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THE THIRD INTEGRAL OF MOTION FOR LOW-VELOCITY STARS

BY H. C. VAN DE HULST

The motion of a star with moderate deviation from a circular orbit can be computed by the method of separation of variables. The suitable co-ordinate system in the meridional plane is uniquely defined by the coefficients of expansion of the gravitation potential around the equilibrium point. It is an elliptic co-ordinate system, the axis of which does not coincide with the axis of rotation. The third integral is quadratic in the velocity components. Application of this method to two sets of initial conditions, for which CONTOPOULOS has computed numerical orbits, shows that the form and size of the envelopes, and the mean periods in both directions all emerge with an accuracy better than one per cent.

1. *Introduction*

The motion of a star in the gravitational field of a galaxy with rotational symmetry has two known integrals, corresponding to the total energy and the component of angular momentum about the axis. There is evidence but no general proof for the existence of a third integral. Small oscillations about the equilibrium orbit occur independently in the vertical and radial directions so that the "vertical energy" (i.e. the sum of the kinetic energy perpendicular to the plane and the potential energy corresponding to the distance from the plane) is a further integral. For oscillations of moderate amplitude CONTOPOULOS (1958, 1960) has shown, first by two numerical orbits, then by series expansion, that such an integral exists. His analytical derivation is complete except for a proof of the convergence. Finally, for large oscillations, which correspond to high-velocity stars, the numerical work of OLLONGREN (1962) and TORGÅRD contains empirical evidence for the existence of a third integral.

The object of this paper is to show that the results of CONTOPOULOS for moderately small amplitudes can be obtained in a simple manner and with ample numerical accuracy by choosing appropriate elliptical co-ordinates and a special form for the potential for which the third integral is known because the Hamilton-Jacobi equation is separable.

The usual reduction to a problem in two dimensions by means of the angular momentum integral is applied. We thus consider only the orbit in the meridional plane, which looks like a distorted Lissajous figure. General theorems about the motions in such orbits, including non-orthogonal distortions, have recently been derived by VAN DE HULST (1962). However, in the present context we consider only a distortion into elliptical co-ordinates, which defines a conformal representation. This is classical theory since EULER solved the "problem of the two fixed centres" about 200 years ago. Possible forms of a galactic potential leading to orbits of this type have been discussed by VAN ALBADA (1952) and by KUZMIN (1953, 1956 a, b). However, their assumption that the foci of the co-ordinate system lie on the rotation axis of the galactic system imposes an unnecessary restriction. This restriction is not introduced in the present paper, so that the co-ordinate surfaces in three dimensions are not ellipsoids and hyperboloids of rotation, but elliptical toroids. The discontinuities at the axis are of no consequence because the orbit does not come there. There are two free parameters, which will be shown to have a precise relation to the coefficients in the series expansion of the potential near the equilibrium point.

2. *Solution in terms of elliptical co-ordinates*

The orthogonal co-ordinates, measured from the equilibrium position (which corresponds to a circular orbit in the complete motion) are [1])

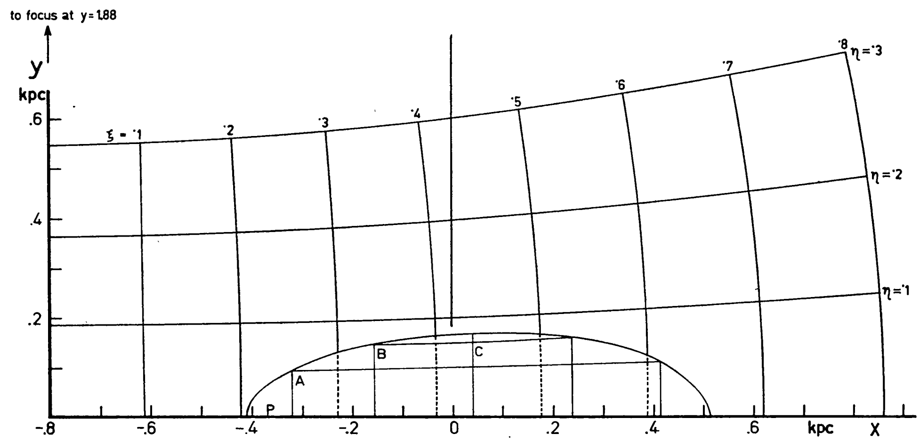
in the galactic plane, outwards: x (usually $\varpi - \varpi_0$)
vertical to the galactic plane : y (usually z)

Elliptical co-ordinates ξ and η are defined by

$$\begin{aligned} x &= c\,\mathrm{sh}\,\xi \cos\eta - k, \\ y &= c\,\mathrm{ch}\,\xi \sin\eta\,. \end{aligned} \tag{1}$$

[1]) At this point the assumption is introduced that there is not only an axis of symmetry, but also a plane of symmetry. This new assumption is obvious in view of the intended application. However, the theory may equally well be developed without it. Instead of two free parameters (c, k) then four parameters, giving the co-ordinates of the two foci in the meridional plane, appear.

FIGURE 1



Envelopes of box orbits shown within the boundary curve. The elliptical co-ordinate system is superposed.

The constants c and k define the position of the two foci: $x = -k, y = \pm c$. The local scale along either of the co-ordinate curves is given by

$$\left(\frac{ds}{d\xi}\right)^2 = \left(\frac{ds}{d\eta}\right)^2 = R = c^2 (\mathrm{ch}^2\xi - \sin^2\eta) \qquad (2)$$

and the angle ε by which the ξ and η curves are turned with respect to the x and y directions, respectively, follows from

$$\tan \varepsilon = \mathrm{th}\, \xi \tan \eta. \qquad (3)$$

In the numerical example (Figure 1) ε is smaller than 2 degrees at any point reached by the orbits.

Introducing the quantities

$$\left.\begin{aligned} x_0 &= c\, \mathrm{sh}\, \xi \quad - k, \\ R_0 &= c^2 \mathrm{ch}^2 \xi = l^2 + 2k\, x_0 + x_0^2, \\ l^2 &= c^2 + k^2, \end{aligned}\right\} \qquad (4)$$

and expanding into powers of η up to η^2 we may write equations (1), (2), (3) in the form:

$$\left.\begin{aligned} x &= x_0 - \tfrac{1}{2}(k + x_0)\, \eta^2, \\ y &= R_0^{\frac{1}{2}} \eta, \\ R &= R_0 - c^2\eta^2, \\ \tan \varepsilon &= (k + x_0)\, \eta\, R_0^{-\frac{1}{2}}. \end{aligned}\right\} \qquad (5)$$

Let α be the total energy, $\Phi(x,y)$ the potential energy, and $K = \alpha - \Phi$ the kinetic energy. It is known (STÄCKEL 1890) that separation of variables is possible if R and $R\Phi$ are sums of functions of ξ and functions of η. We write

$$\begin{aligned} 2R &= \varkappa(\xi) + \lambda(\eta), \\ -2R\Phi &= \mu(\xi) + \nu(\eta). \end{aligned} \qquad (6)$$

With CONTOPOULOS we assume a potential, symmetric to the plane, of the form

$$2\Phi(x,y) = Px^2 + Qy^2 - \frac{2a}{3}x^3 - 2bxy^2 - dx^2y^2 + \ldots, \qquad (7)$$

where the dots stand for terms of higher order in x and/or y^2. We first propose to choose c and k so that (6) is satisfied as closely as possible. This will give an approximation which is very accurate for moderately small amplitudes. We then shall show that the dots in (7) may be completed in such a manner that the solution thus found is an exact solution.

From (5) and (7) we obtain

$$2R\Phi = h_1(x_0) + \eta^2 h_2(x_0) \qquad (8)$$

with

$$\left.\begin{aligned} h_1(x_0) &= -\mu(\xi) = R_0\left(Px_0^2 - \frac{2a}{3}x_0^3\right) + \ldots \\ h_2(x_0) &= -\eta^{-2}\nu(\eta) = (Q - 2b\,x_0 - dx_0^2)\, R_0^2 - \\ &\quad - c^2\left(Px_0 - \frac{2a}{3}x_0^3\right) - \\ &\quad - \left\{Px_0(k + x_0) - ax_0^2(k + x_0)\right\} R_0 + \ldots \end{aligned}\right\} \qquad (9)$$

In order that $\nu(\eta)$ be a function of η alone, $h_2(x_0)$ should be a constant. This requires at least that the coefficients of x_0 and x_0^2 in an expansion of $h_2(x_0)$ into powers of x_0 be 0. After a simple reduction two equations are found:

$$\left.\begin{aligned} &(4Q - P)\, k \;- 2bl^2 = 0, \\ &(4Q - P)\, k^2 + (2Q - 2P)\, l^2 - \\ &\quad - (8b - a)\, kl^2 - dl^4 = 0, \end{aligned}\right\} \qquad (10)$$

which can be solved immediately, giving

$$\left. \begin{aligned} k &= \frac{2(Q-P)}{6b - a + (d/2b)(4Q-P)}, \\ l^2 &= \frac{(4Q-P)(Q-P)}{b(6b-a) + \frac{1}{2}d(4Q-P)}, \end{aligned} \right\} \quad (11)$$

whereas $c = (l^2 - k^2)^{\frac{1}{2}}$.

Referring to equation (6) we may now state that the general form of Φ which satisfies the condition for separability *exactly* and which has the expansion (7) is

$$2\Phi = \frac{R_\circ \,.\, 2\Phi_\circ + Ql^4 \varphi(\eta)}{R_\circ - c^2 \sin^2 \eta}, \quad (12)$$

where $R_\circ$ = the function of $x_\circ$ specified by (4),

$$2\Phi_\circ = Px_\circ^2 - \frac{2a}{3}x_\circ^3 + \text{any higher-order terms in } x_\circ\,,$$

$$\varphi(\eta) = \eta^2 + \text{any higher-order terms in } \eta^2.$$

It may be verified by a direct expansion that (12) has the form (7). This check may be considered as an alternative way of deriving the relations (10). The coefficients of the higher terms in this expansion are fixed, once the higher terms in $\Phi_\circ$ and $\varphi(\eta)$ have been chosen. The coefficients of the terms with xy^4 and x^3y^2 in (7) are the first ones which cannot be chosen at will if the system is to be separable.

In the application we shall adopt $\varphi(\eta) = \sin^2 \eta$ and assume that $2\Phi_\circ$ has no terms beyond the third order. Then

$$\begin{aligned} \varkappa(\xi) &= 2R_\circ & \mu(\xi) &= -2R_\circ \Phi_\circ \\ \lambda(\eta) &= -2c^2 \sin^2 \eta & \nu(\eta) &= -Ql^4 \sin^2 \eta \end{aligned} \quad (13)$$

3. *The "third" integral and the envelope*

One integral of our problem is the total energy

$$\tfrac{1}{2}(\dot{x}^2 + \dot{y}^2) + \Phi = \alpha. \quad (14)$$

The left-hand member, an invariant expression in $x, y, \dot{x}, \dot{y}$, is the integral and the right-hand member is the first integration constant.

Similarly, by classical theory for a potential and co-ordinate system which satisfy (6), we have the further integrals

$$\begin{aligned} R^2\dot{\xi}^2 - \mu - \alpha\varkappa &= -\beta, \\ R^2\dot{\eta}^2 - \nu - \alpha\lambda &= +\beta. \end{aligned} \quad (15)$$

Either one of the left-hand members may be chosen as the independent second integral, which corresponds to the third integral in the 3-dimensional motion. The velocity components along the positive ξ and η directions are

$$\begin{aligned} v_\xi &= \dot{x}\cos\varepsilon + \dot{y}\sin\varepsilon = R^{\frac{1}{2}}\dot{\xi}, \\ v_\eta &= -\dot{x}\sin\varepsilon + \dot{y}\cos\varepsilon = R^{\frac{1}{2}}\dot{\eta}. \end{aligned} \quad (16)$$

Hence the first terms of (15) are $Rv_\xi^{\,2}$ and $Rv_\eta^{\,2}$, respectively, and the sum of both equations (15) is satisfied by (14).

The choice of α defines by (14) a boundary curve (CONTOPOULOS: torus section) inside which the entire orbit must lie and which can be reached only with velocity 0. All orbits with a given α are called a family (OLLONGREN).

The choice of β defines by (15) an envelope in the form of a box, the corners of which are on the boundary curve. The equations for the sides of the box are

left and right sides: $\dot{\xi} = 0$

$\xi = \xi_1$, and $\xi = \xi_2$ are roots of $\mu(\xi) + \alpha\varkappa(\xi) - \beta = 0$;

upper and lower sides: $\dot{\eta} = 0$

$\eta = \pm\eta_1$ are roots of $\nu(\eta) + \alpha\lambda(\eta) + \beta = 0$. (17)

With the specification (13) we have

$$\begin{aligned} &\xi_1 \text{ and } \xi_2 \text{ are roots of } 2R_\circ(\alpha - \Phi_\circ) = \beta, \\ &\eta_1 \text{ is root of } \quad (Ql^4 + 2c^2\alpha)\sin^2\eta = \beta. \end{aligned} \quad (18)$$

The range of possible values of β within a family is

$$0 \leq \beta \leq \beta_{\max}. \quad (19)$$

The choice $\beta = 0$ defines the plane orbit "P" in which η and y remain 0; ξ_1 and ξ_2 then correspond to the points where the boundary cuts the x-axis. The value $\beta_{\max}$ is the maximum of the function $2R_\circ(\alpha - \Phi_\circ)$. It is reached for a value of $x_\circ$ given approximately by

$$x_{\circ,C} = \frac{2\alpha k}{Pl^2 - 2\alpha}. \quad (20)$$

The corresponding orbit "C" (called the central orbit) is a periodic orbit in which the point moves back and forth along the section of an ellipse ξ = constant cutting the axis perpendicularly at $x = x_{\circ,C}$ and reaching the boundary perpendicularly. Orbit "C" does not pass through the equilibrium point.

4. *The oscillation periods*

The mean periods P_ξ and P_η are defined as the limits to which the ratios

$$\frac{\text{total travel time}}{\text{number of trips back and forth}}$$

converge for $t \to \infty$. By classical theory they may be computed as follows; for a rigorous proof, see VINTI (1961). The generalized momenta belonging to the system are by (15)

$$\begin{aligned} p_\xi &= \{\mu(\xi) + \alpha\varkappa(\xi) - \beta\}^{\frac{1}{2}}, \\ p_\eta &= \{\nu(\eta) + \alpha\lambda(\eta) + \beta\}^{\frac{1}{2}}, \end{aligned} \quad (21)$$

and the corresponding action variables are

$$J_\xi(\alpha,\beta) = 2\int_{\xi_1}^{\xi_2} p_\xi\, d\xi, \qquad J_\eta(\alpha,\beta) = 2\int_{-\eta_1}^{\eta_1} p_\eta\, d\eta. \qquad (22)$$

It is possible, by eliminating β from (22), to express α as a function $H(J_\xi, J_\eta)$. The mean periods then follow from the equations

$$P_\xi = \left(\frac{\partial H}{\partial J_\xi}\right)^{-1}, \qquad P_\eta = \left(\frac{\partial H}{\partial J_\eta}\right)^{-1}. \qquad (23)$$

It is not necessary to perform each step of this computation. For we may write

$$\begin{aligned} \frac{\partial H}{\partial \alpha} &= \frac{\partial H}{\partial J_\xi}\frac{\partial J_\xi}{\partial \alpha} + \frac{\partial H}{\partial J_\eta}\frac{\partial J_\eta}{\partial \alpha} = 1\,, \\ \frac{\partial H}{\partial \beta} &= \frac{\partial H}{\partial J_\xi}\frac{\partial J_\xi}{\partial \beta} + \frac{\partial H}{\partial J_\eta}\frac{\partial J_\eta}{\partial \beta} = 0\,. \end{aligned} \qquad (24)$$

By (23) the first factors may be replaced by inverse periods. We may then solve for these periods from (24) thus expressing them in terms of the derivatives of (22), which may be obtained by differentiating under the integral. We state at once the result

$$\begin{aligned} P_\xi &= (N_\xi M_\eta + N_\eta M_\xi)\, M_\eta^{-1}\,, \\ P_\eta &= (N_\xi M_\eta + N_\eta M_\xi)\, M_\xi^{-1}\,, \end{aligned} \qquad (25)$$

where

$$\begin{aligned} M_\xi &= \int_{\xi_1}^{\xi_2} p_\xi^{-1}\, d\xi\,, & M_\eta &= \int_{-\eta_1}^{\eta_1} p_\eta^{-1}\, d\eta\,, \\ N_\xi &= \int_{\xi_1}^{\xi_2} \varkappa(\xi)\, p_\xi^{-1}\, d\xi\,, & N_\eta &= \int_{-\eta_1}^{\eta_1} \lambda(\eta)\, p_\eta^{-1}\, d\eta\,. \end{aligned} \qquad (26)$$

All integrands become ∞ at the integration limits; this reflects the fact that the body in orbit spends a relatively long time close to the sides of the box.

Upon specification of $\varkappa, \lambda, \mu, \nu$ by (13) and omitting terms $\sim \eta^3$ we find

$$\begin{aligned} M_\eta &= \frac{\pi}{(Q l^4 + 2\alpha c^2)^{\frac{1}{2}}}\,, \\ N_\eta &= \frac{-c^2 \beta \pi}{(Q l^4 + 2\alpha c^2)^{3/2}}\,. \end{aligned} \qquad (27)$$

The integrals over ξ cannot be obtained with equal ease. Changing to x_0 as the integration variable we find

$$\begin{aligned} M_\xi &= \int_{x_1}^{x_2} \frac{d x_0}{\{R_0^2(2\alpha - P x_0^2 + \frac{2}{3} a x_0^3) - R_0 \beta\}^{\frac{1}{2}}}\,, \\ N_\xi &= \int_{x_1}^{x_2} \frac{2 R_0\, d x_0}{\{R_0^2(2\alpha - P x_0^2 + \frac{2}{3} a x_0^3) - R_0 \beta\}^{\frac{1}{2}}}\,. \end{aligned} \qquad (28)$$

Since the form within brackets is a polynomial in x_0 of degree 7, numerical or approximative methods are necessary. For instance, either integral may be written and transformed by partial integration as follows.

$$\int_{x_1}^{x_2} \frac{S(x_0)\, d x_0}{\{(x_0 - x_1)(x_2 - x_0)\}^{\frac{1}{2}}} = \frac{\pi}{2}\left\{ S(x_1) + S(x_2) \right\} - \qquad (29)$$
$$- \int_{x_1}^{x_2} \arcsin \frac{x_0 - \frac{1}{2}x_1 - \frac{1}{2}x_2}{\frac{1}{2}(x_2 - x_1)}\, d\, S(x_0)\,.$$

Here S is a function which varies little over the interval, so that the remaining integral forms a small correction term which is smaller than one per cent in the numerical applications.

In the zero-th approximation, in which the term with a may be neglected and R_0 may be put constant ($= l^2$), the result is

$$M_\xi = \frac{\pi}{l^2 P^{\frac{1}{2}}}\,, \quad N_\xi = \frac{2\pi}{P^{\frac{1}{2}}}\,, \quad M_\eta = \frac{\pi}{l^2 Q^{\frac{1}{2}}}\,, \quad N_\eta = 0\,, \qquad (30)$$

from which we obtain the familiar periods for small amplitudes

$$P_\xi = \frac{2\pi}{P^{\frac{1}{2}}}\,, \quad P_\eta = \frac{2\pi}{Q^{\frac{1}{2}}}\,. \qquad (31)$$

It may, finally, be remarked that further distinction between different orbits within a box makes sense only if P_ξ/P_η happens to be a rational number. In that case an appropriately defined phase difference may be regarded as the third isolating integral (corresponding to a fourth integral in 3-dimensional motion).

5. *Numerical application*

The example worked out by CONTOPOULOS (1958, 1960) is based on the unit of length = 1 kpc, of time = 10^7 years, and of velocity = 98 km/sec. We adopt his constants defining the potential field in the neighbourhood of the Sun:

$P = 0.076,\ Q = 0.550,\ a = 0.052,\ b = 0.206,\ d = 0.$

By (11) we find

$$k = 0.8007,\ c = 1.8673,\ l = 2.0317.$$

Full details are given by CONTOPOULOS for two sets of initial conditions, orbit "A" and orbit "B", for which he has performed numerical integration over 500 and 100 time units, respectively. For some reason CONTOPOULOS hesitated to derive from these data the precise mean periods. Estimating the return times by linear interpolation from his tables we find the following results with estimated accuracies

orbit A: $P_\xi = 23.785 \pm .001$, $P_\eta = 8.7294 \pm .0003$,
orbit B, $P_\xi = 23.675 \pm .005$, $P_\eta = 8.6345 \pm .0010$.

TABLE 1

Numerical data on four box orbits within one family

			orbit P	orbit A	orbit B	orbit C
initial conditions, velocities at origin		$\dot{x}$	0.1237	−0.0983	0.0512	no pass
		$\dot{y}$	0	0.0748	0.1126	,,
integration constants		2α	0.01530	0.01526	0.01530	0.01530
		β	0	0.02310	0.05234	0.06366
envelope, left side		ξ_1	0.206	0.252	0.338	0.434
	(at $y=0$)	x_1	−0.4117	−0.3221	−0.1529	0.0412
		d^2x/dy^2	−0.106	−0.127	−0.164	−0.199
right side		ξ_2	0.652	0.609	0.528	0.434
	(at $y=0$)	x_2	0.5126	0.4149	0.2380	0.0412
		d^2x/dy^2	−0.250	−0.243	−0.225	−0.199
top side		η_1	0	0.0495	0.0746	0.0823
	(at $x=0$)	y	0	0.1006	0.1515	no pass
	,,	dy/dx	0	0.0193	0.0291	,,
	,,	d^2y/dx^2	0	0.0204	0.0308	,,
slope	upper left corner	ε	0	0.012	0.024	0.034
	upper right corner	ε	0	0.027	0.036	0.034
auxiliary quantities for periods		M_ξ	2.7794	2.7933	2.8117	2.8194
		N_ξ	23.855	23.769	23.684	23.656
		M_η	1.0233	1.0233	1.0233	1.0233
		N_η	0	−0.0087	−0.0198	−0.0241
periods		P_ξ	23.855	23.665	23.628	23.590
		P_η	8.783	8.699	8.600	8.562

We have added to these two orbits the ones with minimum and maximum β, the plane orbit "P" and the central orbit "C". It follows from (15) and (13) that

$$\beta = l^2\,\dot{y}^2, \tag{32}$$

where $\dot{y}$ is the vertical velocity component at the point of origin ($x=0$, $y=0$).

Using the preceding equations, in particular (18), (20), (4) and (25) we have calculated the values of ξ and η defining the envelopes, the periods, and a few further data relevant to a comparison with CONTOPOULOS' work. These data are given in Table 1 and the forms of the boxes and the boundary curve are illustrated in Figure 1. The agreement is good or excellent throughout, so that it had no interest to quote in the table also the three-figure values given by CONTOPOULOS. The worst difference was the value of x_2 for orbit "A", which we find 0.4149 whereas CONTOPOULOS gives 0.416. The periods found in Table 1 fall 0.2 to 0.5 per cent below those derived from CONTOPOULOS' data. A difference of this order may arise from the fact that, in deriving (27) we have replaced $\sin\eta$ by η and that the potential (12) is not identical to the potential used by CONTOPOULOS.

We may conclude that the theory of motion separable in elliptical co-ordinates is not merely an elegant mathematical possibility but can serve to investigate the precise properties of the orbit of any low-velocity star.

This study arose from many discussions with Dr A. OLLONGREN, to whom I record my thanks.

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6. *Notes added in proof*

From a correspondence with Prof. CONTOPOULOS since this paper was written it appears desirable to add the following notes.

1. CONTOPOULOS has computed the magnitude of the terms in 2Φ which follow from (12) but which were 0 in his computation. The maximum values reached are

	orbit A	orbit B
term in y^4	0.00000	0.00002
term in x^3y^2	0.00013	0.00005
term in xy^4	−0.00001	−0.00002
total	0.00012	0.00005
percentage of 2Φ	0.8	0.3

These form a real difference in the comparison presented in section 5. Moreover, CONTOPOULOS has checked that the replacement of sin η by η in computing M_η and N_η introduces changes up to 0.2 per cent in the periods.

2. The empirical periods given in section 5 were derived on the conjecture that a very good approximation to P_η may be obtained from the time spent in covering an integer number of round trips in the η-direction, provided the number of round trips in the ξ-direction during the same time is very nearly an integer, and conversely. For symmetry reasons also half-periods in the η-direction may be admitted. Points of approximate recurrence to the origin found in CONTOPOULOS' table are:

orbit	pass	t at which $x=0$	number of x-trips	t at which $y=0$	number of y-trips
A	0	0	0	0	0
	a	34.01	1.5	33.80	4.0
	b	81.58	3.5	81.79	9.5
	c	214.03	9.0	213.86	24.5
	d	261.61	11.0	261.86	30.0
	e	295.63	12.5	295.67	34.0
	f	475.70	20.0	475.75	54.5
B	0	0	0	0	0
	a	47.34	2.0	47.48	5.5
	b	61.20	3.5	61.02	7.0
	c	94.70	4.0	94.98	11.0

The acceptable combinations and periods derived are:

intervals	P_ξ	P_η
A 0−c, b−e, d−f	$\frac{214.057}{9} = 23.7841$	$\frac{213.867}{24.5} = 8.7297$
0−d, a−e, c−f	$\frac{261.633}{11} = 23.7848$	$\frac{261.873}{30.0} = 8.7291$
0−f	$\frac{475.700}{20} = 23.7850$	$\frac{475.750}{54.5} = 8.7295$
B 0−a, a−c	$\frac{47.35}{2} = 23.675$	$\frac{47.49}{5.5} = 8.6345$

3. I gratefully acknowledge that Prof. CONTOPOULOS checked some of the numerical data and pointed out an unfortunate computing error in the first version of this paper.