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Conditions for the elements of the scattering matrix

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Summary. A study is made of general conditions for the 16 elements of the scattering matrix that transforms the Stokes parameters of the incident wave into those of the scattered wave. A number of relations is derived for scattering by one particle of arbitrary size, shape and composition. In this case a complete set of nine independent equations is found.

Several methods are described to obtain inequalities for the elements of the scattering matrix for an assembly of particles. A selection of these inequalities is explicitly derived and discussed.

Key words: radiative transfer – light scattering – planetary atmospheres – polarization

1. Introduction

Studying the scattering of electromagnetic radiation by particles is important in many parts of science, especially in astrophysics. A great deal of knowledge may be obtained by analyzing the radiation scattered by particles in the atmospheres of planets and satellites, planetary ring systems, the interplanetary dust cloud, circumstellar matter and the interstellar medium. Taking polarization into account is important but difficult, because even the description of a single scattering event requires a 4 × 4 matrix, the so-called scattering matrix. Generally, we have to deal with 16 different elements, which are functions of wavelength and the directions of incidence and scattering. As pointed out by Van de Hulst (1957), there must be nine relations between the 16 elements for scattering by one particle.

The number of independent elements of the scattering matrix can sometimes be reduced by using symmetry relations. The main purpose of this paper is to seek further simplifications by deriving equalities and inequalities for the elements of the scattering matrix, both for one arbitrary type of particle and for an assembly of particles. The equalities and inequalities considered hold for arbitrary directions of incidence and scattering. Some relations for a single particle (or a collection of identical particles) have been presented earlier by Perrin (1942), Perrin and Abragam (1951), Abhyankar and Fymat (1969) and Fry and Kattawar (1981). For an assembly of non-identical particles some inequalities have been reported by Kuščer and Ribarič (1959), Fry and Kattawar (1981) and Hovenier and Van der Mee (1983). In this paper, however, a comprehensive treatment is given, based on first principles. Several methods are used which are quite general and at the same time elementary. The results may be employed to reduce the amount of computational or experimental labour, and for checking purposes.

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2. Properties of the elements of the scattering matrix of one particle

In this section we will derive relations for quantities that are usually employed in the description of light scattered by a single particle at arbitrary wavelength. In the next section we will consider the implications for scattering of radiation by an assembly of particles.

2.1. The existence of interrelations

The scattering of a simple wave by an arbitrary particle may be described by means of a 2 × 2 amplitude matrix satisfying

$$\begin{pmatrix} E_l \\ E_r \end{pmatrix} = \begin{pmatrix} A_2 & A_3 \\ A_4 & A_1 \end{pmatrix} \begin{pmatrix} E_{l0} \\ E_{r0} \end{pmatrix}. \quad (1)$$

Here E_l and E_r represent the electric field components of the scattered wave parallel and perpendicular to the scattering plane, respectively; in a similar way E_{l0} and E_{r0} relate to the ingoing wave [see e.g. Van de Hulst, 1957]. The elements of the amplitude matrix are, in general, complex functions of the directions of incidence and scattering. They may be written as

$$A_k(\theta, \psi) = r_k \exp(i\rho_k) \quad (2)$$

where $k = 1, 2, 3, 4, \theta$ is the scattering angle (i.e. the angle between the incident and scattered beam), ψ is an azimuthal angle and i is the imaginary unit. In Eq. (2) r_k is nonnegative and ρ_k is real. The dependence on θ and ψ of these quantities and other quantities used later on in this paper has not been written explicitly.

For experimental and observational purposes the matrix of interest is the 4 × 4 scattering matrix containing real elements. This matrix transforms the Stokes parameters of the incident wave $\{I_0, Q_0, U_0, V_0\}$ into the Stokes parameters of the scattered wave $\{I, Q, U, V\}$. Mathematically this may be expressed by the matrix equation

$$\begin{pmatrix} I \\ Q \\ U \\ V \end{pmatrix} = \begin{pmatrix} a_1 & b_1 & b_3 & b_5 \\ c_1 & a_2 & b_4 & b_6 \\ c_3 & c_4 & a_3 & b_2 \\ c_5 & c_6 & c_2 & a_4 \end{pmatrix} \begin{pmatrix} I_0 \\ Q_0 \\ U_0 \\ V_0 \end{pmatrix}. \quad (3)$$

We use Stokes parameters as defined by Chandrasekhar (1950) and Van de Hulst (1957).

As shown by Van de Hulst (1957) the elements of the scattering matrix can be expressed in the elements of the amplitude matrix as follows. We define the auxiliary quantities

$$M_k = A_k A_k^* = r_k^2 \quad (4)$$

$$S_{kj} = \tfrac{1}{2}(A_k A_j^* + A_j A_k^*) = r_k r_j \cos(\rho_k - \rho_j) \quad (5)$$

$$D_{kj} = \tfrac{1}{2} i (A_k A_j^* - A_j A_k^*) = -r_k r_j \sin(\rho_k - \rho_j) \quad (6)$$

where an asterisk denotes complex conjugation and $k, j = 1,2,3,4$. For later use we note that

$$S_{kj} = S_{jk} \quad (7)$$

and

$$D_{kj} = -D_{jk}. \quad (8)$$

From the definition of the Stokes parameters we then find

$$a_1 = \tfrac{1}{2}(M_1 + M_2 + M_3 + M_4) \quad (9)$$

$$b_1 = \tfrac{1}{2}(-M_1 + M_2 - M_3 + M_4) \quad (10)$$

$$b_3 = S_{32} + S_{41} \quad (11)$$

$$b_5 = D_{32} - D_{41} \quad (12)$$

$$c_1 = \tfrac{1}{2}(-M_1 + M_2 + M_3 - M_4) \quad (13)$$

$$a_2 = \tfrac{1}{2}(M_1 + M_2 - M_3 - M_4) \quad (14)$$

$$b_4 = S_{32} - S_{41} \quad (15)$$

$$b_6 = D_{32} + D_{41} \quad (16)$$

$$c_3 = S_{42} + S_{31} \quad (17)$$

$$c_4 = S_{42} - S_{31} \quad (18)$$

$$a_3 = S_{21} + S_{43} \quad (19)$$

$$b_2 = -D_{21} - D_{43} \quad (20)$$

$$c_5 = -D_{42} + D_{31} \quad (21)$$

$$c_6 = -D_{42} - D_{31} \quad (22)$$

$$c_2 = D_{21} - D_{43} \quad (23)$$

$$a_4 = S_{21} - S_{43}. \quad (24)$$

Thus, the 16 elements of the scattering matrix have been expressed in 16 auxiliary quantities which are functions of elements of the amplitude matrix. The latter contains essentially only 7 independent real variables, since Eqs. (4)–(6) show that one phase is irrelevant. Therefore, Van de Hulst (1957) has concluded that there must be 9 relations between the 16 elements of the scattering matrix, but he did not derive these relations. We wish to study this problem here in a systematic way.

We start by observing that the linear relations (9)–(24) can be inverted. The result is

$$M_1 = \tfrac{1}{2}(a_1 + a_2 - b_1 - c_1) \quad (25)$$

$$M_2 = \tfrac{1}{2}(a_1 + a_2 + b_1 + c_1) \quad (26)$$

$$M_3 = \tfrac{1}{2}(a_1 - a_2 - b_1 + c_1) \quad (27)$$

$$M_4 = \tfrac{1}{2}(a_1 - a_2 + b_1 - c_1) \quad (28)$$

$$S_{21} = \tfrac{1}{2}(a_3 + a_4) \quad (29)$$

$$S_{31} = \tfrac{1}{2}(c_3 - c_4) \quad (30)$$

$$S_{41} = \tfrac{1}{2}(b_3 - b_4) \quad (31)$$

$$S_{32} = \tfrac{1}{2}(b_3 + b_4) \quad (32)$$

$$S_{42} = \tfrac{1}{2}(c_3 + c_4) \quad (33)$$

$$S_{43} = \tfrac{1}{2}(a_3 - a_4) \quad (34)$$

$$D_{21} = -\tfrac{1}{2}(b_2 - c_2) \quad (35)$$

$$D_{31} = \tfrac{1}{2}(c_5 - c_6) \quad (36)$$

$$D_{41} = -\tfrac{1}{2}(b_5 - b_6) \quad (37)$$

$$D_{32} = \tfrac{1}{2}(b_5 + b_6) \quad (38)$$

$$D_{42} = -\tfrac{1}{2}(c_5 + c_6) \quad (39)$$

$$D_{43} = -\tfrac{1}{2}(b_2 + c_2). \quad (40)$$

Consequently, we can first seek relations for the auxiliary quantities and then transform these relations into equations for the elements of the scattering matrix.

2.2. *Equations involving the auxiliary quantities*

It is immediately clear from the form of Eqs. (4)–(6) that a number of interrelations can be derived for the auxiliary quantities by using properties of sines and cosines.

First we find by squaring both sides of Eqs. (5) and (6)

$$S_{21}^2 + D_{21}^2 = M_2 M_1 \quad (41)$$

$$S_{31}^2 + D_{31}^2 = M_3 M_1 \quad (42)$$

$$S_{41}^2 + D_{41}^2 = M_4 M_1 \quad (43)$$

$$S_{32}^2 + D_{32}^2 = M_3 M_2 \quad (44)$$

$$S_{42}^2 + D_{42}^2 = M_4 M_2 \quad (45)$$

$$S_{43}^2 + D_{43}^2 = M_4 M_3. \quad (46)$$

An additional set of 12 equations is obtained by using identities of the type

$$\cos(\rho_k - \rho_j) = \cos(\rho_k - \rho_l)\cos(\rho_j - \rho_l) + \sin(\rho_k - \rho_l) \times \sin(\rho_j - \rho_l) \quad (47)$$

where j, k and l are different numbers 1, 2, 3, 4 and $k > j$. Substituting this identity in Eq. (5) yields

$$S_{32} M_1 = S_{31} S_{21} + D_{31} D_{21} \quad (48)$$

$$S_{42} M_1 = S_{41} S_{21} + D_{41} D_{21} \quad (49)$$

$$S_{43} M_1 = S_{41} S_{31} + D_{41} D_{31} \quad (50)$$

$$S_{31} M_2 = S_{32} S_{21} - D_{32} D_{21} \quad (51)$$

$$S_{41} M_2 = S_{42} S_{21} - D_{42} D_{21} \quad (52)$$

$$S_{43} M_2 = S_{42} S_{32} + D_{42} D_{32} \quad (53)$$

$$S_{21} M_3 = S_{32} S_{31} + D_{32} D_{31} \quad (54)$$

$$S_{41} M_3 = S_{43} S_{31} - D_{43} D_{31} \quad (55)$$

$$S_{42} M_3 = S_{43} S_{32} - D_{43} D_{32} \quad (56)$$

$$S_{21} M_4 = S_{42} S_{41} + D_{42} D_{41} \quad (57)$$

$$S_{31} M_4 = S_{43} S_{41} + D_{43} D_{41} \quad (58)$$

$$S_{32} M_4 = S_{43} S_{42} + D_{43} D_{42}. \quad (59)$$

A third set is obtained by substituting

$$-\sin(\rho_k-\rho_j) = -\sin(\rho_k-\rho_l)\cos(\rho_j-\rho_l)+\cos(\rho_k-\rho_l) \times \sin(\rho_j-\rho_l) \quad (60)$$

in Eq. (6). We then find the 12 equations

$$D_{32}M_1 = D_{31}S_{21} - S_{31}D_{21} \quad (61)$$
$$D_{42}M_1 = D_{41}S_{21} - S_{41}D_{21} \quad (62)$$
$$D_{43}M_1 = D_{41}S_{31} - S_{41}D_{31} \quad (63)$$
$$D_{31}M_2 = D_{32}S_{21} + S_{32}D_{21} \quad (64)$$
$$D_{41}M_2 = D_{42}S_{21} + S_{42}D_{21} \quad (65)$$
$$D_{43}M_2 = D_{42}S_{32} - S_{42}D_{32} \quad (66)$$
$$D_{21}M_3 = -D_{32}S_{31} + S_{32}D_{31} \quad (67)$$
$$D_{41}M_3 = D_{43}S_{31} + S_{43}D_{31} \quad (68)$$
$$D_{42}M_3 = D_{43}S_{32} + S_{43}D_{32} \quad (69)$$
$$D_{21}M_4 = -D_{42}S_{41} + S_{42}D_{41} \quad (70)$$
$$D_{31}M_4 = -D_{43}S_{41} + S_{43}D_{41} \quad (71)$$
$$D_{32}M_4 = -D_{43}S_{42} + S_{43}D_{42}. \quad (72)$$

Using well-known formulae for the products of sines and cosines the following three sets of equations without any M_k may be derived, containing 3, 1 and 4 new and non-trivial relations, respectively.

$$S_{21}S_{43} = S_{31}S_{42} + D_{41}D_{32} \quad (73)$$
$$S_{31}S_{42} = S_{41}S_{32} + D_{21}D_{43} \quad (74)$$
$$S_{41}S_{32} = S_{21}S_{43} - D_{31}D_{42} \quad (75)$$
$$D_{41}D_{32} = D_{31}D_{42} - D_{21}D_{43} \quad (76)$$
$$S_{21}D_{43} = S_{31}D_{42} - S_{41}D_{32} \quad (77)$$
$$S_{21}D_{43} = S_{32}D_{41} - S_{42}D_{31} \quad (78)$$
$$S_{31}D_{42} = S_{32}D_{41} - S_{43}D_{21} \quad (79)$$
$$S_{41}D_{32} = S_{42}D_{31} - S_{43}D_{21}. \quad (80)$$

In total we have thus found 38 quadratic three-term relations for the 16 auxiliary quantities, M_k, S_{kj} and D_{kj} with $k>j$.

Apart from the 38 equations considered above others may be found by taking linear combinations. For instance, by combining pairs of Eqs. (73)–(75) we obtain

$$S_{21}S_{43} + D_{21}D_{43} = S_{31}S_{42} + D_{31}D_{42} \quad (81)$$
$$S_{31}S_{42} - D_{31}D_{42} = S_{32}S_{41} - D_{32}D_{41} \quad (82)$$
$$S_{32}S_{41} + D_{32}D_{41} = S_{21}S_{43} - D_{21}D_{43} \quad (83)$$

and similarly from Eqs. (77)–(80)

$$D_{21}S_{43} - S_{21}D_{43} = D_{31}S_{42} - S_{31}D_{42} \quad (84)$$
$$D_{31}S_{42} + S_{31}D_{42} = D_{32}S_{41} + S_{32}D_{41} \quad (85)$$
$$D_{41}S_{32} - S_{41}D_{32} = D_{43}S_{21} + S_{43}D_{21}. \quad (86)$$

These equations may also be proven directly by observing that

$$(\rho_k-\rho_j)-(\rho_l-\rho_i) = (\rho_k-\rho_l)-(\rho_j-\rho_i) \quad (87)$$

and taking cosines and sines, respectively, on both sides [cf. Eqs. (4)–(6)].

We have used distinct indices k and j for S_{kj} and D_{kj} in all equations. They keep their validity, however, if we allow indices to be the same and use

$$S_{ll} = M_l \quad (88)$$
$$D_{ll} = 0. \quad (89)$$

2.3 *Choice of a complete set*

In view of the evident redundancy of the above equations we now seek a minimal number of equations for the auxiliary quantities from which all other relations that are based on Eqs. (4)–(6) can be derived. Suppose $M_1 \neq 0$. We choose 7 quantities as being of primary interest, namely M_1, S_{21}, S_{31}, S_{41}, D_{21}, D_{31} and D_{41}. The others, i.e. M_2, M_3, M_4, S_{32}, S_{42}, S_{43}, D_{32}, D_{42}, and D_{43}, may be uniquely expressed in these by a simple division by M_1 in each of Eqs. (41)–(43), (48)–(50) and (61)–(63), respectively. Since in each equation a quantity appears which is not present in the preceding equations, we may conclude that Eqs. (41)–(43), (48)–(50) and (61)–(63) are independent. Because in a scattering problem not all M_k vanish, we have at least one non-zero M_k. Consequently, if $M_1=0$, another set of 9 independent equations can be chosen carrying this non-zero M_k.

We will now show that Eqs. (41)–(43), (48)–(50) and (61)–(63) are also complete, i.e. all other equations for the same quantities which follow from Eqs. (4)–(6) can be derived from them. Again we shall assume $M_1 \neq 0$. However, the considerations below can easily be modified for some other $M_k \neq 0$. Suppose we have 16 quantities satisfying Eqs. (41)–(43), (48)–(50) and (61)–(63). We may find, for non-zero M_k, unique $r_k > 0$ such that

$$r_k = \sqrt{M_k}. \quad (90)$$

In view of Eqs. (41)–(43) we can find unique δ_{k1} in the range $[0, 2\pi)$ satisfying

$$S_{k1} = r_k r_1 \cos\delta_{k1}, \qquad D_{k1} = -r_k r_1 \sin\delta_{k1} \quad (91)$$

where $k=2, 3, 4$. Substituting Eq. (91) in Eqs. (48)–(50) we obtain

$$S_{32} = r_3 r_2 \cos(\delta_{31}-\delta_{21}) \quad (92)$$
$$S_{42} = r_4 r_2 \cos(\delta_{41}-\delta_{21}) \quad (93)$$
$$S_{43} = r_4 r_3 \cos(\delta_{41}-\delta_{31}). \quad (94)$$

Similarly, we find from Eqs. (61)–(63)

$$D_{32} = -r_3 r_2 \sin(\delta_{31}-\delta_{21}) \quad (95)$$
$$D_{42} = -r_4 r_2 \sin(\delta_{41}-\delta_{21}) \quad (96)$$
$$D_{43} = -r_4 r_3 \sin(\delta_{41}-\delta_{31}). \quad (97)$$

We have thus expressed the 16 quantities in 7 others, namely r_1, r_2, r_3, r_4, δ_{21}, δ_{31}, and δ_{41}. If we now choose an arbitrary $0 \le \rho_1 < 2\pi$ and define

$$\rho_k = \delta_{k_1} + \rho_1 \quad (98)$$

for $k=2,3,4$ we find from Eqs. (90)–(98) that Eqs. (4)–(6) hold. Defining A_k by Eq. (2) we can even completely reconstruct the amplitude matrix. We conclude that all equations stemming from Eqs. (4)–(6) follow from the nine independent Eqs. (41)–(43), (48)–(50) and (61)–(63), or in other words that these latter equations are complete. In the special case that M_k vanishes, we take $r_k=0$ and arbitrary values for δ_{k1} and ρ_k, in analogy with the indeterminacy in the angular polar coordinate of the origin. This does not change the conclusion reached above.

When we have any particular relation for the auxiliary quantities that can be derived from Eqs. (4)–(6) we know that it must be possible to derive it directly from any set of 9 independent and complete equations. To clarify this point we consider by way of example Eqs. (44)–(46), assuming $M_1 \neq 0$. For distinct $k,j = 2,3,4$ we find using Eqs. (41)–(43), (48)–(50) and (61)–(63)

$$\begin{aligned}(S_{kj}^2 + D_{kj}^2)M_1^2 &= (S_{kj}M_1)^2 + (D_{kj}M_1)^2 \\ &= (S_{k1}S_{j1} + D_{k1}D_{j1})^2 + (D_{k1}S_{j1} - S_{k1}D_{j1})^2 \\ &= (S_{k1}^2 + D_{k1}^2)(S_{j1}^2 + D_{j1}^2) = (M_kM_j)M_1^2 \end{aligned} \tag{99}$$

which implies Eqs. (44)–(46).

2.4. *Equations involving the elements of the scattering matrix*

In practice we use the elements of the scattering matrix and not the auxiliary quantities. Therefore, in this subsection we shall (i) translate the relations expressed in terms of the auxiliary quantities into relations expressed in terms of the elements of the scattering matrix, by means of the inverse relations (25)–(40), (ii) seek a complete set having a minimal number of equations, (iii) simplify the equations, (iv) consider alternative proofs, and (v) discuss related work of other authors.

Substituting Eqs. (25)–(40) into Eqs. (41)–(46) we find

$$(a_3 + a_4)^2 + (b_2 - c_2)^2 = (a_1 + a_2)^2 - (b_1 + c_1)^2 \tag{100}$$

$$(c_3 - c_4)^2 + (c_5 - c_6)^2 = (a_1 - b_1)^2 - (a_2 - c_1)^2 \tag{101}$$

$$(b_3 - b_4)^2 + (b_5 - b_6)^2 = (a_1 - c_1)^2 - (a_2 - b_1)^2 \tag{102}$$

$$(b_3 + b_4)^2 + (b_5 + b_6)^2 = (a_1 + c_1)^2 - (a_2 + b_1)^2 \tag{103}$$

$$(c_3 + c_4)^2 + (c_5 + c_6)^2 = (a_1 + b_1)^2 - (a_2 + c_1)^2 \tag{104}$$

$$(a_3 - a_4)^2 + (b_2 + c_2)^2 = (a_1 - a_2)^2 - (b_1 - c_1)^2 . \tag{105}$$

Transforming the 12 Eqs. (48)–(59) one obtains

$$(b_3 \pm b_4)(a_1 + a_2 \mp b_1 \mp c_1) = (a_3 + a_4)(c_3 \mp c_4) - (b_2 - c_2)(c_5 \mp c_6) \tag{106}$$

$$(b_3 \pm b_4)(a_1 - a_2 \pm b_1 \mp c_1) = (a_3 - a_4)(c_3 \pm c_4) + (b_2 + c_2)(c_5 \pm c_6) \tag{107}$$

$$(a_3 \pm a_4)(a_1 \mp a_2 - b_1 \pm c_1) = (b_3 \pm b_4)(c_3 - c_4) \pm (b_5 \pm b_6)(c_5 - c_6) \tag{108}$$

$$(a_3 \pm a_4)(a_1 \mp a_2 + b_1 \mp c_1) = (b_3 \mp b_4)(c_3 + c_4) \pm (b_5 \mp b_6)(c_5 + c_6) \tag{109}$$

$$(c_3 \pm c_4)(a_1 \pm a_2 \mp b_1 - c_1) = (a_3 \pm a_4)(b_3 - b_4) + (b_2 \mp c_2)(b_5 - b_6) \tag{110}$$

$$(c_3 \pm c_4)(a_1 \mp a_2 \mp b_1 + c_1 = (a_3 \mp a_4)(b_3 + b_4) + (b_2 \pm c_2)(b_5 + b_6) \tag{111}$$

where each $\pm$ pair represents two equations. Similarly, the 12 Eqs. (61)–(72) yield

$$(b_2 \pm c_2)(a_1 \pm a_2 - b_1 \mp c_1) = (b_5 \mp b_6)(c_3 - c_4) \pm (b_3 \mp b_4)(c_5 - c_6) \tag{112}$$

$$(b_2 \pm c_2)(a_1 \pm a_2 + b_1 \pm c_1) = (\pm b_3 + b_4)(c_5 + c_6) + (c_3 + c_4)(b_5 \pm b_6) \tag{113}$$

$$(b_5 \pm b_6)(a_1 + a_2 \mp b_1 \mp c_1) = (a_3 + a_4)(c_5 \mp c_6) + (b_2 - c_2)(c_3 \mp c_4) \tag{114}$$

$$(b_5 \pm b_6)(a_1 - a_2 \pm b_1 \mp c_1) = (b_2 + c_2)(c_3 \pm c_4) - (a_3 - a_4)(c_5 \pm c_6) \tag{115}$$

$$(c_5 \pm c_6)(a_1 + a_2 \mp b_1 \mp c_1) = (a_3 + a_4)(b_5 \mp b_6) - (b_2 - c_2)(b_3 \mp b_4) \tag{116}$$

$$(c_5 \pm c_6)(a_1 - a_2 \mp b_1 \pm c_1) = (b_2 + c_2)(b_3 \pm b_4) - (a_3 - a_4)(b_5 \pm b_6) . \tag{117}$$

Finally, we derive from Eqs. (73)–(80)

$$a_3^2 - a_4^2 = c_3^2 - c_4^2 - b_5^2 + b_6^2 \tag{118}$$

$$c_3^2 - c_4^2 = b_3^2 - b_4^2 + b_2^2 - c_2^2 \tag{119}$$

$$b_3^2 - b_4^2 = a_3^2 - a_4^2 + c_5^2 - c_6^2 \tag{120}$$

$$b_5^2 - b_6^2 = b_2^2 - c_2^2 + c_5^2 - c_6^2 \tag{121}$$

$$(a_3 + a_4)(b_2 + c_2) = (c_3 - c_4)(c_5 + c_6) + (b_3 - b_4)(b_5 + b_6) \tag{122}$$

$$(a_3 + a_4)(b_2 + c_2) = (b_3 + b_4)(b_5 - b_6) + (c_3 + c_4)(c_5 - c_6) \tag{123}$$

$$(c_3 - c_4)(c_5 + c_6) = (b_3 + b_4)(b_5 - b_6) - (a_3 - a_4)(b_2 - c_2) \tag{124}$$

$$(b_3 - b_4)(b_5 + b_6) = (c_3 + c_4)(c_5 - c_6) + (a_3 - a_4)(b_2 - c_2). \tag{125}$$

Consequently, we have found 38 quadratic relations for the 16 elements of the scattering matrix.

In discussing completeness we first make the assumption $M_1 \neq 0$, which is now translated by Eq. (25) into $b_1 + c_1 \neq a_1 + a_2$. It then follows from the independence and completeness of Eqs. (41)–(43), (48)–(50) and (61)–(63) and from the existence of Eqs. (25)–(40) that the 9 Eqs. (100)–(102), (106$^+$), (108$^-$), (110$^+$), (112$^+$), (114$^+$), and (116$^+$), where + and − refer to upper and lower signs, respectively, are independent and complete, i.e. that all other relations for the same quantities based on Eqs. (4)–(6) can be derived from them. If $b_1 + c_1 = a_1 + a_2$, we can obtain an alternative set of 9 independent and complete equations for the scattering matrix, by selecting another set of 9 independent and complete equations for the auxiliary quantities.

We shall now show that the 38 relations (100)–(125) between products and squares of the elements of the scattering matrix can be reduced to simpler equations by linear combination. First we consider relations involving squares. By adding Eqs. (100) and (105), (101) and (104), and (102) and (103), respectively, we obtain

$$a_3^2 + a_4^2 + b_2^2 + c_2^2 = a_1^2 + a_2^2 - b_1^2 - c_1^2 \tag{126}$$

$$c_3^2 + c_4^2 + c_5^2 + c_6^2 = a_1^2 + b_1^2 - a_2^2 - c_1^2 \tag{127}$$

$$b_3^2 + b_4^2 + b_5^2 + b_6^2 = a_1^2 + c_1^2 - a_2^2 - b_1^2 . \tag{128}$$

It may be observed that Eq. (127) involves the squares of all elements in the first two columns of the scattering matrix and likewise Eq. (128) for the first two rows. It proved useful to define sums of squares on each row and each column with the particular sign convention shown below.

$$A = a_1^2 - c_1^2 - c_3^2 - c_5^2 \tag{129}$$

$$B = -b_1^2 + a_2^2 + c_4^2 + c_6^2 \tag{130}$$

$$C = -b_3^2 + b_4^2 + a_3^2 + c_2^2 \quad (131)$$

$$D = -b_5^2 + b_6^2 + b_2^2 + a_4^2 \quad (132)$$

$$P = a_1^2 - b_1^2 - b_3^2 - b_5^2 \quad (133)$$

$$Q = -c_1^2 + a_2^2 + b_4^2 + b_6^2 \quad (134)$$

$$R = -c_3^2 + c_4^2 + a_3^2 + b_2^2 \quad (135)$$

$$S = -c_5^2 + c_6^2 + c_2^2 + a_4^2 . \quad (136)$$

We may now rewrite Eqs. (126)–(128) as

$$-P - Q + C + D = -A - B + R + S = 0 \quad (137)$$

$$-A + B = 0 \quad (138)$$

$$-P + Q = 0 . \quad (139)$$

Similarly we obtain from Eqs. (118)–(121)

$$R - D = 0 \quad (140)$$

$$-R + C = 0 \quad (141)$$

$$S - C = 0 \quad (142)$$

$$S - D = 0 . \quad (143)$$

Collecting all these results we find from (137)–(139) that

$$A = B = \tfrac{1}{2}(R + S) \quad (144)$$

$$P = Q = \tfrac{1}{2}(C + D) , \quad (145)$$

and from Eqs. (140)–(143) that

$$C = D = R = S . \quad (146)$$

These together give 7 equations that can be written in the surprisingly simple form

$$A = B = C = D = P = Q = R = S . \quad (147)$$

The value of the constant also equals d^2, where d is the absolute value of the determinant of the amplitude matrix. This may be checked by simple algebra. An immediate corollary is

$$-A + B + C + D - 2P = 0 \quad (148)$$

which proves that the sum of the squares of all 16 elements is $4a_1^2$, a result earlier expounded by Fry and Kattawar (1981).

We found that a graphical code may be helpful in visualizing the relations. Let a 4×4 array of dots represent the scattering matrix, a full circle at the appropriate place the square of an element and an open circle the negative square of the element and adopt the convention that the sum of these (positive and negative) squares in a pictogram is zero. Then Fig. 1 presents in seven separate pictograms the seven equations shown. Many more can easily be drawn by combining 2 or 4 rows and/or columns.

We shall now turn our attention to relations involving products. By subtracting Eqs. (100) and (105), (101) and (104), and (102) and (103), respectively, we find

$$a_1 a_2 - b_1 c_1 - a_3 a_4 + b_2 c_2 = 0 \quad (149)$$

$$a_1 b_1 - a_2 c_1 - c_3 c_4 - c_5 c_6 = 0 \quad (150)$$

$$a_1 c_1 - a_2 b_1 - b_3 b_4 - b_5 b_6 = 0 . \quad (151)$$

This exhausts the equations in which products occur that have both factors in the upper left 2×2 square (a_1, a_2, b_1, c_1). If we extend the graphic code by indicating a positive product by a solid connecting line and a negative product by a dashed connection while all $\pm$ products indicated have to be added to get zero, the preceding equations read as the top set of Fig. 2.

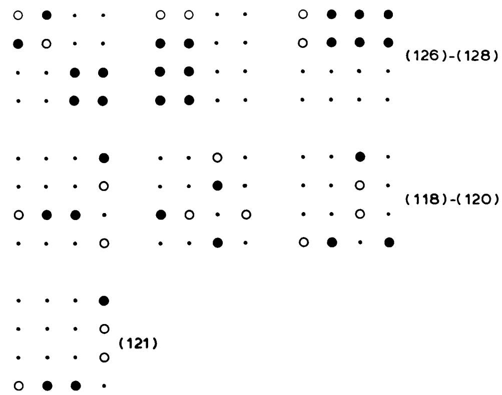

Fig. 1. Graphic code representing seven separate equations involving squares of the elemente of the scattering matrix

The next 24 equations of this type follow from Eqs. (106)–(117) with + or − signs by repeated addition or subtraction. They are shown in graphic code in the next eight sets of Fig. 2 and read as follows:

$$a_1 b_3 - b_4 c_1 - a_3 c_3 - c_2 c_5 = 0 \quad (152)$$

$$a_2 b_3 - b_1 b_4 - a_4 c_3 + b_2 c_5 = 0 \quad (153)$$

$$a_1 b_4 - b_3 c_1 - b_2 c_6 + a_4 c_4 = 0 \quad (154)$$

$$b_1 b_3 - a_2 b_4 - a_3 c_4 - c_2 c_6 = 0 \quad (155)$$

$$a_1 a_3 - a_2 a_4 - b_3 c_3 + b_6 c_6 = 0 \quad (156)$$

$$a_3 b_1 - a_4 c_1 - b_3 c_4 + b_6 c_5 = 0 \quad (157)$$

$$a_1 a_4 - a_2 a_3 + b_4 c_4 - b_5 c_5 = 0 \quad (158)$$

$$a_4 b_1 - a_3 c_1 + b_4 c_3 - b_5 c_6 = 0 \quad (159)$$

$$a_1 c_3 - b_1 c_4 - a_3 b_3 - b_2 b_5 = 0 \quad (160)$$

$$c_1 c_3 - a_2 c_4 - a_3 b_4 - b_2 b_6 = 0 \quad (161)$$

$$a_1 c_4 - b_1 c_3 + a_4 b_4 - b_6 c_2 = 0 \quad (162)$$

$$c_1 c_4 - a_2 c_3 + a_4 b_3 - b_5 c_2 = 0 \quad (163)$$

$$a_1 b_2 - b_5 c_3 + a_2 c_2 - b_4 c_6 = 0 \quad (164)$$

$$b_1 b_2 - b_5 c_4 + c_1 c_2 - b_4 c_5 = 0 \quad (165)$$

$$a_1 c_2 - b_3 c_5 + a_2 b_2 - b_6 c_4 = 0 \quad (166)$$

$$b_1 c_2 - b_3 c_6 + b_2 c_1 - b_6 c_3 = 0 \quad (167)$$

$$a_1 b_5 - b_6 c_1 - a_4 c_5 - b_2 c_3 = 0 \quad (168)$$

$$a_2 b_5 - b_1 b_6 - a_3 c_5 + c_2 c_3 = 0 \quad (169)$$

$$a_1 b_6 - b_5 c_1 + a_3 c_6 - c_2 c_4 = 0 \quad (170)$$

$$b_1 b_5 - a_2 b_6 - b_2 c_4 - a_4 c_6 = 0 \quad (171)$$

$$a_1 c_5 - b_1 c_6 - b_3 c_2 - a_4 b_5 = 0 \quad (172)$$

$$a_2 c_5 - c_1 c_6 + b_2 b_3 - a_3 b_5 = 0 \quad (173)$$

$$a_1 c_6 - b_1 c_5 + a_3 b_6 - b_2 b_4 = 0 \quad (174)$$

$$c_1 c_5 - a_2 c_6 - c_2 b_4 - a_4 b_6 = 0 . \quad (175)$$

The final 12 products of the possible 120 ones between pairs of distinct elements of the scattering matrix occur in 3 more

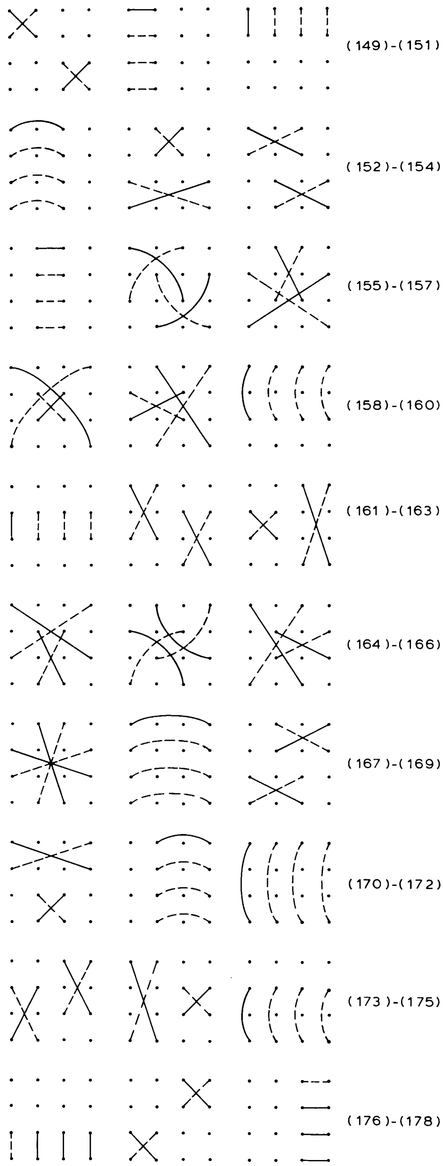

Fig. 2. Graphic code representing Eqs. (149)–(178). Each pictogram represents one equation, as numbered consecutively in the margin

relations, shown in code by the bottom set of Fig. 2. In full notation they read

$$a_3c_2 + a_4b_2 - c_3c_5 + c_4c_6 = 0 \quad (176)$$

$$c_3c_6 - c_4c_5 + b_3b_6 - b_4b_5 = 0 \quad (177)$$

$$-b_3b_5 + b_4b_6 + a_3b_2 + c_2a_4 = 0. \quad (178)$$

They follow from Eqs. (122)–(125) by linear combination of pairs of elements of the scattering matrix or from Eqs. (84)–(86).

In summary of Fig. 2 we may state that there are 30 equations, each containing four terms, which are products of two elements, and no product occurring twice. Jointly they exhaust all 120 possible products between distinct elements of the scattering matrix. The 30 equations subdivide into two types. The 12 equations of type 1 carry corresponding products of any two chosen rows or columns; they follow a simple sign rule. The equations of type 2 express that the sum or difference of any chosen pair of complementary subdeterminants vanishes; the sign rule here is not obvious.

The graphic code provides a nice survey of equations and facilitates the derivation of corollaries by addition and subtraction. It is immediately clear now that sums and differences of the elements in the first and second column obey the two equations

$$-(a_1 \pm b_1)^2 + (c_1 \pm a_2)^2 + (c_3 \pm c_4)^2 + (c_5 \pm c_6)^2 = 0, \quad (179)$$

and similarly for all 6 combinations involving the first row or the first column. However, the sums and differences of the elements in the third and fourth column obey

$$-(b_3 \pm b_5)^2 + (b_4 \pm b_6)^2 + (a_3 \pm b_2)^2 + (c_2 \pm a_4)^2 = 2d^2 \quad (180)$$

and similarly for all 6 combinations of rows or columns not involving the first row or column.

Hitherto we have derived a plethora of equations for the elements of the scattering matrix by using equations for the auxiliary quantities. There is, however, a different way. It is physically clear (cf. Van de Hulst, 1957) that scattering of a fully polarized wave by a single particle results in a completely polarized beam. If we denote by $\{I_0, Q_0, U_0, V_0\}$ and $\{I, Q, U, V\}$ the Stokes parameters of the incident and scattered waves, respectively, this physical property means that we must have

$$I^2 - Q^2 - U^2 - V^2 = 0 \quad (181)$$

for any incident fully polarized wave. On employing Eq. (3) we obtain

$$\begin{aligned} I^2 - Q^2 - U^2 - V^2 = &(a_1I_0 + b_1Q_0 + b_3U_0 + b_5V_0)^2 \\ &- (c_1I_0 + a_2Q_0 + b_4U_0 + b_6V_0)^2 \\ &- (c_3I_0 + c_4Q_0 + a_3U_0 + b_2V_0)^2 \\ &- (c_5I_0 + c_6Q_0 + c_2U_0 + a_4V_0)^2 . \quad (182) \end{aligned}$$

By taking incident beams with Stokes parameters $\{1, \pm 1, 0, 0\}$, $\{1, 0, \pm 1, 0\}$ and $\{1, 0, 0, \pm 1\}$, respectively, and requiring Eq. (181) to hold we first derive the 6 equations

$$a_1^2 - c_1^2 - c_3^2 - c_5^2 = -b_1^2 + a_2^2 + c_4^2 + c_6^2 \quad (183)$$

$$a_1b_1 = c_1a_2 + c_3c_4 + c_5c_6 \quad (184)$$

$$a_1^2 - c_1^2 - c_3^2 - c_5^2 = -b_3^2 + b_4^2 + a_3^2 + c_2^2 \quad (185)$$

$$a_1b_3 = c_1b_4 + c_3a_3 + c_5c_2 \quad (186)$$

$$a_1^2 - c_1^2 - c_3^2 - c_5^2 = -b_5^2 + b_6^2 + b_2^2 + a_4^2 \quad (187)$$

$$a_1b_5 = c_1b_6 + c_3b_2 + c_5a_4. \quad (188)$$

If we now use Eqs. (183)–(188) to simplify Eq. (182) and take incident beams with Stokes parameters $\{1, 1/\sqrt{2}, 1/\sqrt{2}, 0\}$, $\{1, 1/\sqrt{2}, 0, 1/\sqrt{2}\}$ and $\{1, 0, 1/\sqrt{2}, 1/\sqrt{2}\}$, respectively, we obtain the 3 additional equations

$$b_1b_3 = a_2b_4 + c_4a_3 + c_6c_2 \quad (189)$$

$$b_1b_5 = a_2b_6 + c_4b_2 + c_6a_4 \quad (190)$$

$$b_3b_5 = b_4b_6 + a_3b_2 + c_2a_4 . \quad (191)$$

Equations (183)–(191) imply that Eq. (182) can be rewritten as

$$I^2 - Q^2 - U^2 - V^2 = d^2(I_0^2 - Q_0^2 - U_0^2 - V_0^2). \quad (192)$$

Consequently, if the incident light is fully polarized, so is the scattered light, and if the incident light is not fully polarized, the scattered light is only fully polarized if d^2 vanishes, i.e. if the amplitude matrix has zero determinant.

The 9 Eqs. (183)–(191) can also be verified by transforming them into relations for the auxiliary quantities and they can thus be based on Eqs. (4)–(6). Consequently, they follow from the 9 independent and complete equations (100)–(102), (106^+), (108^-), (110^+), (112^+), (114^+) and (116^+). However, it can be shown that Eqs. (183)–(191) are not complete. For example, the elements of the matrix

$$\begin{pmatrix} 1 & -q & 0 & 0 \\ -1 & q & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{193}$$

with $-1 < q < 1$ satisfy Eqs. (183)–(191), but although $b_1 + c_1 \neq a_1 + a_2$ the elements do not satisfy Eqs. (102) and (147). This shows that the matrix (193) cannot be a proper scattering matrix, even though Eqs. (183)–(191) are true. In fact, for this simple case it is easily verified that there is no corresponding amplitude matrix, since $M_1 \neq 0$, $M_4 \neq 0$, $S_{41} = D_{41} = 0$ and therefore Eq. (43) does not hold [cf. Eqs. (25), (28), (31) and (37)].

On comparing our results for interrelations between the elements of the scattering matrix with related work of other authors we wish to make the following comments.

Perrin (1942) and Perrin and Abragam (1951) have reported a few relations for the special cases of scattering by spherical particles and Rayleigh scattering.

Abhyankar and Fymat (1969) have found 9 equations for the elements of the scattering matrix (their Eqs. (10c)–(15c) and (19c)–(21c)). It is readily verified that these equations remain unaltered if the sign convention for the Stokes parameter V is chosen to be the same as in this paper. Six of the equations are identical to our Eqs. (100)–(105) and the other three can be derived from our Eqs. (118)–(120) and (176)–(178). However, the authors did not make clear whether their equations are complete. Fry and Kattawar (1981) presented two sets of 9 equations for the elements of the scattering matrix [cf. their Eqs. (4a)–(4i) or their Eqs. (4a)–(4f) and (5a)–(5c)]. Their first set of equations consists of our Eqs. (100)–(105) and (176)–(178); their second set contains our Eqs. (100)–(105) and combinations of our Eqs. (118)–(120). However, a thorough analysis of the three sets of 9 equations published by Abhyankar and Fymat (1969) and by Fry and Kattawar (1981) shows that none of these sets is complete. This may be verified, for example, by observing that the elements of the matrix

$$\begin{pmatrix} 4 & 0 & \sqrt{3} & 1 \\ 0 & 0 & \sqrt{3} & -3 \\ 2 & 0 & -2\sqrt{3} & -2 \\ 0 & 2\sqrt{3} & 0 & 0 \end{pmatrix}$$

obey each one of their three sets of equations but do not satisfy our Eqs. (106^+), (108^-), (110^+), (112^+), (114^+) and (116^+), nor e.g. Eq. (152).

2.5. Inequalities

In preparation of the discussion on an assembly of particles [cf. Sect. 3] we shall now show how it is possible to derive inequalities for the elements of the scattering matrix of a single particle. We mention the following sources for obtaining inequalities.

Equations (4)–(6) provide

$$M_k \geq 0 \tag{194}$$

$$|S_{kj}| \leq (M_k M_j)^{1/2} \tag{195}$$

$$|D_{kj}| \leq (M_k M_j)^{1/2} \tag{196}$$

$$|S_{kj} D_{kj}| \leq \tfrac{1}{2} M_k M_j \tag{197}$$

$$|S_{kj} \pm D_{kj}| \leq (2 M_k M_j)^{1/2} \tag{198}$$

where some well-known properties of sines and cosines have been used. We may further write

$$|S_{kl} \pm S_{jm}| \leq |S_{kl}| + |S_{jm}| \leq (M_k M_l)^{1/2} + (M_j M_m)^{1/2} \leq \tfrac{1}{2}(M_k + M_l + M_j + M_m) \tag{199}$$

where we have used the triangle inequality and the fact that the geometric mean of two positive numbers does not exceed their arithmetic mean. Similarly, we find

$$|D_{kl} \pm D_{jm}| \leq \tfrac{1}{2}(M_k + M_l + M_j + M_m). \tag{200}$$

On using Eqs. (9)–(24) or (25)–(40) these inequalities yield inequalities for the elements of the scattering matrix.

An important source of inequalities is provided by equations (100)–(125) and similar equalities. They may be transformed into inequalities e.g. by omitting terms which are always positive or by using the inequality $a^2 + b^2 \geq 2ab$ to replace the sum of two squares. We can further employ the well-known inequality

$$(x_1 y_1 + x_2 y_2 + \cdots + x_n y_n)^2 \leq (x_1^2 + \cdots + x_n^2)(y_1^2 + \cdots + y_n^2) \tag{201}$$

for real numbers x_k, y_k, $k = 1, 2, \ldots, n$, where the equality sign holds if and only if x_k/y_k is constant. This inequality is called Schwarz's or Cauchy's inequality (Arfken, 1970; Hardy et al., 1934). Taking $|y_1| = |y_2| = \cdots = |y_n| = 1$ we find that

$$\sum_{k=1}^{n} x_k^2 \leq r \tag{202}$$

implies

$$\sum_{k=1}^{n} |x_k| \leq (rn)^{1/2} \tag{203}$$

where the equality sign holds if and only if it holds in Eq. (202) and all $|x_k|$ are the same.

It can be shown (see e.g. Chandrasekhar, 1950) that Stokes parameters always admit the inequality

$$I \geq (Q^2 + U^2 + V^2)^{1/2}, \tag{204}$$

where the equality sign holds if and only if the beam is fully polarized. Applying this rule to the scattered beams for different incident beams, satisfying inequality (204), yields inequalities for the elements of the scattering matrix [cf. Eq. (3)].

Evidently, a certain inequality may sometimes be obtained from more than one source, but the amount of algebra may be much less using one source than another.

We do not aim here at a comprehensive list of inequalities. Instead, we will give some examples to indicate the possibilities in this field.

(i) Equations (9), (10), (13), (14) and (194) show that

$$a_1 \geq 0 \tag{205}$$

and that $|b_1|$, $|c_1|$ and $|a_2|$ do not exceed a_1. If we further use Eqs. (11), (12), (15)–(24), (199) and (200) we can infer that *no element of the scattering matrix has an absolute value larger than* a_1.

(ii) It follows directly from Eqs. (25), (26) and (194) that

$$|b_1 + c_1| \leq a_1 + a_2 \,. \tag{206}$$

Similarly we have

$$|b_1 - c_1| \leq a_1 - a_2 \tag{207}$$

$$|a_2 \pm c_1| \leq a_1 \pm b_1 \tag{208}$$

$$|a_2 \pm b_1| \leq a_1 \pm c_1, \tag{209}$$

where Eqs. (208) and (209) represent two inequalities each.

(iii) Equations (100)–(125) form a particularly rich source of inequalities. From these we may derive that

$$|b_1 + c_1| \leq a_1 + a_2 \tag{210}$$

$$|b_2 - c_2| \leq a_1 + a_2 \tag{211}$$

$$|a_3 + a_4| \leq a_1 + a_2 \tag{212}$$

$$(a_3 + a_4)^2 + (b_2 - c_2)^2 \leq (a_1 + a_2)^2. \tag{213}$$

Thus, Eq. (206) is also implied by Eq. (100). Using Eqs. (100) and (203) we find

$$|a_3 + a_4| + |b_2 - c_2| \leq 2^{1/2}\{(a_1 + a_2)^2 - (b_1 + c_1)^2\}^{1/2} \leq 2^{1/2}(a_1 + a_2) \tag{214}$$

$$|a_3 + a_4| + |b_2 - c_2| + |b_1 + c_1| \leq 3^{1/2}(a_1 + a_2) \,. \tag{215}$$

By summing Eqs. (100)–(105) we find that

$$\sum_{k=1}^{15} |x_k| \leq a_1 \sqrt{45} \tag{216}$$

where $x_k \neq a_1$ denotes one of the elements of the scattering matrix.

(iv) According to Eq. (3) an incident wave with Stokes parameters $\{1, 0, 0, 0\}$ yields a scattered wave with Stokes parameters $\{a_1, c_1, c_3, c_5\}$. Hence, Eq. (204) gives

$$a_1^2 \geq c_1^2 + c_3^2 + c_5^2 \,. \tag{217}$$

If we apply Eq. (204) to other incident waves, alternative inequalities may be found.

2.6. *The simple case* $A_3 = A_4 = 0$

Hitherto we have considered a particle with arbitrary size, shape and composition. We will now consider the special case $A_3 = A_4 = 0$. This holds for all angles e.g. if the particle is small compared to the wavelength and the polarizability is isotropic or for a homogeneous sphere made of some optically inactive substance [see Van de Hulst, 1957]. From Eqs. (9)–(24) we find that the scattering matrix is of the type

$$\begin{pmatrix} a_1 & b_1 & 0 & 0 \\ b_1 & a_1 & 0 & 0 \\ 0 & 0 & a_3 & b_2 \\ 0 & 0 & -b_2 & a_3 \end{pmatrix}. \tag{218}$$

There are three independent parameters now. The entire collection of relations expressed in terms of the elements of the scattering matrix reduces to only one non-trivial equation, viz.

$$a_3^2 + b_2^2 = a_1^2 - b_1^2 \,. \tag{219}$$

Equation (192) now takes the simple form

$$I^2 - Q^2 - U^2 - V^2 = (a_1^2 - b_1^2)(I_0^2 - Q_0^2 - U_0^2 - V_0^2). \tag{220}$$

From the inequalities of the preceding subsection or directly from Eq. (219) we may derive some useful inequalities, such as

$$|b_1| \leq a_1 \,, \quad |b_2| \leq a_1 \,, \quad |a_3| \leq a_1 \tag{221}$$

$$\left.\begin{aligned} &|a_3| + |b_2| \leq 2^{1/2}(a_1^2 - b_1^2)^{1/2} \\ &|a_3| + |b_1| \leq 2^{1/2}(a_1^2 - b_2^2)^{1/2} \\ &|b_1| + |b_2| \leq 2^{1/2}(a_1^2 - a_3^2)^{1/2} \end{aligned}\right\} \tag{222}$$

$$|a_3| + |b_2| + |b_1| \leq 3^{1/2} a_1 \,. \tag{223}$$

3. Scattering by an assembly of particles

In this section we consider an assembly of many independently scattering particles, each of them characterized by an individual amplitude matrix. Since the scattered waves are essentially incoherent, we should not add the individual amplitude matrices. Instead, we have to add the Stokes parameters and therefore the elements of the scattering matrices of the individual particles to find the scattering matrix of the assembly. Hence, a formula of the same type as Eq. (3) can be used for the assembly with

$$a_1 = \sum_g a_1^g \tag{224}$$

where the upper index g pertains to an individual particle. Similar summations must be made for the other elements.

3.1. *Relations involving the elements of the scattering matrix*

It is clear that two particular elements of the scattering matrix which are equal for each particle are also equal in the scattering matrix of the assembly. Further, a matrix element vanishing for each particle also vanishes for the assembly.

The quadratic equations between the elements of the scattering matrix of a single particle are, generally, lost in the summation process. To study the implications of this fact in more detail we first consider [cf. Eq. (41)]

$$\begin{aligned} &\sum_g M_2^g \cdot \sum_g M_1^g - \{\sum_g S_{21}^g\}^2 - \{\sum_g D_{21}^g\}^2 \\ &= \sum_g (r_2^g)^2 \sum_g (r_1^g)^2 - \{\sum_g r_2^g r_1^g \cos(\rho_2^g - \rho_1^g)\}^2 - \{\sum_g r_2^g r_1^g \sin(\rho_2^g - \rho_1^g)\}^2 \\ &= \sum_g (r_2^g r_1^g)^2 + \sum_{g \neq h} (r_2^g)^2 (r_1^h)^2 - \sum_g \{r_2^g r_1^g \cos(\rho_2^g - \rho_1^g)\}^2 \\ &\quad - \sum_{g \neq h} \{r_2^g r_1^g \cos(\rho_2^g - \rho_1^g)\}\{r_2^h r_1^h \cos(\rho_2^h - \rho_1^h)\} \\ &\quad - \sum_g \{r_2^g r_1^g \sin(\rho_2^g - \rho_1^g)\}^2 - \sum_{g \neq h} \{r_2^g r_1^g \sin(\rho_2^g - \rho_1^g)\} \\ &\quad \times \{r_2^h r_1^h \sin(\rho_2^h - \rho_1^h)\} \\ &= \sum_{g \neq h} (r_2^g)^2 (r_1^h)^2 - \sum_{g \neq h} r_2^g r_1^g r_2^h r_1^h \cos\{(\rho_2^g - \rho_1^g) - (\rho_2^h - \rho_1^h)\} \end{aligned} \tag{225}$$

where $g, h = 1, 2, 3, \ldots$ each label the constituent particles of the assembly. The expression (225) reaches a minimum if

$$\rho_2^g - \rho_1^g = \rho_2^h - \rho_1^h \tag{226}$$

for all g and h. This minimum may be written as

$$\sum_{g<h} (r_2^g r_1^h - r_2^h r_1^g)^2, \tag{227}$$

which is positive unless

$$r_2^g / r_1^g = r_2^h / r_1^h \tag{228}$$

for all g and h. In this case, the minimum is zero. On using Eqs. (25), (26), (29) and (35) we can now conclude that for an assembly of particles

$$(a_3 + a_4)^2 + (b_2 - c_2)^2 \leq (a_1 + a_2)^2 - (b_1 + c_1)^2 \tag{229}$$

where the equality sign holds if and only if $(\rho_2 - \rho_1)$ and r_2/r_1 are the same for all particles. Similarly, we can prove that

$$(c_3 - c_4)^2 + (c_5 - c_6)^2 \leq (a_1 - b_1)^2 - (a_2 - c_1)^2 \tag{230}$$

$$(b_3 - b_4)^2 + (b_5 - b_6)^2 \leq (a_1 - c_1)^2 - (a_2 - b_1)^2 \tag{231}$$

$$(b_3 + b_4)^2 + (b_5 + b_6)^2 \leq (a_1 + c_1)^2 - (a_2 + b_1)^2 \tag{232}$$

$$(c_3 + c_4)^2 + (c_5 + c_6)^2 \leq (a_1 + b_1)^2 - (a_2 + c_1)^2 \tag{233}$$

$$(a_3 - a_4)^2 + (b_2 + c_2)^2 \leq (a_1 - a_2)^2 - (b_1 - c_1)^2 . \tag{234}$$

It is clear from the above discussion that for an assembly of particles with proportional amplitude matrices (with real or complex constants of proportionality) the inequalities (229)–(234) turn out to be equalities, as is the case for a single particle. This occurs, in particular, for an assembly of identical particles with the same orientation or for a cloud of identical spheres.

Using analogs of Eq. (225) one may show that the single particle equations (48)–(59), (61)–(72) and (73)–(80) generally turn into inequalities if one considers an assembly of particles, and that the same is true for Eqs. (106)–(125). However, it is impossible to state in a general rule whether the equality sign in a particular equation should be replaced by $\geq$ or by $\leq$, since this depends on the amplitudes and phases involved. The equality signs are preserved under the same general conditions as mentioned above, but also in some other accidental cases, when the terms relating to the individual particles balance each other.

3.2. *Inequalities: The general case*

For an assembly of particles we have the following possibilities of generating inequalities:

(i) Using inequalities for the individual particles and performing summations such as in Eq. (224).

(ii) Using Eqs. (229)–(234) and their implications.

(iii) Using the inequality (204) for the scattered light with several beams of incident light.

We refrain from giving a multitude of examples, but wish to point out a few particularly useful inequalities. By summation and using the triangle inequality we obtain Eq. (205) [cf. Eq. (205) for a single particle and Eq. (224)], while

$$|b_1| \leq a_1 \tag{235}$$

and similarly for the other elements of the scattering matrix. Hence, *no element of the scattering matrix of an assembly of particles has an absolute value exceeding* a_1. Using Eqs. (206)–(209) and summing over constituent particles yield

$$|b_1 + c_1| \leq a_1 + a_2 \tag{236}$$

and similar expressions. From Eqs. (203) and (229) we obtain directly Eqs. (214) and (215) for the assembly. Applying the inequality (204) gives Eq. (217) and related inequalities for the assembly.

3.3. *Inequalities: A special case*

Usually the scattering matrix of an assembly of particles contains less than 16 different elements, since symmetry properties often reduce this number and cause various elements to vanish (see Van de Hulst, 1957). A fairly general type of scattering matrix which is often considered in radiative transfer studies has the form

$$\begin{pmatrix} a_1 & b_1 & 0 & 0 \\ b_1 & a_2 & 0 & 0 \\ 0 & 0 & a_3 & b_2 \\ 0 & 0 & -b_2 & a_4 \end{pmatrix}. \tag{237}$$

This is valid for an assembly consisting of:

(i) randomly oriented particles having a plane of symmetry, like ellipsoids, or

(ii) particles and their mirror particles in equal numbers and with random orientation.

Equations (229)–(234) now simplify to the four inequalities

$$(a_3 + a_4)^2 + 4b_2^2 \leq (a_1 + a_2)^2 - 4b_1^2 \tag{238}$$

$$|a_3 - a_4| \leq |a_1 - a_2| \tag{239}$$

$$|a_2 - b_1| \leq |a_1 - b_1| \tag{240}$$

$$|a_2 + b_1| \leq |a_1 + b_1| . \tag{241}$$

From these or the more general relations of the preceding subsection we obtain

$$|a_2| \leq a_1, \quad |a_3| \leq a_1, \quad |a_4| \leq a_1, \quad |b_1| \leq a_1, \quad |b_2| \leq a_1 \tag{242}$$

$$|b_1| \leq \tfrac{1}{2}(a_1 + a_2) \tag{243}$$

$$|a_3 + a_4| + 2|b_2| \leq 2^{1/2}\{(a_1 + a_2)^2 - 4b_1^2\}^{1/2} \leq 2^{1/2}(a_1 + a_2) \tag{244}$$

$$|a_3 + a_4| + 2|b_1| \leq 2^{1/2}\{(a_1 + a_2)^2 - 4b_2^2\}^{1/2} \leq 2^{1/2}(a_1 + a_2) \tag{245}$$

$$2(|b_2| + |b_1|) \leq 2^{1/2}\{(a_1 + a_2)^2 - (a_3 + a_4)^2\}^{1/2} \leq 2^{1/2}(a_1 + a_2) \tag{246}$$

$$|a_3 + a_4| + 2|b_1| + 2|b_2| \leq 3^{1/2}(a_1 + a_2) . \tag{247}$$

Summing Eq. (238) and the squares of Eqs. (239)–(241) we get

$$|a_2| + |a_3| + |a_4| + 2^{1/2}|b_1| + 2^{1/2}|b_2| \leq a_1\sqrt{15} \tag{248}$$

$$|a_2| + |a_3| + |a_4| + 2|b_1| + 2|b_2| \leq a_1\sqrt{21} . \tag{249}$$

On exploiting the inequality (204) we see that incident light with Stokes parameters $\{1, -1, 0, 0\}$ and $\{1, 1, 0, 0\}$ yields Eqs. (240) and (241), respectively. Taking the incident beams $\{1, 0, 1, 0\}$ and $\{1, 0, 0, 1\}$ we find

$$b_1^2 + b_2^2 + a_3^2 \leq a_1^2 \tag{250}$$

and

$$b_1^2 + b_2^2 + a_4^2 \leq a_1^2 . \quad (251)$$

The last two equations may also be derived from Eqs. (238)–(241), since we have

$$b_1^2 + b_2^2 + [\max(|a_3|,|a_4|)]^2 = b_1^2 + b_2^2 + [\tfrac{1}{2}|a_3 + a_4| + \tfrac{1}{2}|a_3 - a_4|]^2 \leq [\tfrac{1}{2}(a_1 + a_2)]^2 + [\tfrac{1}{2}(a_1 - a_2)]^2 + \tfrac{1}{2}|a_1 + a_2| \cdot |a_1 - a_2| = [\max(|a_1|,|a_2|)]^2 = a_1^2 . \quad (252)$$

Equations (250) and (251) yield

$$|b_1| + |b_2| \leq 2^{1/2}(a_1^2 - a_3^2)^{1/2} \leq a_1\sqrt{2} \quad (253)$$
$$|b_1| + |a_k| \leq 2^{1/2}(a_1^2 - b_2^2)^{1/2} \leq a_1\sqrt{2} \quad (254)$$
$$|b_2| + |a_k| \leq 2^{1/2}(a_1^2 - b_1^2)^{1/2} \leq a_1\sqrt{2} \quad (255)$$
$$|b_1| + |b_2| + |a_k| \leq a_1\sqrt{3} \quad (256)$$

where $k = 3, 4$.

The scattering matrix of a collection of optically inactive homogeneous spheres may be considered as a special case of the matrix (237). The same is true in the case of Rayleigh scattering with or without depolarization effects for an ensemble of randomly oriented particles.

The inequalities given by Kuščer and Ribarič (1959), Fry and Kattawar (1981) and Hovenier and Van der Mee (1983) are contained in the inequalities given above. Kuščer and Ribarič (1959) first exploited Eq. (204) to obtain inequalities for the elements of the scattering matrix. Equation (204), which is the requirement that the degree of polarization, p, of a beam of light satisfies $0 \leq p \leq 1$, leads to necessary conditions on the scattering matrix. For scattering matrices of the type (237) this simple criterion does not lead to sufficient conditions as exemplified by the matrix (237) having $a_1 = 8$, $a_2 = 6$, $a_3 = 4$, $a_4 = 0$, $b_1 = 2\sqrt{6}$, $b_2 = 0$. It is readily verified that this matrix always transforms Stokes vectors satisfying Eq. (204) into vectors of the same type but does not obey Eq. (239).

When light scattered by a single particle is observed, an integration over a (small) solid angle is performed. Since the amplitude matrix will, in general, not be constant over this solid angle, it is clear from Eq. (225) that strict equalities will not be found. Instead one will find the same type of inequalities as for an assembly of particles, as noted by Fry and Kattawar (1981). We observe that the same remark applies to a spread in wavelength.

In summary, we have given a fairly complete description of equalities and inequalities valid for the scattering matrix for a single particle as well as for an assembly of particles. In doing so we have systematically developed a minimal but complete set of equations for the elements of the scattering matrix of a single particle or a collection of identical particles with the same orientation. We have also presented and used a variety of sources for obtaining inequalities for the elements of the scattering matrix both of a single particle and of an assembly of particles.

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References

Abhyankar, K.D., Fymat, A.L.: 1969, *J. Math. Phys.* **10**, 1935
Arfken, G.: 1970, Mathematical Methods for Physicists, Academic Press, New York
Chandrasekhar, S.: 1950, Radiative Transfer, Oxford University Press, London and New York; also Dover, New York, 1960
Fry, E.S., Kattawar, G.W.: 1981, *Appl. Optics* **20**, 2811
Hardy, G.H., Littlewood, J.E., Pólya, G.: 1934, Inequalities, Cambridge University Press, Cambridge
Hovenier, J.W., van der Mee, C.V.M.: 1983, *Astron. Astrophys.* **128**, 1
van de Hulst, H.C.: 1957, Light Scattering by Small Particles, Wiley, New York; also Dover, New York, 1981
Kuščer, I., Ribarič, M.: 1959, *Optica Acta* **6**, 42
Perrin, F.: 1942, *J. Chem. Phys.* **10**, 415
Perrin, F., Abragam, A.: 1951, *J. Phys. Rad.* **12**, 69