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COMMUNICATIONS FROM THE OBSERVATORY AT LEIDEN.

On the random motion of material particles in the expanding universe. Explanation of a paradox, by *G. Lemaître*.

In *B. A. N.* 193 Professor DE SITTER pointed out a contradiction between two results for the average motions of material particles, or extragalactic nebulae, in the expanding universe; the integration of the equations of motion of the inertial field (i. e. the equations of the geodesic) giving the result that the velocity is inversely proportional to the radius of the universe, whilst from the integration of the equation of energy it is found that the ratio between the pressure and the material density, (which is proportional to the *square* of the velocity of random motion) is inversely proportional to the radius. The first result is entirely independent of the integration of the field equations and the equation of energy, the only assumption involved being that the line-element is

$$(1) \qquad ds^2 = -R^2\, d\sigma^2 + dt^2,$$

R being a function of t only. Geodesics are straight lines described with a velocity which can be directly calculated from a variation of σ, giving

$$ds \,.\, \delta ds = -R^2\, d\sigma \,.\, \delta d\sigma$$

and

$$\delta \int ds = -\int R^2 \frac{d\sigma}{ds} \,.\, \delta d\sigma = \int d\left(R^2 \frac{d\sigma}{ds}\right) .\, \delta\sigma = 0.$$

Therefore

$$(2) \qquad \frac{d\sigma}{ds} = \frac{\varphi_0}{R^2},$$

where φ_0 is a constant.

Writing $u^\mu = dx^\mu / ds$, the material energy tensor is

$$(3) \qquad T^{\mu\nu} = \rho_0\, u^\mu u^\nu.$$

We have therefore

$$3p = -\sum_1^3 T_i^{\,i} = -\sum_1^3 g_{i\varkappa}\, T^{i\varkappa} = -\rho_0 \sum_1^3 g_{i\varkappa}\, u^i u^\varkappa = \\ = \rho_0 R^2 \left(\frac{d\sigma}{ds}\right)^2,$$

and by (2)

$$(4) \qquad \frac{3p}{\rho_0} = \frac{\varphi_0^{\,2}}{R^2}.$$

The second result is a consequence of the assumption introduced in the integration of the equation of energy, viz:

$$(5) \qquad \varkappa \rho_0 R^3 = \alpha = \text{constant},$$

from which follows

$$(6) \qquad \varkappa p R^4 = \beta = \text{constant},$$

and consequently

$$(7) \qquad \frac{p}{\rho_0} = \frac{\beta}{\alpha R}.$$

This assumption is intended to express that the total mass of the universe remains constant.

The explanation of the apparent contradiction between (4) and (7) must be sought in a discussion of this assumption.

The equations of conservation are by (3)

$$\sqrt{g}\; u_\mu\, T^{\mu\nu}{}_{|\nu} \equiv \frac{\partial}{\partial x^\nu} (\sqrt{g}\, \rho_0\, u^\nu) = 0.$$

Integrating over a tube of universe such that the particle never crosses the spatial cylindrical boundary, we have by Green's theorem, the integrals vanishing on the spatial boundary,

$$\int \frac{\partial}{\partial t} (\sqrt{g}\, \rho_0\, u^4)\, dx^1\, dx^2\, dx^3\, dt = 0.$$

Consequently

$$\frac{\partial}{\partial t} (\sqrt{g}\, \rho_0\, u^4) = 0,$$

which can be written

$$(8) \qquad \varkappa \rho_0 R^3 \frac{dt}{ds} = \alpha_1 = \text{constant}.$$