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High-order Scattering in Diffuse Reflection from a Semi-infinite Atmosphere

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An asymptotic formula representing the terms with a relative error 0 (n^{-2}) is found for the radiation intensity diffusely reflected from a semi-infinite atmosphere after n successive scatterings. The method, valid for an arbitrary phase function, is to transform a power series in k, the diffusion exponent, into one in $t = (1 - a)^{1/2}$ and then into one in a, the albedo for single scattering. Four examples, including isotropic and highly anisotropic scattering, show the method to be accurate and convenient. If exact values for the orders $n = 1$ to 6 are known, values for all higher orders can be estimated to within a percent by this method.

Key words: radiative transfer — diffuse reflection

1. The Problem

The most direct approach to multiple scattering problems is first to compute the radiation field comprising the quanta which have not been scattered at all (zero order), then the field of quanta which have suffered one scattering (first order), etc. We shall call this the "method of successive orders". The series summing all orders forms the total radiation field.

Let a be the "albedo for single scattering" (astrophysics) or "number of secondaries" (neutron scattering). Then the intensity of the radiation field of the n-th order contains the factor a^n. Hence the intensity of the total field is a power series in a. It is irrelevant for our purpose whether (as suggested above) it has been derived from a probabilistic reasoning, or (as often is done) from the Neuman series for solving an integral equation, or from still different means (as illustrated below).

The convergence of this power series is very fast if the cloud, or layer, is optically rather thin, or if the albedo a of the individual scatterers is low. The convergence is slow for thick layers and for large a. For this reason the method of successive orders has never been advocated as a good general method of computation, although its useful range has certainly been extended by the fast computers.

Let the atmosphere, or plane parallel layer, be homogeneous with optical thickness b, and be illuminated by internal or external sources. Then the asymptotic behaviour of the series summing all orders is as a geometric series with ratio

$$\eta_1(b) \cdot a \, ,$$

where $\eta_1(b)$ is the largest eigenvalue of an integral equation, the values of which have been solved for several types of scattering (Mullikin, 1962; van de Hulst, 1963; van de Hulst and Irvine, 1962; Leonard and Mullikin, 1964); $\eta_1 = 0$ for $b = 0$, $\eta_1 = 1$ for $b = \infty$.

To any finite non-zero value of b, a value $a = \{\eta_1(b)\}^{-1} > 1$ corresponds, for which the series starts to diverge. This corresponds to the critical condition of a nuclear reactor with slab geometry.

Limiting the discussion to the astrophysically interesting range of values $0 < a \leqq 1$ we easily see that the worst convergence is obtained in the limit of a conservative semi-infinite atmosphere, $a = 1$, $b = \infty$. In this limit the series still converges, but more slowly than a geometric series.

It has been pointed out repeatedly that it would be useful to have an accurate theory of the asymptotic behaviour of the series in this unfavourable case. For instance, in the theory of absorption lines seen in the diffusely reflected light of a thick planetary atmosphere, it is necessary to vary a continuously, which can be done by means of this series (Belton, 1968).

Uesugi and Irvine (1969, 1970) set themselves to solve this problem and did so. However, it is possible by using more direct methods to arrive at more complete asymptotic forms, as we shall show below. The different aims of these papers may be expressed as follows. Irvine and Uesugi state that they would be happy to compute 50 orders numerically by successive scattering and to replace the rest by an asymptotic formula, reaching an accuracy of one percent. Our aim rather was to limit the successive scattering method to only 3 — 10 terms and still to achieve a safe transition to the asymptotic form and a total accuracy of 0.1% or better.

2. Derivation of the Asymptotic Expansion

Our simple starting point is that, if we know the sum of the series for all a, then we know all coefficients of its power expansion (by a Taylor series). The high coefficients, i.e. the high orders of scattering, come effectively into play only when a gets close to 1. Conversely, the known behaviour of the result near $a = 1$ should suffice to find the asymptotic behaviour of the high orders. This idea is developed into practical formulae below.

Let f be the physical quantity we wish to discuss as a function of a. It may, for instance be the intensity of diffusely reflected light for given cosines μ_0 and μ of the angles of incidence and emergence; or it may be a moment or a bi-moment obtained upon integrating the reflected intensity over one or both of these arguments.

It is known (e.g. van de Hulst, 1968a) that such a quantity has a "near-conservative" expansion of the form

$$f = F_0 + F_1 k + F_2 k^2 + \dots \qquad (1)$$

where k is the diffusion exponent or inverse diffusion length, which occurs in the combination $\exp(\pm k\tau)$ with the optical depth τ in the asymptotic solution of the radiation field in deep layers. It is also known that a itself has a similar expansion

$$a = 1 - Ak^2 + Bk^4 + \dots \qquad (2)$$

Defining

$$t = (1 - a)^{1/2} \qquad (3)$$

we find from (2) that

$$k^2 = t^2/A + Bt^4/A^3 + \dots \qquad (4)$$

and, thereby, can convert (1) into a power series in t:

$$f = G_0 + G_1 t + G_2 t^2 + \dots \qquad (5)$$

where

$$G_0 = F_0,\ G_1 = F_1/A^{1/2},\ G_2 = F_2/A\ ,$$
$$G_3 = F_3/A^{3/2} + BF_1/2A^{5/2},\ G_4 = F_4/A^2 + BF_2/A^3. \qquad (6)$$

The expansions (1) and (5) are about equally useful. Expansions in the form (5) have also been derived to the linear term by Kolesov and Sobolev (1969) and to the quadratic term by Sobolev (1969).

The further task is twofold. 1. How to find the coefficients F_1 to F_4, A and B, and hence G_1 to G_4; this we shall postpone to the next Section. 2. How to use this knowledge to find the asymptotic expression for f_n in

$$f = f_0 + f_1 a + f_2 a^2 + \dots + f_n a^n + \dots \qquad (7)$$

We perform the second task by using the binonium expansion

$$-t = -(1 - a)^{1/2} = \sum_{n=0}^{\infty} y_n a^n \qquad (8)$$

where for $N \geqq 2$ exactly

$$y_n = \frac{1.3.5.\dots(2n-3)}{2.4.6.\dots(2n-2)\,2n}\,. \qquad (9)$$

By the Wallis formula (Abramowitz and Stegun, 1965, p. 258) y_n has the asymptotic form

$$y_n \sim (4\pi n^3)^{-1/2}[\,1 + 3/8\,n^{-1} + \dots] \qquad (10)$$

and consequently

$$y_n - y_{n-1} \sim (4\pi n^3)^{-1/2}[\,-3/2\ n^{-1} + \dots]\,. \qquad (11)$$

The even terms in (5) do not contribute to the asymptotic forms[1]) of f_n but the odd terms give

$$f_n = -G_1(y_n) - G_3(y_n - y_{n-1})$$
$$- G_5(y_n - 2y_{n-1} + y_{n-2}) - \dots \qquad (12)$$

Inserting (10) and (11) into (12) we find at once

$$f_n \sim (4\pi n^3)^{-1/2}[-G_1 + 3/2\,(G_3 - 1/4\,G_1)n^{-1} + \dots]\,. \qquad (13)$$

For practical purposes we shall use the equivalent form

$$f_n = -G_1\{4\pi(n+c)^3\}^{-1/2}\{1 + O(n^{-2})\} \qquad (14)$$

with

$$c = G_3/G_1 - 1/4\,. \qquad (15)$$

[1]) It is assumed that the terms of very high orders (even and odd) in (5), which do contribute in principle to the asymptotic form of f_n, vanish sufficiently rapidly.

3. Numerical Examples

The convenience of these formulae for numerical work has been tested by four examples. One function tested was the reflection function $R(1,1)$ for perpendicular incidence and reflection. The other was the bi-moment, in our notation called

$$URU = \int_0^1\int_0^1 R(\mu,\mu_0)2\mu d\mu\, 2\mu_0 d\mu_0 \qquad (16)$$

which is identical with the "Bond Albedo" of a planet covered with this atmosphere. Both functions were taken for isotropic scattering ($g=0$) and Henyey-Greenstein scattering with anisotropy factor $g=0.75$.

Table 1 shows the results of this test. Since the numerical data were partially extracted from machine output made for different purposes, the origin of the values is not uniform. First, we have simple, exact expressions (van de Hulst, 1968a) for the coefficients

$$A = \{3(1-g)\}^{-1},$$

$$B = -\{4-9g+5gh\}/\{45(1-g)^3(1-h)\}, \qquad (17)$$

where $g=\omega_1/3$ and $h=\omega_2/5$ are coefficients of the expansion of the phase function as

$$\Phi(\cos\alpha) = 1 + 3gP_1(\cos\alpha) + 5hP_2(\cos\alpha) + \dots .$$

Further exact expressions are

quantity $R(1,1)$	*quantity* URU
$F_0 = R_0(1,1)$	$F_0 = UR_0U$
$F_1 = \frac{-4}{3(1-g)}\{K_0(1)\}^2$	$F_1 = \frac{-4}{3(1-g)}$
	$F_2 = \frac{4q_0}{3(1-g)}$ (18)

where the suffix 0 has been used for "value valid for $k=0$, $a=1$". The reflection function $R(\mu,\mu_0)$, its bi-moment URU, the escape function $K(\mu)$, and the extrapolation length q are all known with five-figure accuracy. These numbers are based on classical expressions in terms of the H-function and its moments for isotropic scattering and have been found by "asymptotic fitting" for the Henyey-Greenstein phase function with $g=0.75$ (van de Hulst 1968b). Inserting (17) and (18) into (6) we obtain the first few coefficients G_n with high accuracy.

A further set of G_n values was found by performing the expansion (5) numerically, starting from the values of f for $a=0.99$, 0.95, 0.9 and 0.8, known by asymptotic fitting. Defining

$$z_0(t) = f, \quad z_{n+1}(t) = \{z_n(t) - G_n\}/t \qquad (19)$$

we simply plotted $z_n(t)$ against t and found $z_n(0) = G_n$ as the intercept of the curve, smoothly extrapolated to $t=0$. The maximum uncertainties introduced by

Table 1. *Numerical test of asymptotic formula*

Quantity f anisotropy		$R(1,1)$ $g=0$		$R(1,1)$ $g=0.75$		URU $g=0$		URU $g=0.75$	
Coefficients of	G_0	1.057		1.119		1		1	
expansions (5)	G_1	− 3.661		− 7.329		−2.309		− 4.619	
	G_2	7.84		23.6 ± 0.1		2.842		11.414	
	G_3	−14.8 ± 0.2		−51 ± 3		−3.233		−19.7 ± 0.2	
Derived constant (15)	c	3.76		6.75		1.15 (exact!)		4.00	
left:	$n=1$	0.1250	1.298	0.0102	0.220	0.2046	0.645	0.0472	0.528
Values of low-order terms f_n	2	0.0866	1.197	0.0129	0.334	0.1157	0.647	0.0445	0.655
right:	3	0.0653	1.147	0.0155	0.470	0.0767	0.648	0.0409	0.757
Products $f_n(n+c)^{3/2}$	4	0.0517	1.117	0.0176	0.620			0.0372	0.841
	5	0.0423	1.098	0.0192	0.774			0.0338	0.911
	6	0.0356	1.084	0.0203	0.925			0.0307	0.970
	8	0.0265	1.068	0.0210	1.190			0.0255	1.060
	12	0.0168	1.052	0.0190	1.543			0.0182	1.166
	16	0.0119	1.045	0.0159	1.722			0.0136	1.219
	20	0.0090	1.041	0.0131	1.816			0.0106	1.247
	∞	0	1.033	0	2.068	0	0.651	0	1.303

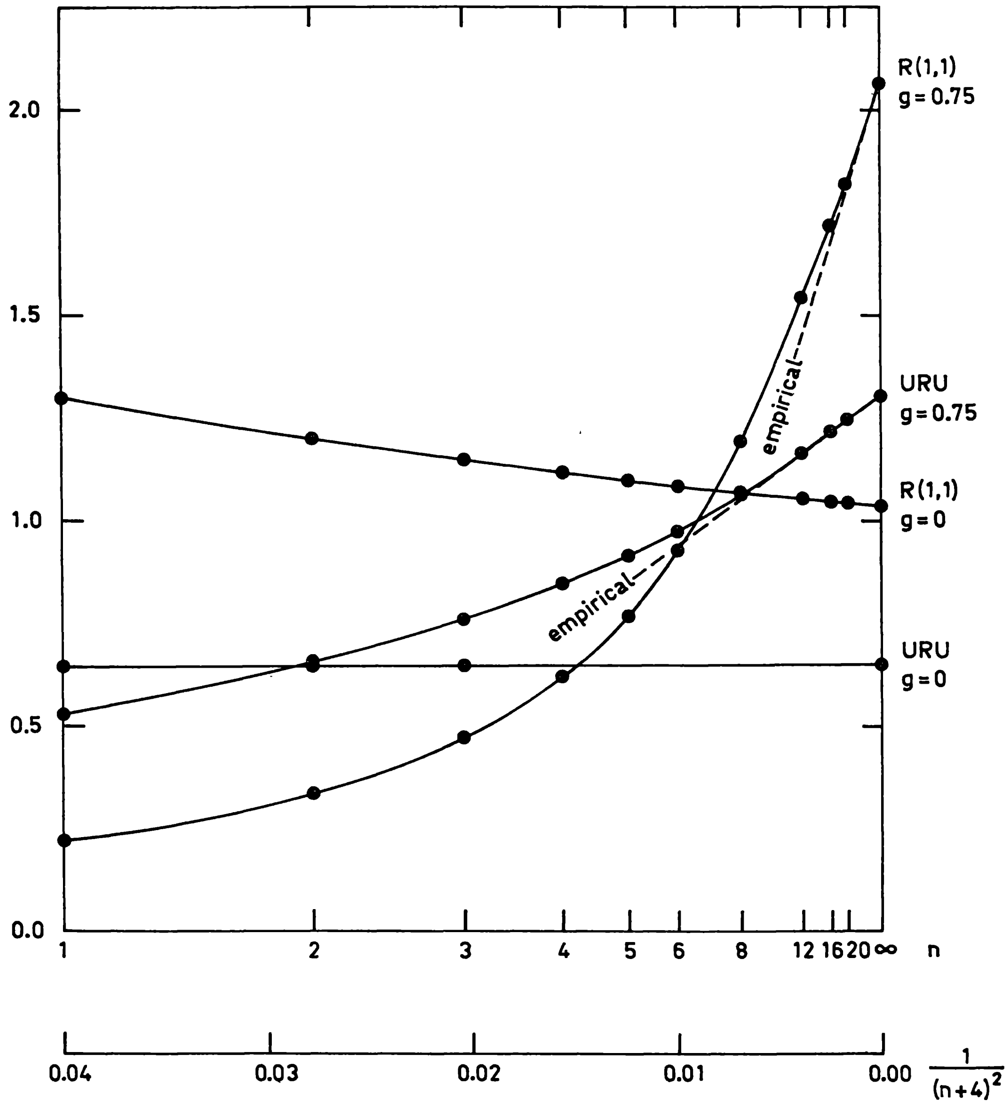

Fig. 1. Approach to asymptotic values in four examples

this process are indicated in Table 1. They do not greatly affect the values of c, found by Eq. (15).

The example URU for isotropic scattering is exceptional in having exact expressions for G_0 through G_3. Expanding the n-th moment of the H-function as

$$\alpha_n = \sum_{j=0}^{\infty} a_{nj} t^j . \tag{20}$$

And writing the Hopf function for $\tau = \infty$ as $q_\infty = 0.710446$, we have the well known relations

$$a_{n1} = -\sqrt{3}\, a_{n+1,\,0} \tag{21}$$

$$a_{00} = 2,\ a_{10} = 2/\sqrt{3},\ a_{20} = 2q_\infty/\sqrt{3},\ a_{11} = -2q_\infty . \tag{22}$$

A well known relation between the moments

$$t\alpha_4 = \frac{1}{5} + \frac{1-t^2}{4}(\alpha_2^2 - 2\alpha_1\alpha_3) \tag{23}$$

in the limit $t = 0$ now gives

$$a_{30} = \left(\frac{1}{5} + \frac{1}{3} q_\infty^2\right)\sqrt{3}, \tag{24}$$

an expression also given by Sobolev (1967). The value

$$a_{21} = -\left(\frac{3}{5} + q_\infty^2\right) \tag{25}$$

now follows by (21). Equating coefficients of t^2 in another well known relation

$$t\alpha_2 = \frac{1}{3} - \frac{1-t^2}{4}\alpha_1^2, \tag{26}$$

we can now step to

$$a_{12} = \frac{14}{15}\sqrt{3}\,. \tag{27}$$

Finally, the exact expression

$$URU = 1 - 2t\alpha_1 \tag{28}$$

gives in combination with (20), (22), (27) the exact values of G_0, G_1, G_2, G_3. By (15) this leads to the unexpectedly simple result that for this example

$$c = 23/20, \text{ exactly}\,. \tag{29}$$

The lower part of Table 1 contains the normal "orders" f_n defined by Eq. (7) and computed by successive scattering. These values were taken from computer output kindly provided by Dr. K. Grossman of the NASA Institute for Space Studies, New York. In order to check the asymptotic relation in the form (14) we have also tabulated the products $f_n(n+c)^{3/2}$ and these are the ordinates of the points plotted in Fig. 1. The last line in Table 1 shows the values of $-G_1(4\pi)^{-1/2}$, which by Eq. (14) should be the limiting value for $n \to \infty$, a result derived along a quite different way by Uesugi and Irvine (1970).

4. Discussion

In Fig. 1 all numbers have been plotted against $(n+4)^{-2}$. Here 4 is an arbitrary choice to make the picture look nice; if we would have taken a different number, the curves would still approach the point $n = \infty$ along the same tangent. The value of the slope at this endpoint is not specified in (14), but it should be possible by (9) and (12) to express it in a form, which will involve G_5. Conversely, the empirical slope shown in the graph might be used to find the value of G_5.

If we should have taken the wrong value of c, or should have worked directly from Eq. (13), the same endpoint would have been found, but it would have been necessary to plot against $(n+d)^{-1}$, d arbitrary, in order to approach this endpoint with a finite slope, so that the approach would have been much slower.

The differences among the curves in Fig. 1 are entirely as expected. The case URU for isotropic scattering should show a rapid approach to the asymptotic behaviour. The zero-order (incident) light already has a wide distribution in directions and the isotropic scattering cannot but help to spread the radiation. However, it was a surprise that the approach is so fast that the entire rise in Table 1 from $n = 1$ to $n = \infty$ is less than one percent! The absence of numbers for $n = 4$ to 20 is due to an accidental program failure; it seemed superfluous to remedy this for the present example. In contrast, the example $R(1, 1)$ for $g = 0.75$ shows the slowest approach. The incident radiation goes straight into the atmosphere. The strongly forward directed phase function (with 93.3 percent of the scattering in the forward hemisphere and a ratio 343 of exact forward to exact backward scatter) tends to keep it this way for the low orders of scattering. Table 1 shows that f_1 is very small indeed and that the coefficients f_n go on rising to $n = 8$ and that only for still higher orders the tendency for convergence appears. The value of c is largest in this case, and the curve in Fig. 1 shows a substantial rise. The two other examples duly are between the extreme examples just mentioned.

Although the computations of f_n presented here were extended to $n = 20$, Fig. 1 shows that fewer terms would suffice for a smooth graphical interpolation between the highest n computed and the known limit for $n = \infty$. Counting 1, 2, 3, ... ∞ would not leave more than 10 percent uncertainty (in the range near $n = 10$) even in the least favourable case. Known values for $n = 1$ through 6 would suffice for a 1 percent accuracy in the interpolated values of all higher terms.

5. Summation

Having thus found, in principle, a way to derive accurate values for all high-order terms f_n, the next step, logically, should be to choose a method to perform the summation of the series (7). Here a variety of options exists, which we have not explored in detail. The choice may depend on the quantity and accuracy of results wanted, the range of a-values, the type of computer available, etc. In the study of absorption lines in diffusely reflecting atmospheres, it may be useful to have approximation formulae in which a can be varied continuously. We mention several possibilities:

(A) If direct values, say to $n = 20$, can be computed, it should be simple to approximate the remainder by an integral. It seems best to start from (13) instead of (14). The separate terms yield incomplete Γ-functions of order $-1/2$, $-3/2$, $-5/2$ etc., each of which can be reduced by simple recurrence relations to the incomplete Γ-function of order $+1/2$, i.e. to the error function. This method (limited to the dominant term) was proposed by Uesugi and Irvine (1969, 1970). (B) All worries about slowly converging series can be avoided by simply subtracting from the series sought the term $G_1 t$ in (5), the separate orders of which are known exactly by (8) and (9). Going one or two steps further we can also subtract the exact expansions of $G_3 t^3$ and $G_5 t^5$. The remaining power series in a converges rapidly and can be summed numerically.

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