *disk* (Rood 1974*a*; Dickel and Rood 1975, 1978). (*b*) If Doppler redshifts in the Coma cluster were in fact small, then two-body relaxation would produce much larger radial mass segregation in the age of the Coma cluster (assumed to be at least the age of the Earth or our Galaxy) than is observed (Rood 1974*a*). (*c*) The temperature of the ionized intergalactic gas in the Coma, Perseus, and Virgo clusters, derived from the shape of the X-ray spectrum, corresponds to the observed velocity dispersion of the cluster galaxies (Kellogg, Baldwin, and Koch 1975; Bahcall and Sarazin 1977). (*d*) The virial mass of groups and other systems of galaxies is correlated with other mass indicators (e.g., total luminosity and possibly the mass of ionized intergalactic gas). (*e*) The presence of radio trial galaxies in clusters (Miley *et al.* 1972) and the correlations of the spiral/S0 ratio with radial distance from the cluster center (Gregory 1975; Melnick and Sargent 1977) and decreasing cluster X-ray luminosity (Bahcall 1977*b*) are also consistent with the Doppler interpretation of redshift.

3. *Unbound groups.*—The groups are bound because their crossing times are smaller than the Hubble time and the composite probability function of their velocities (relative to the group barycenter) is well approximated by a Gaussian function.

The actual mass of a bound group could differ significantly from the virial mass if the group is not virialized. Recent *N*-body experiments by Ross and analytical considerations by King (private communication) indicate that it takes not ∼2 but several crossing times for a group to become virialized, which leads King to ask if this might be the cause of the observed virial properties. In partial answer to this question we note that (*a*) the values of M_{VT}/L_T for the STV and TG groups are uncorrelated with crossing time (Table 4 of Paper I), and (*b*) the distribution in a plot of log M_{VT}/L_T versus log V (Fig. 5 of Paper I) of the points for the 14 STV groups and seven TG groups with the largest crossing times suggests that these groups may actually have a smaller range in log M_{VT}/L_T and a weaker correlation between M_{VT}/L_T and V than the groups with smaller crossing times.

4. *Missing mass.*—(*a*) It may be in the most massive groups with high velocity dispersions that increased inter-actions cause more stripping of stars or other material from the outer parts of galaxies or the formation of more faint stars in their outer reaches so that more nonluminous mass is present. (*b*) If the difference between the average mass-to-light ratio of groups and individual galaxies is caused by the presence of the missing mass, and our estimates of uncertainties are valid, then the observed *range* in M_{VT}/L_T implies a ±2 σ range of a factor of 30 in cosmic M_{VT}/L_T. This means that some groups could have no missing mass but others could have up to 30 times more missing mass than visible mass. Mass-to-light ratios of rich clusters, which tend to be more accurate than M_{VT}/L_T values for individual groups, range from about $M_{VT}/L_v \approx 30$ to $M_{VT}/L_v \approx 400$ solar units (data from Oemler 1974; Dressler 1978; van den Bergh 1978). If the fraction of missing mass in a group or cluster ranges from 0.0 to more than 0.9, then intuitively we might expect other observable group or cluster properties to vary systematically with mass-to-light ratio, but such variations appear to be small or nonexistent.

5. *Missing dynamical physics.*—Groups and clusters of galaxies are about 10^{10} times larger than the solar system, so a considerable extrapolation is involved in the application of Newtonian theory to such systems. Nevertheless, Newtonian theory predicts the observed isothermal density distribution of a rich cluster (Zwicky 1957; Ostriker 1977). If a different theory were in fact valid, then it must also predict an isothermal density distribu-tion, and the identical Newtonian prediction would then be a remarkable coincidence!

So we see that all of the above hypotheses encounter difficulties, but one of these effects or some other effect must be present. New insights are clearly needed. The history of other dynamical problems, illustrated by some in the solar system, might provide some perspective: There are many examples of observed discrepancies and effects which later turned out to be caused by underestimates of errors or even blunders, but there are also notable exceptions which required new ideas. Kepler, by a careful analysis of relatively uncertain data, was able to extract the three laws which played a central role in the formulation of Newtonian dynamical theory (see, e.g., Pannekoek 1961). Newtonian theory triumphed with the observational detection of the missing mass (Neptune) resolving problems with the orbit of Uranus (see, e.g., Smart 1953). Later, and in contrast, the excess precession of the perihelion of Mercury was resolved with the discovery of missing dynamical theory (general relativity), which then predicted the existence of intrinsic redshifts in gravitational fields (see, e.g., Weinberg 1972).

APPENDIX I

PROJECTION EFFECTS

By definition, the standard deviation of $\log_{10} V$ (about the average) caused by projection effects alone is

$$\sigma_p(\log V) = \frac{1}{2C} \frac{\sigma_p \langle v'^2 \rangle}{\langle v'^2 \rangle} , \tag{A1}$$

where C is 2.303 (the natural logarithm of 10) and v' is the ratio of the line-of-sight velocity (relative to the group barycenter) to the space velocity.

For randomly oriented velocity vectors, the probability function of v'^2 is

$$f(v'^2) = \frac{1}{2v'}, \qquad 0 \le v'^2 \le 1 \,. \tag{A2}$$

The variance of v'^2 about its average value ($\frac{1}{3}$ derived by Limber and Mathews 1960) is

$$\sigma_p{}^2(v'^2) = \int_0^1 (v'^2 - \tfrac{1}{3})^2 f(v'^2) d(v'^2) = 4/45 \,. \tag{A3}$$

If N galaxies are randomly sampled from a group, then, according to the central limit theorem (see, e.g., Brunk 1960, p. 156), the sample $\langle v'^2 \rangle$ has an approximately normal distribution with mean $\frac{1}{3}$ and standard deviation

$$\sigma_p(\langle v'^2 \rangle) = N^{-1/2}\sigma(v'^2) = \tfrac{2}{3}(5N)^{-1/2} \,. \tag{A4}$$

Substituting into equation (A1), we obtain the relation for $\sigma_p(\log V)$, equation (1).

Similarly, $\sigma_p(\log R_l)$, equation (4), follows from the frequency distribution of r'^2 (the ratio of the projected separation to the true separation),

$$f(r'^2) = \frac{1}{2(1 - r'^2)^{1/2}} \,, \tag{A5}$$

and the variance of r'^2 about its average value ($\frac{2}{3}$),

$$\sigma_p(r'^2) = \int_0^1 (r'^2 - \tfrac{2}{3})^2 f(r'^2) d(r'^2) = \frac{2}{3\sqrt{5}} \,. \tag{A6}$$

And finally, $\sigma_p(\log R)$, equation (2), follows from the relations involving $y = r'^{-1}$,

$$f(y) = \frac{y_M}{(y_M{}^2 - 1)^{1/2}} \frac{1}{y^2(y^2 - 1)^{1/2}} \,, \qquad 1 \le y \le y_M \,, \tag{A7}$$

$$\langle y \rangle = \frac{y_M}{(y_M{}^2 - 1)^{1/2}} \sec^{-1} y_M = \alpha \frac{\pi}{2} \,, \tag{A8}$$

and

$$\sigma^2(y) = \frac{y_M}{(y_M{}^2 - 1)^{1/2}} \left\{ C \log \left[y_M + (y_M{}^2 - 1)^{1/2}\right] - \alpha\pi \cos^{-1}\left(\frac{1}{y_M}\right) + \left(\alpha \frac{\pi}{2}\right)^2 \frac{(y_M{}^2 - 1)^{1/2}}{y_M} \right\} \,, \tag{A9}$$

where y_M is the maximum value of y set by the criterion that two galaxies separated on the celestial sphere by a distance smaller than some chosen value (r_s), typically their average diameter, are treated as a single galaxy in virial calculations; for the 42 STV groups (deV 8 and deV 45 omitted because they may be spurious) and 19 TG groups of Paper I, we find $1 \lesssim y \lesssim 100$ with a median value $y_M \sim 25$, so that

$$\alpha = 0.975 \quad \text{and} \quad \sigma(y) = 1.2 \,. \tag{A10}$$

Table 4 lists α and $\sigma(y)$ as a function of y_M. Note that as y_M approaches infinity, (a) α approaches unity so that $\langle y \rangle$ reduces to the result obtained by Limber and Mathews (1960), and (b) $\sigma(y)$ diverges logarithmically. For *compact groups*, as y_M approaches unity, y_M approaches 0.6 and $\sigma(y)$ approaches zero.

TABLE 4

α AND $\sigma(y)$ AS A FUNCTION OF y_M

y_M	α	$\sigma(y)$
1..........	0.6366	0.00
2..........	0.7698	0.24
5..........	0.8898	0.62
10..........	0.9409	0.91
25..........	0.9753	1.25
50..........	0.9875	1.48
100..........	0.9937	1.69

Because v and r are assumed to be uncorrelated with each other, $\sigma_p(\log V)$ and $\sigma_p(\log R)$ are independent parameters, so that $\sigma_p(\log M_{VT}/L_T)$ is given by equation (3).

APPENDIX II

QUASI-EQUILIBRIUM EFFECTS

a) Two-Body Mass Distribution

A first approximation to the evaluation of quasi-equilibrium effects for a group can be obtained by considering the limiting case where a sample galaxy of mass m is assumed to move under the gravitational attraction of a point mass, $M - m$, where M is the total mass of the group (i.e., the sum of the masses of its galaxies if intergalactic material can be neglected). We denote the separation between the two bodies by r and their velocity difference by v. If we assume initially that all the sample galaxies have the same mass, then the standard deviations of R, V, and M_{VT}/L_T caused by quasi-equilibrium effects alone are the standard deviations of r, v, and v^2r/G, respectively. The effect of a mass spectrum of sample galaxies is discussed in Appendix III.

By applying the basic equations governing two-body motion (e.g., see McCuskey 1963), we can evaluate the variance

$$\sigma_e^2(r) = \int_0^p \frac{(r - \langle r \rangle)^2}{p} \, dt, \tag{A11}$$

where p is an orbital period. For a sample of N galaxies in a group, it then follows that

$$\sigma_e(\log R) = \sigma_e(\log R_I) = \frac{e(1 - \frac{1}{2}e^2)^{1/2}}{C(2N)^{1/2}}, \tag{A12}$$

where e is a characteristic orbital eccentricity. Similarly,

$$\sigma_e(\log V) = \frac{e[U(e)]^{1/2}[1 + 2e^2U(e)]}{C(2N)^{1/2}}, \tag{A13}$$

where

$$U(e) = 1 + \tfrac{3}{4}e^2 + \frac{3 \cdot 5}{4 \cdot 6} e^4 + \frac{3 \cdot 5 \cdot 7}{4 \cdot 6 \cdot 8} e^6 + \cdots. \tag{A14}$$

And finally,

$$\sigma_e\left(\log \frac{M_{VT}}{L_T}\right) = \frac{e(1 - \frac{1}{2}e^2)^{1/2}}{C(2N)^{1/2}} (1 + e^2 y). \tag{A15}$$

The two-body model of a group represents the limiting case of extreme central density. Differences between results obtained with a two-body mass distribution and results obtained with a less approximate mass distribution provide an indication of the sensitivity of results to mass distributions.

b) Homogeneous Mass Distribution

In Paper I, we found that the *average* surface number density of galaxies within the main body of a group is approximately constant. Within the observational uncertainties, this result suggests that a sphere with a homogeneous mass distribution is a more realistic model of a typical group than is a two-body model. By applying the easily derivable equations governing the motion of a galaxy in a homogeneous sphere, we find

$$\sigma_e(\log R) = \sigma_e(\log R_I) = \frac{\{1 - \frac{1}{2}e^2 - [2\pi^{-1}E(e)]^2\}^{1/2}2\pi^{-1}K(e)}{C\sqrt{N}}, \tag{A16}$$

$$\sigma_e(\log V) = \frac{e^2[2\pi^{-1}K(e) + e^2/4]}{4C(2N)^{1/2}}, \tag{A17}$$

and

$$\sigma_e\left(\log \frac{M_{VT}}{L_T}\right) = \frac{\sigma(v^2r/ka^3)}{C\sqrt{N}} \left\langle \frac{1}{v^2r/ka^3} \right\rangle, \tag{A18}$$

where k is the gravitational constant times $4\pi/3$ of the mass density of the sphere, e is the orbital eccentricity, a

is the orbital semimajor axis (the orbit is an ellipse with center at the center of the sphere), $E(e)$ is the complete elliptic integral of the second kind, $K(e)$ is the complete elliptic integral of the first kind, and

$$\left\langle \frac{1}{v^2 r/ka^3} \right\rangle = 1 + 0.75e^2 + 0.578125e^4 + 0.464844e^6 + 0.380595e^8 + 0.335861e^{10} + \cdots , \tag{A19}$$

$$\sigma\left(\frac{v^2 r}{ka^3}\right) = (1 - \langle v^2 r/ka^3 \rangle - \tfrac{3}{2}e^2 + \tfrac{5}{8}e^4 - \tfrac{1}{16}e^6)^{1/2} , \tag{A20}$$

$$\langle v^2 r/ka^3 \rangle = 1 - \frac{e^2}{4}(3 - e^2) - \frac{3e^4(5 - e^2)}{64} - \frac{5e^6}{256}(3 - e^2) - \frac{105e^8}{49152}(13 - 5e^2)$$

$$- \frac{14175e^{10}}{14745600}(17 - 7e^2) - \cdots . \tag{A21}$$

APPENDIX III

THEORETICAL CONTRIBUTIONS TO STANDARD DEVIATIONS

1. Contributions by projection effects, $\sigma_p(x)$, can be evaluated from relations in § II if a value for N, the effective number of galaxies sampled in a group, is selected. Because, in the evaluation of virial parameters, velocities are galactic mass weighted and galactic separations are weighted by mass products, N is the actual number of galaxies sampled (usually about 6) only when all galaxies have the same mass. At the other extreme, if all but a tiny fraction of the mass of a group were in a single galaxy, then $N = 1$ regardless of the actual number of galaxies sampled. In general, we adopt $N = m_1^{-1}(m_1 + m_2 + \cdots + m_{AN})$, where m_1 is the mass of the most massive group member, m_2 is the mass of the second most massive member, ..., m_{AN} is the mass of the least massive member sampled. $\langle N \rangle = 3.25$ and $N_{\mathrm{median}} = 2.71$ for the 42 STV group sample, and $\langle N \rangle = 3.61$, $N_{\mathrm{median}} = 2.84$ for the 19 TG group sample. In our calculations, we adopt $N = 3$. The groups in Turner's N-body simulations used in Paper I have $N = 2.67$.

2. Contributions by quasi-equilibrium effects, $\sigma_e(x)$, can be evaluated from relations in § III, if a model and a characteristic orbital eccentricity, e, are selected. The homogeneous model is generally a better approximation than the two-body model except possibly for galaxies with small orbital eccentricity in the extreme outer regions of a group. Values of $\sigma_e(x)$ as a function of orbital eccentricity of galaxies in the two-body and homogeneous models are presented in Table 1. For a given eccentricity, the two-body model produces larger values of $\sigma_e(x)$ than does the homogeneous model. For $e \gtrsim 0.6$ in the two-body model, $\sigma_e(\log V)$ exceeds the total observed standard deviation, $\sigma(\log V)$, for the STV and TG samples. Hence, $\sigma_e(\log M_{\mathrm{VT}}/L_T) < 0.15$ for the minority of galaxies in the extreme outskirts of a group. The velocity vectors are oriented isotropically—i.e., orthogonal velocity dispersions are equal—if $e = 0.9$ for both the two-body and the homogeneous models. For our calculations, we adopt the values of $\sigma_e(x)$ corresponding to $e = 0.9$ of the homogeneous model.

3. Contributions by uncertainties in galactic radial velocities, $\sigma_v(x)$, can be evaluated from the mean errors they produce in V and V^2, i.e., $\sigma_v(V)$ and $\sigma_v(V^2)$, which are evaluated for each STV and TG group in Paper I. It follows that

$$\sigma_v(\log R) = \sigma_v(\log R_I) = 0 , \tag{A22}$$

$$\sigma_v{}^2(\log V) = \frac{1}{C^2 n} \sum_{i=1}^{n} \frac{\sigma_v{}^2(V)}{V^2} , \tag{A23}$$

and

$$\sigma_v{}^2\left(\log \frac{M_{\mathrm{VT}}}{L_T}\right) = \frac{1}{C^2 n} \sum_{i=1}^{n} \frac{\sigma_v{}^2(V^2)}{V^4} , \tag{A24}$$

where the summation is over the $n = 42$ (STV) (or $n = 19$) TG groups.

4. Contributions by uncertainties in galactic masses, $\sigma_m(x)$, can be evaluated by comparing the observed virial parameters, x, derived from galactic masses, m_i, with the values x' calculated using galactic masses larger or smaller (randomly chosen for each galaxy) than m_i by the average absolute fractional mean errors estimated by Dickel and Rood (1978). It follows that

$$\sigma_m{}^2(x) = \sum_{i=1}^{n} \frac{(\log x/x')^2}{n - 1} , \tag{A25}$$

where x can equal $\log R$, $\log R_I$, $\log V$, or $\log M_{\mathrm{VT}}/L_T$.

5. Contributions by uncertainties in galactic luminosities, $\sigma_{L_T}(x)$, are zero when $x = \log R$, $\log R_I$, and $\log V$. The quantity $\sigma_{L_T}(\log M_{VT}/L_T)$ can be evaluated from

$$\sigma_{L_T}{}^2\left(\log \frac{M_{VT}}{L_T}\right) = \sum_{i=1}^{n} \frac{(\log L_T/L_T')^2}{n-1}\,, \tag{A26}$$

where L_T is the adopted total luminosity of a group. We set L_T equal to the average of the values obtained by extrapolating the sum of the luminosities of galaxies as bright or brighter than (i) the faintest and (ii) the second faintest sampled group member with a Schechter luminosity function. L_T' is the luminosity estimated from case (i) [or case (ii)] alone. The observational errors in the apparent luminosities of the brighter galaxies in groups are usually a few hundredths of a magnitude—photoelectric accuracy, so the contribution to $\sigma_{L_T}(x)$ by this source can be neglected. The uncertainty in galactic extinction is typically ~ 0.15 mag, and the uncertainty in internal extinction is typically ~ 0.2 mag, but these errors are primarily systematic, and we will neglect them. Errors in luminosities due to errors in distances are incorporated in the discussion of distance uncertainties below.

6. Contributions by uncertainties in the distances to groups, $\sigma_d(x)$, are evaluated from the mean errors in the logarithm of the redshift distances, d, of individual groups caused by their velocity dispersions, V, i.e.,

$$\sigma_d(\log R) = \sigma_d(\log R_I) = \sigma_d\left(\log \frac{M_{VT}}{L_T}\right) = \sum_{i=1}^{n} \frac{[\sigma(\log d)]^2}{n-1}\,, \tag{A27}$$

$$\sigma_d(\log V) = 0\,, \tag{A28}$$

where

$$\sigma(\log d) = \frac{V}{3^{1/2} C H d N^{1/2}}\,. \tag{A29}$$

7. Total deviations: The individual p, e, v, m, L_T, and d contributions and the total cumulative $pevmL_Td$ contributions to $\sigma(x)$, calculated as explained above, are listed in Table 2. These values contain the following uncertainties:

a) The quantity $\sigma_p(\log R)$ is overestimated because the galactic separation vectors are actually correlated with one another. But because $N = 3$, there are only three separation vectors, and two can have any orientation. If the contribution by the third separation vector were neglected entirely, then an extreme lower limit to the actual value of $\sigma_p(\log R)$ would be $\sqrt{(\frac{2}{3})} = 0.82$ of the adopted value.

b) We have adopted the homogeneous sphere $e = 0.9$ values of $\sigma_e(x)$ in our calculations. Homogeneous sphere $e < 0.9$ values are smaller (see Table 3). An overestimate to $\sigma_e(x)$ is obtained by assuming that half the galaxies in a group have the homogeneous sphere $e = 0.9$ values but the other half have the maximum permissible (set by the observed value of $\log V$) two-body values. Then $\sigma_e(\log M_{VT}/L_T) = [(0.05)^2 + (0.15)^2]^{1/2} = 0.11$, so that $\sigma_{pemvL_Td}(\log M_{VT}/L_T)$ would be increased by a factor of only 1.05.

The contributions $\sigma_{pe}(x) = [\sigma_p{}^2(x) + \sigma_e{}^2(x)]^{1/2}$ derived from our analytical relations agree well with the estimates obtained in Paper I from Turner's N-body simulations. This suggests that $\sigma_{pe}(x)$ is insensitive to the simplifying assumptions adopted in our derivations.

c) To derive $\sigma_v(x)$, we adopted the mean errors in radial velocity determinations derived by de Vaucouleurs, de Vaucouleurs, and Corwin (1976) from a comprehensive comparison of the values obtained by different workers for galaxies held in common. Dickel and Rood (1978) have some evidence that the de Vaucouleurs et al. estimates may be too small by an average factor of 1.5; but even if this were the case, $\sigma_{pemvL_Td}(\log M_{VT}/L_T)$ would be increased by a factor of only 1.15.

If we increase our samples of groups by including those with $\sigma_v(V^2)/V^2 > 0.5$, then the 61 resulting STV groups have $\sigma_v(\log V) = 0.10$, which would increase the value of $\sigma_{pemvL_Td}(\log M_{VT}/L_T)$ by a factor of only 1.12. For the TG groups, inclusion of the "high velocity uncertainty" groups gives a total sample of 29, but they are totally dominated by radial velocity uncertainties, so we can use this total sample only after more accurate radial velocities are obtained.

d) In our analysis, the primary indicator of galactic mass is assumed to be the observed internal dynamical structure of a galaxy. Masses determined by this means were estimated to have an average random error of a factor of 1.4 (or 1.2 for a small number of especially well observed galaxies) (Dickel and Rood 1978). A secondary galactic-mass indicator, calibrated against the primary indicator, is taken to be the Holmberg luminosity of a galaxy. Masses determined from Holmberg luminosities were estimated to have an average random error of a factor of 1.8. These estimated uncertainties are valid to the extent that the observed internal dynamical structure and Holmberg luminosities are actual indicators of the total mass of a galaxy. If they are not, then the weighting of velocities and separations needed to estimate virial parameters would be unknown, and estimates in the literature would be correspondingly uncertain by an unknown amount.

In what follows, we assume that the observed internal dynamical structure and Holmberg luminosities are, in fact, reliable indicators of total galactic mass, and their random uncertainties are approximately as given above. Then the statistically estimated values of $\sigma_m(x)$ are, in effect, the additional cumulative contribution to $\sigma_{pevd}(x)$

caused by the uncertainties in the effective number of group members N. It is important to recognize that the estimates of $\sigma_m(x)$ remain valid even if our adopted galactic masses are all *systematically in error by the same factor*.

e) About three-fourths (average and median value) of the total luminosity of each STV and TG group is contained in the galaxies sampled. Consequently, (i) the derived value of L_T is relatively insensitive to the particular functional approximation of the luminosity function adopted to extrapolate from the observed to the total group luminosity, and (ii) if we retain the assumption that luminosities are mass indicators, then, on the average, the observed properties of three-fourths of the mass of a group are used in our virial calculations, making sampling uncertainties small.

f) Our results are completely independent of the value of the Hubble constant; i.e., $\sigma(x)$ and $\sigma_d(x)$ remain unchanged when H is changed. Our distance estimates, however, rest on the assumption that the velocity of the center of mass of a group is part of the Hubble flow. The parameter $\sigma_d(x)$ is larger if that is not the case, but Sandage and Tammann (1975) present evidence that it *is* the case.

We conclude that if present estimates of galactic masses are at least approximately correct or systematically in error by a common factor, then uncertainties in $\sigma_{pevmL_Td}(x)$ are so small that the standard deviations contained in Table 3 are realistic and meaningful.

APPENDIX IV

THEORETICAL CONTRIBUTIONS TO CORRELATION COEFFICIENTS

The correlation coefficients caused by each type of uncertainty acting alone are listed in Table 2. They have been estimated from the following considerations.

1. *Projection effects.*—To study projection effects in § II, we assumed that velocity and separation vectors are independent of one another. Accordingly, if $\log R$ changes due to projection effects by an amount $\Delta_p(\log R)$, then $\log V$ is unaffected but $\Delta_p(\log M_{VT}/L_T)_V = \Delta_p(\log R)$. (The subscript V means that V is held constant as R changes.) Similarly, if $\log V$ changes by $\Delta_p(\log V)$, then $\Delta_p(\log M_{VT}/L_T)_R = 2\Delta p(\log V)$.

2. *Quasi-equilibrium effects.*—We assume that the homogeneous sphere model of a group applies. Then changes in R will be a weighted sum of changes in r, the orbital radius of a galaxy, changes in V will similarly follow from changes in the orbital velocity v, and changes in M_{VT}/L_T will follow from changes in v^2r. Consequently, the various correlation coefficients can be estimated from derivatives of equations governing the orbital motion. Time average values are listed in Table 2.

3. *Radial velocity uncertainties.*—A fractional change in $\log V$ due to this effect does not change R but does change $\log M_{VT}/L_T$ by $2\Delta_V(\log V)$.

4. *Galactic mass uncertainty.*—The correlation coefficients induced by this effect are estimated in strict analogy with the case of projection effects.

5. *Group luminosity uncertainties.*—These affect neither V nor R.

6. *Distance errors.*—A fractional change in distance produces a corresponding fractional change in R and $(M_{VT}/L_T)^{-1}$, but does not affect V.

APPENDIX V

THEORETICAL NET CORRELATION COEFFICIENTS

The displacements Δx_i and Δy_i in the definition of the correlation coefficient, equation (6), are the sums

$$\Delta x_i = (\Delta x_p + \Delta x_e + \Delta x_v + \Delta x_m + \Delta x_{L_T} + \Delta x_d + \Delta x_A)_i,$$ (A30)

$$\Delta y_i = (\Delta y_p + \Delta y_e + \Delta y_v + \Delta y_m + \Delta y_{L_T} + \Delta y_d + \Delta y_A)_i,$$ (A31)

where $\Delta x_j, \Delta y_j$ ($j = p, e, v, m, L_T, d$) are the displacements induced by each effect acting alone, and $\Delta x_A, \Delta y_A$ represent the net displacements by all effects in addition to the *pevmL_Td* effects. Inserting equations (A30) and (A31) into equation (6), we find

$$\rho(y, x) = \frac{1}{n\sigma_x\sigma_y} \sum_{i=1}^{n} (\Delta x_p\Delta y_p + \Delta x_e\Delta y_e + \Delta x_v\Delta y_v + \Delta x_m\Delta y_m + \Delta x_{L_T}\Delta y_{L_T} + \Delta x_d\Delta y_d + \Delta x_A\Delta y_A)_i.$$ (A32)

Sums of mixed effects such as

$$\sum_{i=1}^{n} \Delta x_p\Delta y_e$$

are equal to zero because different effects are statistically independent, i.e., uncorrelated.[1]

[1] Recall that the contributions to standard deviations by errors in luminosities due to distance errors are incorporated entirely in the contributions by distance errors.

Consider the jth effect. The standard deviations in the sample of n groups produced if the jth effect acted alone are $\sigma_j(x)$ and $\sigma_j(y)$. The corresponding correlation coefficient is

$$\rho_j(y, x) = \left[\sum_{i=1}^{n} (\Delta x_j \Delta y_j)_i \right] \Big/ [n\sigma_j(x)\sigma_j(y)] . \tag{A33}$$

It follows from equation (A33) that if $\rho_j(y, x) = 0$, then $\sum_{i=1}^{n} (\Delta x_j \Delta y_j)_i = 0$; if $\rho_j(y, x) = 1$, then $\sum_{i=1}^{n} (\Delta x_j \Delta y_j)_i = n\sigma_j(x)\sigma_j(y)$; and if $\rho_j(y, x) = -1$, then $\sum_{i=1}^{n} (\Delta x_j \Delta y_j)_i = -n\sigma_j(x)\sigma_j(y)$. Applying this result to each p, e, v, m, L_T, and d effect by using the correlation coefficients for each effect derived in Appendix IV and tabulated in Table 2, we obtain the relations for the net correlation coefficients, equations (8)–(10).

The results of the discussion on the uncertainties in the standard deviations of virial parameters due to the p, e, v, m, L_T, d effects (Appendix III) apply as well to the correlation coefficients. If galactic mass estimates are at least roughly correct or systematically in error by a common factor, then the derived net correlation coefficients are realistic and meaningful.

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