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**Effective mass and tricritical point for lattice fermions localized by a random mass**

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This is a numerical study of quasiparticle localization in symmetry class  $BD$  (realized, for example, in chiral  $p$ -wave superconductors), by means of a staggered-fermion lattice model for two-dimensional Dirac fermions with a random mass. For sufficiently weak disorder, the system size dependence of the average (thermal) conductivity  $\sigma$  is well described by an effective mass  $M_{\text{eff}}$ , dependent on the first two moments of the random mass  $M(r)$ . The effective mass vanishes linearly when the average mass  $\bar{M} \rightarrow 0$ , reproducing the known insulator-insulator phase boundary with a scale invariant dimensionless conductivity  $\sigma_c = 1/\pi$  and critical exponent  $\nu = 1$ . For strong disorder a transition to a metallic phase appears, with larger  $\sigma_c$  but the same  $\nu$ . The intersection of the metal-insulator and insulator-insulator phase boundaries is identified as a *repulsive* tricritical point.

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**I. INTRODUCTION**

Superconductors with neither time-reversal symmetry nor spin-rotation symmetry (for example, having chiral  $p$ -wave pairing) still retain one fundamental symmetry: the charge-conjugation (or particle-hole) symmetry of the quasiparticle excitations. Because of this symmetry, quasiparticle localization in a disordered chiral  $p$ -wave superconductor is in a different universality class than in a normal metal.<sup>1</sup> The difference is particularly interesting in two dimensions, when the quantum Hall effect governs the transport properties. The electrical quantum Hall effect in a normal metal has the thermal quantum Hall effect as a superconducting analog,<sup>2-4</sup> with different scaling properties because of the particle-hole symmetry.

The thermal quantum Hall transition is analogous to the electrical quantum Hall transition at the center of a Landau level but the scaling of the thermal conductivity  $\sigma$  near the phase boundary is different from the scaling of the electrical conductivity because of the particle-hole symmetry. A further difference between these two problems appear if the superconducting order parameter contains vortices.<sup>2,5,6</sup> A vortex contains a Majorana bound state at zero excitation energy in the weak-pairing regime.<sup>7,8</sup> A sufficiently large density of Majorana bound states allows for extended states at the Fermi level, with a thermal conductivity increasing  $\propto \ln L$  with increasing system size  $L$ .<sup>3</sup> This so-called thermal metal has no counterpart in the electronic quantum Hall effect.

The Bogoliubov-De Gennes Hamiltonian of a disordered chiral  $p$ -wave superconductor can be approximated at low energies by a Dirac Hamiltonian with a random mass (see Sec. II). For that reason, it is convenient to parameterize the phase diagram in terms of the average mass  $\bar{M}$  and the fluctuation strength  $\delta M$ . As indicated in Fig. 1, there are two types of phase transitions,<sup>10,11</sup> a metal-insulator (M-I) transition upon decreasing  $\delta M$  at constant  $\bar{M}$  and an insulator-insulator (I-I) transition upon decreasing  $\bar{M}$  through zero at constant (not too large)  $\delta M$ . The I-I transition separates phases with a different value of the thermal Hall conductance while the M-I transition separates the thermal metal from the

thermal insulator. Only the I-I transition remains if there are no vortices, or more generally, if there are no Majorana bound states.<sup>2,5,6</sup> In the nomenclature of Ref. 5, the symmetry class is called  $BD$  with Majorana bound states and  $D$  without.

The *primary purpose* of our paper is to investigate, by numerical simulation, to what extent the scale dependence of localization by a random mass can be described in terms of an effective nonfluctuating mass:  $\sigma(L, \bar{M}, \delta M) = \sigma(L, M_{\text{eff}}, 0)$ , for some function  $M_{\text{eff}}(\bar{M}, \delta M)$ . Because there is no other length scale in the problem at zero energy,  $\sigma(L, M_{\text{eff}}, 0)$  can only depend on  $L$  and  $M_{\text{eff}}$  through the dimensionless combination  $LM_{\text{eff}}v/\hbar \equiv L/\xi$ . The effective-mass hypothesis thus implies one-parameter scaling:  $\sigma(L, \bar{M}, \delta M) = \sigma_0(L/\xi)$ . Two further implications concern the critical conductivity  $\sigma_c$  (which is the scale invariant value of  $\sigma$  on the phase boundary  $\bar{M}=0$ ) and the critical exponent  $\nu$  (governing the divergence of the localization length  $\xi \propto \bar{M}^{-\nu}$ ).

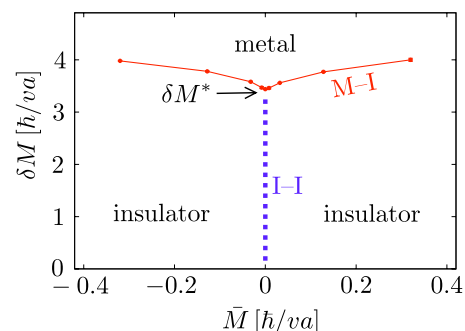


FIG. 1. (Color online) Phase diagram in symmetry class  $BD$ , calculated numerically from the lattice model of staggered fermions described in Sec. III. [A qualitatively similar phase diagram was calculated for a different model (Ref. 9) in Refs. 10 and 11]. The thermal conductivity decays exponentially  $\propto e^{-L/\xi}$  in the localized phase and increases  $\propto \ln L$  in the metallic phase. The thermal conductivity is scale invariant on the M-I phase boundary (solid line), as well as on the I-I phase boundary (dashed line). The M-I and I-I phase boundaries meet at the tricritical point  $\delta M^*$ .

Both  $\sigma_c$  and  $\nu$  follow directly from the effective-mass hypothesis. By construction, the scaling function  $\sigma_0$  is the conductivity of ballistic massless Dirac fermions, which has been calculated in the context of graphene. For a system with dimensions  $L \times W$ , and periodic boundary conditions in the transverse direction, it is given by<sup>12,13</sup>

$$\begin{aligned} \sigma_0(L/\xi) &= G_0 \frac{L}{W} \sum_{n=-\infty}^{\infty} \cosh^{-2} \sqrt{(2\pi n L/W)^2 + (L/\xi)^2} \\ &\xrightarrow{W \gg L} G_0 \frac{1}{\pi} \int_0^{\infty} dq \cosh^{-2} \sqrt{q^2 + (L/\xi)^2}. \end{aligned} \quad (1.1)$$

A scale invariant conductivity

$$\lim_{\xi \rightarrow \infty} \sigma_0(L/\xi) \equiv \sigma_c = G_0 \frac{L}{W} \sum_{n=-\infty}^{\infty} \cosh^{-2}(2\pi n L/W) \quad (1.2)$$

is reached for vanishing effective mass. In the limit of a large aspect ratio  $W/L \gg 1$  we recover the known value  $\sigma_c = G_0/\pi$  of the critical conductivity for a random mass with zero average.<sup>14</sup> The critical exponent  $\nu=1$  follows by comparing the expansion of the conductivity

$$\sigma(L, \bar{M}, \delta M) = \sigma_c + [L^{1/\nu} \bar{M} f(\delta M)]^2 + \mathcal{O}(\bar{M})^4 \quad (1.3)$$

in (even) powers of  $\bar{M}$  with the expansion of the scaling function [Eq. (1.1)] in powers of  $L$ . This value for  $\nu$  is aspect-ratio independent and agrees with the known result for the I-I transition.<sup>1</sup>

The description in terms of an effective mass breaks down for strong disorder. We find that the scaling function at the M-I transition differs appreciably from  $\sigma_0$ , with an aspect-ratio independent critical conductivity  $\sigma_c \approx 0.4G_0$ . The critical exponent remains close to or equal to  $\nu=1$  (in disagreement with earlier numerical simulations<sup>11</sup>).

The *secondary purpose* of our paper is to establish the nature of the tricritical point  $\delta M^*$  at which the two insulating phases and the metallic phase meet. The existence of such a fixed point of the scaling flow is expected on the basis of general arguments<sup>5</sup> but whether it is a repulsive or attractive fixed point has been a matter of debate. From the scale dependence of  $\sigma$  near this tricritical point, we conclude that it is a *repulsive* fixed point (in the sense that  $\sigma$  scales with increasing  $L$  to larger values for  $\delta M > \delta M^*$  and to smaller values for  $\delta M < \delta M^*$ ). An *attractive* tricritical point had been suggested as a possible scenario,<sup>15,16</sup> in combination with a repulsive critical point at some  $\delta M^{**} < \delta M^*$ . Our numerics does not support this scenario.

The outline of this paper is as follows. In the next two sections we introduce the Dirac Hamiltonian for chiral  $p$ -wave superconductors and the lattice fermion model that we use to simulate quasiparticle localization in symmetry class  $BD$ . We only give a brief description, referring to the Appendix and Ref. 17 for a more detailed presentation of the model. The scaling of the thermal conductivity and the localization length near the insulator-insulator and metal-insulator transitions are considered separately in Secs. IV and V, re-

spectively. The tricritical point, at which the two phase boundaries meet, is studied in Sec. VI. We conclude in Sec. VII.

## II. CHIRAL $p$ -WAVE SUPERCONDUCTORS

The quasiparticles in a superconductor have electron and hole components  $\psi_e, \psi_h$  that are eigenstates, at excitation energy  $\varepsilon$ , of the Bogoliubov-De Gennes equation

$$\begin{pmatrix} H_0 - E_F & \Delta \\ \Delta^\dagger & -H_0^* + E_F \end{pmatrix} \begin{pmatrix} \psi_e \\ \psi_h \end{pmatrix} = \varepsilon \begin{pmatrix} \psi_e \\ \psi_h \end{pmatrix}. \quad (2.1)$$

In a chiral  $p$ -wave superconductor the order parameter  $\Delta = \frac{1}{2} \{ \chi(\mathbf{r}), p_x - ip_y \}$  depends linearly on the momentum  $\mathbf{p} = -i\hbar \partial / \partial \mathbf{r}$ , so the quadratic terms in the single-particle Hamiltonian  $H_0 = p^2/2m + U(\mathbf{r})$  may be neglected near  $p=0$ .

For a uniform order parameter  $\chi(\mathbf{r}) = \chi_0$ , the quasiparticles are eigenstates of the Dirac Hamiltonian

$$H_{\text{Dirac}} = v(p_x \sigma_x + p_y \sigma_y) + v^2 M(\mathbf{r}) \sigma_z \quad (2.2)$$

with velocity  $v = \chi_0$  and mass  $M = (U - E_F)/\chi_0^2$  (distinct from the electron mass  $m$ ). The Pauli matrices are

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (2.3)$$

The particle-hole symmetry for the Dirac Hamiltonian is expressed by

$$\sigma_x H_{\text{Dirac}}^* \sigma_x = -H_{\text{Dirac}}. \quad (2.4)$$

Randomness in the electrostatic potential  $U(\mathbf{r})$  translates into randomness in the mass  $M(\mathbf{r}) = \bar{M} + \delta M(\mathbf{r})$  of the Dirac fermions. The sign of the average mass  $\bar{M}$  determines the thermal Hall conductance,<sup>2-4</sup> which is zero for  $\bar{M} > 0$  (strong pairing regime) and quantized at  $G_0 = \pi^2 k_B^2 T / 6h$  for  $\bar{M} < 0$  (weak-pairing regime).

The Dirac Hamiltonian [Eq. (2.2)] provides a generic low-energy description of the various realizations of chiral  $p$ -wave superconductors proposed in the literature: strontium ruthenate,<sup>18</sup> superfluids of fermionic cold atoms,<sup>19,20</sup> and ferromagnet-semiconductor-superconductor heterostructures.<sup>21-23</sup> What these diverse systems have in common is that they have superconducting order with neither time-reversal nor spin-rotation symmetry. Each of these systems is expected to exhibit the thermal quantum Hall effect, described by the phase diagram studied in this work.

## III. STAGGERED FERMION MODEL

Earlier numerical investigations<sup>10,11,15,16</sup> of the class  $BD$  phase diagram were based on the Cho-Fisher network model.<sup>9</sup> Here we use a staggered-fermion model in the same symmetry class, originally developed in the context of lattice gauge theory<sup>24,25</sup> and recently adapted to the study of transport properties in graphene.<sup>17</sup> An attractive feature of the lattice model is that, by construction, it reduces to the Dirac

Hamiltonian on length scales large compared to the lattice constant  $a$ .

The model is defined on a square lattice in a strip geometry, extending in the longitudinal direction from  $x=0$  to  $x=L=N_x a$  and in the transverse direction from  $y=0$  to  $y=W=N_y a$ . We use periodic boundary conditions in the transverse direction. The transfer matrix  $T$  from  $x=0$  to  $x=L$  is derived in Ref. 17, and we refer to that paper and to the Appendix for explicit formulas.

The dispersion relation of the staggered fermions,

$$\tan^2(k_x a/2) + \tan^2(k_y a/2) + \left(\frac{Mav}{2\hbar}\right)^2 = \left(\frac{\varepsilon a}{2\hbar v}\right)^2 \quad (3.1)$$

has a Dirac cone at wave vectors  $|\mathbf{k}|a \ll 1$  which is gapped by a nonzero mass. Staggered fermions differ from Dirac fermions by the pole at the edge of Brillouin zone ( $|k_x| \rightarrow \pi/a$  or  $|k_y| \rightarrow \pi/a$ ), which is insensitive to the presence of a mass. We do not expect these large-wave-number modes to affect the large-length scaling of the conductivity because they preserve the electron-hole symmetry.

The energy is fixed at  $\varepsilon=0$  (corresponding to the Fermi level for the superconducting quasiparticles). The transfer matrix  $T$  is calculated recursively using a stable QR decomposition algorithm.<sup>26</sup> An alternative stabilization method<sup>17</sup> is used to recursively calculate the transmission matrix  $t$ . Both algorithms give consistent results but the calculation of  $T$  is more accurate than that of  $t$  because it preserves the electron-hole symmetry irrespective of round-off errors.

The random mass is introduced by randomly choosing values of  $M$  on each site uniformly in the interval  $(\bar{M} - \delta M, \bar{M} + \delta M)$ . Variations in  $M(\mathbf{r})$  on the scale of the lattice constant introduce Majorana bound states, which place the model in the  $BD$  symmetry class.<sup>27</sup> In principle, it is possible to study also the class  $D$  phase diagram (without Majorana bound states), by choosing a random mass landscape that is smooth on the scale of  $a$ . Such a study was recently performed,<sup>28</sup> using a different model,<sup>29</sup> to demonstrate the absence of the M-I transition in class  $D$ .<sup>2,5,6</sup> Since here we wish to study both the I-I and M-I transitions, we do not take a smooth mass landscape.

#### IV. SCALING NEAR THE INSULATOR-INSULATOR TRANSITION

##### A. Scaling of the conductivity

In Fig. 2 we show the average (thermal) conductivity  $\sigma = (L/W)\langle \text{Tr } tt^\dagger \rangle$  (averaged over some  $10^3$  disorder realizations) as a function of  $L$  for a fixed  $\delta M$  in the localized phase. Data sets with different  $\bar{M}$  collapse on a single curve upon rescaling with  $\xi$ . (In the logarithmic plot this rescaling amounts simply to a horizontal displacement of the entire data set.) The scaling curve (solid line in Fig. 2) is the effective-mass conductivity [Eq. (1.1)], with  $M_{\text{eff}} = \hbar/v\xi$ . Figure 3 shows the linear scaling of  $\sigma$  with  $(\bar{M}L)^2$  for small  $\bar{M}$ , as expected from Eq. (1.3) with  $\nu=1$ .

We have studied the aspect-ratio dependence of the critical conductivity  $\sigma_c$ . As illustrated in Fig. 4, the convergence

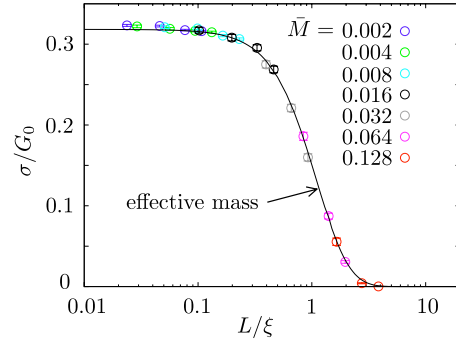


FIG. 2. (Color online) Average conductivity  $\sigma$  (with error bars indicating the statistical uncertainty) at fixed disorder strength  $\delta M = 2.5\hbar/va$ , as a function of system size  $L$ . The aspect ratio of the disordered strip is fixed at  $W/L=5$ . Data sets at different values of  $\bar{M}$  (listed in the figure in units of  $\hbar/va$ ) collapse upon rescaling by  $\xi$  onto a single curve (solid line), given by Eq. (1.1) in terms of an effective mass  $M_{\text{eff}} = \hbar/v\xi$ .

for  $W/L \rightarrow \infty$  is to the value  $\sigma_c = 1/\pi$  expected from Eq. (1.1). The conductivity of ballistic massless Dirac fermions also has an aspect-ratio dependence,<sup>13</sup> given by Eq. (1.2) (for periodic boundary conditions). The comparison in Fig. 4 of  $\sigma_c$  with Eq. (1.2) shows that  $\sigma_c$  at the I-I transition follows quite closely this aspect-ratio dependence (unlike at the M-I transition discussed in Sec. V A).

##### B. Scaling of the Lyapunov exponent

The transfer matrix  $T$  provides an independent probe of the critical scaling through the Lyapunov exponents. The transfer-matrix product  $TT^\dagger$  has eigenvalues  $e^{\pm\mu_n}$  with  $0 \leq \mu_1 \leq \mu_2 \leq \dots$ . The  $n$ th Lyapunov exponent  $\alpha_n$  is defined by

$$\alpha_n = \lim_{L \rightarrow \infty} \frac{\mu_n}{L}. \quad (4.1)$$

The dimensionless product  $W\alpha_1 \equiv \Lambda$  is the inverse of the MacKinnon-Kramer parameter.<sup>30</sup> We obtain  $\alpha_1$  by increasing  $L$  at constant  $W$  until convergence is reached (typically for  $L/W \approx 10^3$ ). The large- $L$  limit is self-averaging but some improvement in statistical accuracy is reached by averaging

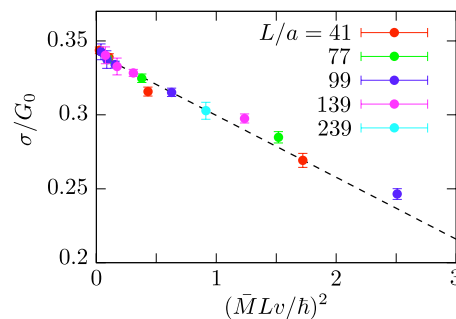


FIG. 3. (Color online) Plot of the average conductivity  $\sigma$  versus  $(\bar{M}L)^2$ , for fixed  $\delta M = 2.5\hbar/va$  and  $W/L=3$ . The dashed line is a least-square fit through the data, consistent with critical exponent  $\nu=1$ .

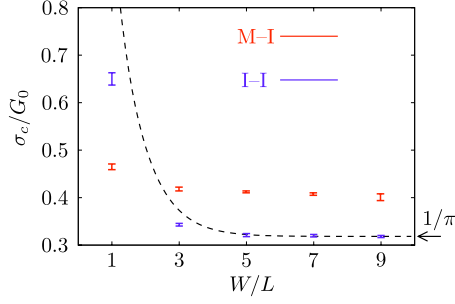


FIG. 4. (Color online) Dependence on the aspect ratio  $W/L$  of the critical conductivity at the I-I transition ( $\bar{M}=0$ ,  $\delta M=2.5\hbar/va$ ) and at the M-I transition ( $\bar{M}=0.032\hbar/va$ ,  $\delta M$  tuned to the transition). The dashed curve is the aspect-ratio dependence of the conductivity of ballistic massless Dirac fermions [Eq. (1.2)]. It describes the I-I transition quite well but not the M-I transition.

over a small number (10–20) of disorder realizations.

We seek the coefficients in the scaling expansion

$$\Lambda = \Lambda_c + c_1 W^{1/\nu} (\bar{M} - M_c) + \mathcal{O}(\bar{M} - M_c)^2, \quad (4.2)$$

for fixed  $\delta M$ . The fit in Fig. 5 gives  $\Lambda_c=0.03$ ,  $\nu=1.05$ , and  $M_c=7 \cdot 10^{-4}$  consistent with the expected values<sup>10</sup>  $\Lambda_c=0$ ,  $\nu=1$ , and  $M_c=0$ .

## V. SCALING NEAR THE METAL-INSULATOR TRANSITION

### A. Scaling of the conductivity

To investigate the scaling near the metal-insulator transition, we increase  $\delta M$  at constant  $\bar{M}$ . Results for the conductivity are shown in Fig. 6. In the metallic regime  $\delta M > \delta M_c$  the conductivity increases logarithmically with system size  $L$ , in accord with the theoretical prediction<sup>1,3</sup>

$$\sigma/G_0 = \frac{1}{\pi} \ln L + \text{const.} \quad (5.1)$$

(See the dashed line in Fig. 6, upper panel.)

In the insulating regime  $\delta M < \delta M_c$  the conductivity decays exponentially with system size while it is scale indepen-

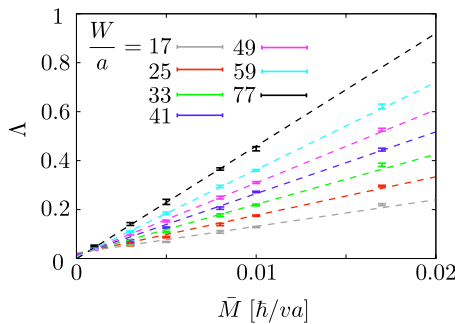


FIG. 5. (Color online) Plot of  $\Lambda = W\alpha_1$  (with  $\alpha_1$  the first Lyapunov exponent) as a function of  $\bar{M}$  near the insulator-insulator transition, for fixed  $\delta M=2.5\hbar/va$  and different values of  $W$ . The dashed lines are a fit to Eq. (4.2).

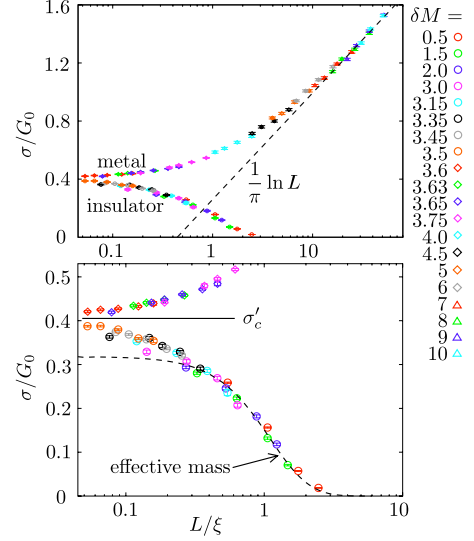


FIG. 6. (Color online) Average conductivity  $\sigma$  at fixed average mass  $\bar{M}=0.032\hbar/va$ , as a function of system size  $L$ . (The two panels show the same data on a different scale.) The aspect ratio of the disordered strip is fixed at  $W/L=5$ . Data sets at different values of  $\delta M$  (listed in the figure in units of  $\hbar/va$ ) collapse upon rescaling by  $\xi$  onto a pair of curves in the metallic and insulating regimes. The metal-insulator transition has a scale invariant conductivity  $\sigma'_c$ , larger than the value  $G_0/\pi$  which follows from the effective-mass scaling (dashed curve in the lower panel). The upper panel shows that the conductivity in the metallic regime follows the logarithmic scaling [Eq. (5.1)].

dent at the critical point  $\delta M = \delta M_c$ . Data sets for different  $\delta M$  collapse onto a single function of  $L/\xi$  but this function is different from the effective-mass scaling  $\sigma_0(L/\xi)$  of Eq. (1.1). (See the dashed curve in Fig. 6, lower panel.) This indicates that the effective-mass description, which applies well near the insulator-insulator transition, breaks down at large disorder strengths near the metal-insulator transition. The two transitions therefore have a different scaling behavior and can have different values of critical conductivity and critical exponent (which we denote by  $\sigma'_c$  and  $\nu'$ ).

Indeed, the critical conductivity  $\sigma'_c=0.41G_0$  is significantly larger than the ballistic value  $G_0/\pi=0.32G_0$ . Unlike at the insulator-insulator transition, we found no strong aspect-ratio dependence in the value of  $\sigma'_c$  (red data points in Fig. 4). To obtain the critical exponent  $\nu'$  we follow Ref. 31 and fit the conductivity near the critical point including terms of second order in  $\delta M - \delta M_c$ ,

$$\sigma = \sigma'_c + c_1 L^{1/\nu'} [\delta M - \delta M_c + c_2 (\delta M - \delta M_c)^2] + c_3 L^{2/\nu'} (\delta M - \delta M_c)^2. \quad (5.2)$$

Results are shown in Fig. 7, with  $\nu' = 1.02 \pm 0.06$ . The quality of the multiparameter fit is assured by a reduced chi-squared value close to unity ( $\chi^2=0.95$ ). Within error bars, this value of the critical exponent is the same as the value  $\nu=1$  for the insulator-insulator transition.



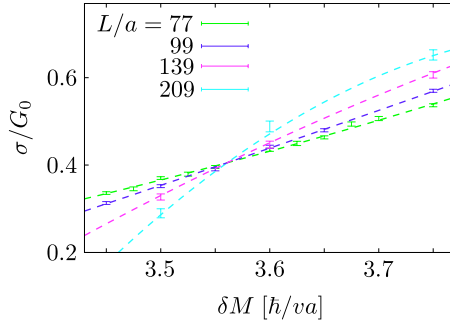


FIG. 7. (Color online) Plot of the average conductivity  $\sigma$  as a function of  $\delta M$  near the metal-insulator transition, for fixed  $\bar{M} = 0.032\hbar/va$ . The length  $L$  is varied at fixed aspect ratio  $W/L=3$ . The dashed curves are a fit to Eq. (5.2).

### B. Scaling of the Lyapunov exponent

As an independent measurement of  $\nu'$ , we have investigated the finite-size scaling of the first Lyapunov exponent. Results are shown in Fig. 8. Within the framework of single-parameter scaling, the value of  $\nu'$  should be the same for  $\sigma$  and  $\Lambda$  but the other coefficients in the scaling law may differ,

$$\Lambda = \Lambda_c + c'_1 L^{1/\nu'} [\delta M - \delta M'_c + c'_2 (\delta M - \delta M'_c)^2] + c'_3 L^{2/\nu'} (\delta M - \delta M'_c)^2. \quad (5.3)$$

Results are shown in Fig. 8, with  $\nu' = 1.06 \pm 0.05$ . The chi-squared value for this fit is relatively large,  $\chi^2 = 5.0$ , but the value of  $\nu'$  is consistent with that obtained from the conductivity (Fig. 7).

### VI. TRICRITICAL POINT

As indicated in the phase diagram of Fig. 1, the tricritical point at  $\bar{M}=0$ ,  $\delta M = \delta M^*$  is the point at which the insulating phases at the two sides of the I-I transition meet the metallic phase. We have searched for this tricritical point by calculating the scale dependence of the conductivity  $\sigma$  on the line  $\bar{M}=0$  for different  $\delta M$ . Results are shown in Fig. 9.

The calculated scale dependence is consistent with the identification of the point  $\delta M^* = 3.44 \hbar/va$  as a repulsive

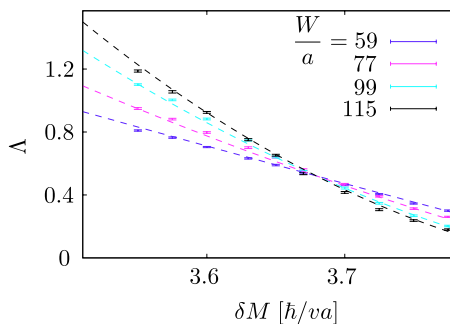


FIG. 8. (Color online) Plot of  $\Lambda = W\alpha_1$  (with  $\alpha_1$  the first Lyapunov exponent) as a function of  $\delta M$  near the metal-insulator transition, for fixed  $\bar{M} = 0.032\hbar/va$  and different values of  $W$ . The dashed curves are a fit to Eq. (5.3).

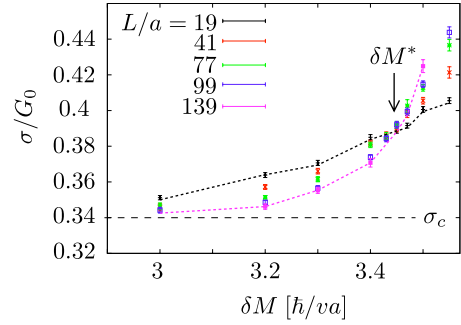


FIG. 9. (Color online) Conductivity  $\sigma$  as a function of  $\delta M$  on the critical line  $\bar{M}=0$ , for different values of  $L$  at fixed aspect ratio  $W/L=3$ . (The dotted lines through data points are guides to the eyes.) The tricritical point  $\delta M^*$  is indicated, as well as the scale invariant large- $L$  limit  $\sigma_c$  for  $\delta M < \delta M^*$ .

fixed point. The conductivity increases with increasing  $L$  for  $\delta M > \delta M^*$  while for  $\delta M < \delta M^*$  it decreases toward the scale invariant large- $L$  limit  $\sigma_c$ .

### VII. DISCUSSION

We have studied quasiparticle localization in symmetry class  $BD$ , by means of a lattice fermion model.<sup>17</sup> The thermal quantum Hall effect<sup>2-4</sup> in a chiral  $p$ -wave superconductor at weak disorder is in this universality class, as is the phase transition to a thermal metal<sup>3</sup> at strong disorder.

For weak disorder our lattice model can also be used to describe the localization of Dirac fermions in graphene with a random gap<sup>28,32,33</sup> (with  $\sigma$  the electrical, rather than thermal, conductivity and  $G_0 = 4e^2/h$  the electrical conductance quantum). The metallic phase at strong disorder requires Majorana bound states,<sup>2,5,6</sup> which do not exist in graphene (symmetry class  $D$  rather than  $BD$ ). We therefore expect the scaling analysis in Sec. IV at the I-I transition to be applicable to chiral  $p$ -wave superconductors as well as to graphene while the scaling analysis of Sec. V at the M-I transition applies only in the context of superconductivity. (Here we disagree with Refs. 32 and 33, which maintain that the M-I transition exists in graphene as well.)

Our lattice fermion model is different from the network model<sup>9</sup> used in previous investigations<sup>10,11,15,16</sup> but it falls in the same universality class so we expect the same critical conductivity and critical exponent. For the I-I transition analytical calculations<sup>1,14</sup> give  $\sigma_c = G_0/\pi$  and  $\nu = 1$ , in agreement with our numerics. There are no analytical results for the M-I transition. We find a slightly larger critical conductivity ( $\sigma'_c = 0.4G_0$ ), which has the qualitatively more significant consequence that the effective-mass scaling which we have demonstrated at the I-I transition breaks down at the M-I transition (compare Figs. 2 and 6, lower panel).

We conclude from our numerics that the critical exponents  $\nu$  at the I-I transition and  $\nu'$  at the M-I transition are both equal to unity within a 5% error margin, which is significantly smaller than the result  $\nu = \nu' = 1.4 \pm 0.2$  of an earlier numerical investigation<sup>11</sup> but close to the value found in later work by these authors.<sup>16</sup> The logarithmic scaling [Eq.

(5.1)] of the conductivity in the thermal metal phase, predicted analytically,<sup>1,3</sup> is nicely reproduced by our numerics (Fig. 6, upper panel).

The nature of the tricritical point has been much debated in the literature.<sup>15,16</sup> Our numerics indicates that this is a repulsive critical point (Fig. 9). This finding lends support to the simplest scaling flow along the I-I phase boundary,<sup>14</sup> toward the free-fermion fixed point at  $\bar{M}=0$  and  $\delta M=0$ .

In conclusion, we hope that this investigation brings us closer to a complete understanding of the phase diagram and scaling properties of the thermal quantum Hall effect. We now have two efficient numerical models in the  $BD$  universality class, the Cho-Fisher network model<sup>9</sup> studied previously and the lattice fermion model<sup>17</sup> studied here. There is a consensus on the scaling at weak disorder, although some disagreement on the scaling at strong disorder remains to be resolved.

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### APPENDIX: TRANSFER MATRIX FOR STAGGERED FERMIONS

To make this paper self-contained, we give the staggered-fermion transfer matrix derived in Refs. 17, 24, and 25. The values  $\Psi_{m,n}=\Psi(x_m,y_n)$  of the wave function at a lattice point are collected into a set of  $N_y$ -component vectors  $\Psi_m=(\Psi_{m,1},\Psi_{m,2},\dots,\Psi_{m,N_y})$ , one for each  $m=1,2,\dots,N_x$ . The  $N_y\times N_y$  transfer matrix  $\mathcal{T}_m$  is defined by

$$\Psi_{m+1}=\mathcal{T}_m\Psi_m. \quad (\text{A1})$$

The transfer matrix  $\mathcal{T}$  through the entire strip is then the product of the  $\mathcal{T}_m$ 's.

The differential operators in the Dirac Hamiltonian [Eq. (2.2)] are discretized by

$$\partial_x\Psi\rightarrow\frac{1}{2a}(\Psi_{m+1,n}+\Psi_{m+1,n+1}-\Psi_{m,n}-\Psi_{m,n+1}), \quad (\text{A2})$$

$$\partial_y\Psi\rightarrow\frac{1}{2a}(\Psi_{m,n+1}+\Psi_{m+1,n+1}-\Psi_{m,n}-\Psi_{m+1,n}), \quad (\text{A3})$$

and the mass term is replaced by

$$M\sigma_z\Psi\rightarrow\frac{1}{4}M_{m,n}\sigma_z(\Psi_{m+1,n}+\Psi_{m+1,n+1}+\Psi_{m,n}+\Psi_{m,n+1}) \quad (\text{A4})$$

with  $M_{m,n}=M(x_m+a/2,y_n+a/2)$ . The zero-energy Dirac equation  $H_{\text{Dirac}}\Psi=0$  is applied at the points  $(x_m+a/2,y_n)$  by averaging the terms at the two adjacent points  $(x_m$

$+a/2,y_n\pm a/2)$ . (This is the staggered lattice construction introduced by Kogut and Susskind to avoid the fermion doubling problem.<sup>34</sup>)

The resulting finite difference equation can be written in a compact form with the help of the  $N_y\times N_y$  tridiagonal matrices  $\mathcal{J}$ ,  $\mathcal{K}$ , and  $\mathcal{M}^{(m)}$ , defined by the following nonzero elements:

$$\mathcal{J}_{n,n}=1, \quad \mathcal{J}_{n,n+1}=\mathcal{J}_{n,n-1}=\frac{1}{2}, \quad (\text{A5})$$

$$\mathcal{K}_{n,n+1}=\frac{1}{2}, \quad \mathcal{K}_{n,n-1}=-\frac{1}{2}, \quad (\text{A6})$$

$$\begin{aligned} \mathcal{M}_{n,n}^{(m)} &= \frac{1}{2}(M_{m,n}+M_{m,n-1}), & \mathcal{M}_{n,n+1}^{(m)} &= \frac{1}{2}M_{m,n}, \\ \mathcal{M}_{n,n-1}^{(m)} &= \frac{1}{2}M_{m,n-1}. \end{aligned} \quad (\text{A7})$$

In accordance with the periodic boundary conditions in the transverse direction, the indices  $n\pm 1$  should be evaluated modulo  $N_y$ .

The discretized Dirac equation is expressed in terms of the matrices [Eqs. (A5)–(A7)] by

$$\begin{aligned} \frac{1}{2a}\mathcal{J}(\Psi_{m+1}-\Psi_m) &= \left(-\frac{i}{2a}\sigma_z\mathcal{K}-\frac{1}{4}v^2\sigma_y\mathcal{M}^{(m)}\right) \\ &\quad \times (\Psi_m+\Psi_{m+1}). \end{aligned} \quad (\text{A8})$$

Rearranging Eq. (A8) we arrive at Eq. (A1) with the transfer matrix

$$\begin{aligned} \mathcal{T}_m &= \left(\mathcal{J}+i\sigma_z\mathcal{K}+\frac{1}{2}v^2a\sigma_y\mathcal{M}^{(m)}\right)^{-1} \\ &\quad \times \left(\mathcal{J}-i\sigma_z\mathcal{K}-\frac{1}{2}v^2a\sigma_y\mathcal{M}^{(m)}\right). \end{aligned} \quad (\text{A9})$$

Particle-hole symmetry for the zero-energy-transfer matrix requires

$$\sigma_x\mathcal{T}_m^*\sigma_x=\mathcal{T}_m, \quad (\text{A10})$$

which is satisfied by Eq. (A9). Current conservation requires

$$\mathcal{T}_m^\dagger J_x \mathcal{T}_m = J_x, \quad (\text{A11})$$

which holds for the discretized current operator

$$J_x = \frac{1}{2}v\sigma_x\mathcal{J}. \quad (\text{A12})$$

For a uniform mass  $M_{mn}=M$ , we may calculate the eigenvalues  $e^{ik_x a}$  of  $\mathcal{T}_m$  analytically. This gives the dispersion relation

$$\tan^2(k_x a/2) + \tan^2(k_y a/2) + (Mav/2\hbar)^2 = 0 \quad (\text{A13})$$

with  $k_y=2\pi l/N_y$ ,  $l=1,2,\dots,N_y$ , in accord with Eq. (3.1) at zero energy.

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