

# Localization in a disordered multi-mode waveguide with absorption or amplification

T.Sh. Misirpashaev<sup>a,b,\*</sup>, J.C.J. Paasschens<sup>a,c</sup>, C.W.J. Beenakker<sup>a</sup>

<sup>a</sup> *Instituut-Lorentz, University of Leiden, P O Box 9506, 2300 RA Leiden, The Netherlands*

<sup>b</sup> *Landau Institute for Theoretical Physics, 2 Kosygin Street, Moscow 117334, Russia*

<sup>c</sup> *Philips Research Laboratories, 5656 AA Eindhoven The Netherlands*

Received 9 September 1996

---

## Abstract

An analytical and numerical study of transmission of radiation through a multi-mode waveguide containing a random medium with a complex dielectric constant  $\varepsilon = \varepsilon' + i\varepsilon''$  is presented. Depending on the sign of  $\varepsilon''$ , the medium is absorbing or amplifying. The transmitted intensity decays exponentially  $\propto \exp(-L/\xi)$  as the waveguide length  $L \rightarrow \infty$ , regardless of the sign of  $\varepsilon''$ . The localization length  $\xi$  is computed as a function of the mean free path  $l$ , the absorption or amplification length  $|\sigma|^{-1}$ , and the number of modes in the waveguide  $N$ . The method used is an extension of the Fokker–Planck approach of Dorokhov, Mello, Pereyra and Kumar to non-unitary scattering matrices. Asymptotically exact results are obtained for  $N \gg 1$  and  $|\sigma| \gg 1/N^2 l$ . An approximate interpolation formula for all  $\sigma$  agrees reasonably well with numerical simulations.

*PACS.* 78.45.+h, 42.25.Bs; 72.15.Rn; 78.20.Cj

---

## 1. Introduction

Localization of waves in one-dimensional random media has been studied extensively, both for optical and for electronic systems [1, 2]. An analytical solution for the case of weak disorder (mean free path  $l$  much greater than the wavelength  $\lambda$ ) was obtained as early as 1959 by Gertsenshtein and Vasil'ev [3]. The transmittance  $T$  (being the ratio of transmitted and incident intensity) has a log-normal distribution for large lengths  $L$  of the system, with a mean  $\langle \ln T \rangle = -L/\xi$  characterized by a localization length  $\xi$  equal to the mean free path.

---

\* Corresponding author. Fax: +31 71 527 5404, e-mail. [missu@lorentz.leidenuniv.nl](mailto:missu@lorentz.leidenuniv.nl)

This early work was concerned with the propagation of classical waves, and hence included also the effect of absorption. In the presence of absorption the transmittance decays faster, according to [4,5]  $\langle \ln T \rangle = (\sigma - l^{-1})L$ , where  $|\sigma|$  is the inverse absorption length ( $\sigma < 0$ ). Absorption is the result of a positive imaginary part  $\epsilon''$  of the (relative) dielectric constant  $\epsilon = \epsilon' + i\epsilon''$ . For a homogeneous  $\epsilon''$  one has

$$\sigma = -2k \operatorname{Im} \sqrt{1 + i\epsilon''} \approx -k\epsilon'' \quad \text{if } |\epsilon''| \ll 1, \quad (1.1)$$

where  $k$  is the wavenumber. A negative  $\epsilon''$  corresponds to amplification by stimulated emission of radiation, with inverse amplification length  $\sigma > 0$ . Propagation of waves through amplifying one-dimensional random media has been studied in [6–11]. In the limit  $L \rightarrow \infty$  amplification also leads to a faster decay of the transmittance, according to  $\langle \ln T \rangle = (-|\sigma| - l^{-1})L$  [8,9].

A natural extension of these studies is to waveguides which contain more than a single propagating mode. Localization in such “quasi-one-dimensional” systems has been studied on the basis of a scaling theory [12], a supersymmetric field theory [13], or a Fokker–Planck equation [14,15]. It is found that the localization length for  $N$  modes is enhanced by a factor of order  $N$  relative to the single-mode case. These investigations were concerned with quantum mechanical, rather than classical waves, and therefore did not include absorption. It is the purpose of the present paper to extend the Fokker–Planck approach of Dorokhov, Mello, Pereyia and Kumar [14,15] (DMPK) to include the effects on the transmittance of a non-zero imaginary part of the dielectric constant.

According to the general duality relation [9], the localization length is an even function of  $\sigma$  for any  $N$ ,

$$\xi(\sigma) = \xi(-\sigma) \quad (1.2)$$

It follows that both absorption and amplification lead to a faster decay of the transmittance for large  $L$ . For  $N \gg 1$  we find that, to a good approximation,

$$\frac{1}{\xi} = \frac{2}{(N+1)l} + (\sigma^2 + 2|\sigma|/l)^{1/2} \quad (1.3)$$

This result becomes exact in the two limits  $|\sigma| \gg 1/N^2 l$  and  $|\sigma| \ll 1/N^2 l$ . We compare with numerical solutions of the Helmholtz equation, and find reasonably good agreement over the whole range of  $\sigma$ .

The outline of the paper is as follows. In Section 2 we formulate the scattering problem and summarize the duality relation of [9]. In Section 3 we derive a Fokker–Planck equation for the transmission and reflection eigenvalues  $\mathcal{T}_n, \mathcal{R}_n, n = 1, 2, \dots, N$ . These are eigenvalues of the matrix products  $tt^\dagger$  and  $rr^\dagger$ , respectively, where  $t$  and  $r$  are the transmission and reflection matrices of the waveguide. For  $\sigma = 0$  the Fokker–Planck equation is the DMPK equation [14,15]. A reduced Fokker–Planck equation, containing only the  $\mathcal{R}_n$ 's, was previously obtained and studied in [16,17]. To obtain the localization length one needs to include also the  $\mathcal{T}_n$ 's, which are no longer related to the  $\mathcal{R}_n$ 's when  $\sigma \neq 0$ . We find that a closed Fokker–Planck equation containing

$\mathcal{R}_n$ 's and  $\mathcal{T}_n$ 's exists only for  $N=1$ . If  $N > 1$  there appears an additional set of “slow variables,” consisting of eigenvectors of  $rr^\dagger$  in a basis where  $tt^\dagger$  is diagonal. (These new variables do not appear when  $\sigma = 0$ , because then  $rr^\dagger$  and  $tt^\dagger$  commute.) Because of these additional relevant variables we have not been able to make as much progress in the solution of the Fokker–Planck equation for  $\sigma \neq 0$  as one can for  $\sigma = 0$  [18]. In Section 4 we show that a closed evolution equation for  $\langle \ln T \rangle$  can be obtained if  $|\sigma| \gg 1/N^2 l$ , which leads to the second term in Eq. (1.3). (This term could also have been obtained from the incoherent radiative transfer theory for  $\sigma < 0$ , but not for  $\sigma > 0$ .) To contrast the multi-mode and single-mode cases, we also briefly discuss in Section 4 the derivation of the localization length for  $N=1$ . (Our  $N=1$  results were given without derivation in [9].) Finally, in Section 5 we compare the analytical results for the multi-mode case with numerical simulations.

## 2. Formulation of the scattering problem

We consider a random medium of length  $L$  with a spatially fluctuating dielectric constant  $\varepsilon = \varepsilon' + i\varepsilon''$ , embedded in an  $N$ -mode waveguide with  $\varepsilon = 1$ . The scattering matrix  $S$  is a  $2N \times 2N$  matrix relating incoming and outgoing modes at some frequency  $\omega$ . It has the block structure

$$S = \begin{pmatrix} r' & t' \\ t & r \end{pmatrix}, \tag{2.1}$$

where  $t, t'$  are the transmission matrices and  $r, r'$  the reflection matrices. We introduce the sets of transmission and reflection eigenvalues  $\{\mathcal{T}_n\}, \{\mathcal{T}'_n\}, \{\mathcal{R}_n\}, \{\mathcal{R}'_n\}$ , being the eigenvalues of  $tt^\dagger, t't'^\dagger, rr^\dagger, r'r'^\dagger$ , respectively. Total transmittances and reflectances are defined by

$$T = N^{-1} \text{Tr } tt^\dagger, \quad R = N^{-1} \text{Tr } rr^\dagger, \tag{2.2a}$$

$$T' = N^{-1} \text{Tr } t't'^\dagger, \quad R' = N^{-1} \text{Tr } r'r'^\dagger \tag{2.2b}$$

Here  $T$  and  $R'$  are the transmitted and reflected intensity divided by the incident intensity from the left. Similarly,  $T'$  and  $R$  correspond to incident intensity from the right. By taking the trace in Eq. (2.2) we are assuming diffuse illumination, i.e. that the incident intensity is equally distributed over the  $N$  modes. Two systems which differ only in the sign of  $\varepsilon''(r)$  are called dual. Scattering matrices of dual systems are related by [9]

$$S(\varepsilon'')S^\dagger(-\varepsilon'') = 1 \tag{2.3}$$

This duality relation takes the place of the unitarity constraint when  $\varepsilon'' \neq 0$ .

An optical system usually possesses time-reversal symmetry, as a result of which  $S(\varepsilon'')S^*(-\varepsilon'') = 1$ . Combining this relation with Eq. (2.3), we find that  $S = S^T$  is a symmetric matrix. Hence  $\mathcal{T}_n = \mathcal{T}'_n$  and  $T = T'$ . (The reflectances  $R$  and  $R'$  may differ.) The case of broken time-reversal symmetry might also be physically relevant [19], and

is included here for completeness. In the absence of time-reversal symmetry  $S$  is an arbitrary complex matrix.

The duality relation (2.3) has consequences for the reflection and transmission eigenvalues of two dual systems [9]. If  $N = 1$  the relation

$$T(\varepsilon'')/R(\varepsilon'') = T'(-\varepsilon'')/R'(-\varepsilon'') \quad (2.4)$$

holds for all  $L$ . If  $N \geq 1$  we have two relations for  $L \rightarrow \infty$ ,

$$\lim_{L \rightarrow \infty} \mathcal{R}_n(\varepsilon'') = \lim_{L \rightarrow \infty} \mathcal{R}_n^{-1}(-\varepsilon''), \quad (2.5)$$

$$\lim_{L \rightarrow \infty} L^{-1} \ln \mathcal{T}_n(\varepsilon'') = \lim_{L \rightarrow \infty} L^{-1} \ln \mathcal{T}_n(-\varepsilon'') \quad (2.6)$$

The transmittance  $T = N^{-1} \sum_n \mathcal{T}_n$  is dominated by the largest transmission eigenvalue, hence

$$\lim_{L \rightarrow \infty} L^{-1} \ln T(\varepsilon'') = \lim_{L \rightarrow \infty} L^{-1} \ln T(-\varepsilon'') \quad (2.7)$$

In other words, two dual systems have the same localization length, as stated in Eq. (1.2).

### 3. Fokker–Planck equation

We derive a Fokker–Planck equation for the evolution of the distribution of scattering matrices with increasing length  $L$  of the waveguide. In the absence of gain or loss ( $\sigma = 0$ ), the evolution equation is known as the Dorokhov–Mello–Percy–Kumar (DMPK) equation [14,15]. Original derivations of this equation relied on the unitarity of the scattering matrix, making use of the invariant measure on the unitary group and the polar decomposition of a unitary matrix. These derivations cannot readily be generalized to the case  $\sigma \neq 0$ , in particular because the scattering matrix no longer admits a polar decomposition (This means that the matrix products  $v^\dagger$  and  $u^\dagger$  do not commute.) The alternative derivation of the DMPK equation [18] does not use the polar decomposition and is suitable for our purpose.

Without loss of generality we can write the transmission and reflection submatrices of the scattering matrix as follows,

$$S = \begin{pmatrix} r' & t' \\ t & r \end{pmatrix} = \begin{pmatrix} U\sqrt{\mathbf{R}'}W & U'\sqrt{\mathbf{T}'}Z \\ V\sqrt{\mathbf{T}}W' & -V'\sqrt{\mathbf{R}}Z' \end{pmatrix} \quad (3.1)$$

Here  $U, U', V, V', W, W', Z, Z'$  are  $N \times N$  unitary matrices, while  $\mathbf{R}, \mathbf{R}', \mathbf{T}, \mathbf{T}'$  are diagonal matrices whose elements are the reflection and transmission eigenvalues  $\{\mathcal{R}_n\}, \{\mathcal{R}'_n\}, \{\mathcal{T}_n\}, \{\mathcal{T}'_n\}$ . For  $\sigma = 0$ , the unitarity constraint  $SS^\dagger = 1$  implies  $U = U', V = V', W = W', Z = Z'$ , and  $\mathbf{R} = \mathbf{R}' = 1 - \mathbf{T} = 1 - \mathbf{T}'$ . Eq. (3.1) then constitutes the polar decomposition of the scattering matrix. In this case one can derive a Fokker–Planck equation for the evolution of only transmission or only reflection eigenvalues. If  $\sigma \neq 0$ ,

the Fokker-Planck equation contains both the transmission and reflection eigenvalues, as well as elements of the matrix  $Q = V^\dagger V'$  relating eigenvectors of  $tt^\dagger$  and  $rr^\dagger$ . The only constraint on the scattering matrix if  $\sigma \neq 0$  is imposed by time-reversal symmetry, which requires  $S = S^T$ , hence  $W = U^T$ ,  $Z = V^T$ ,  $W' = U'^T$ ,  $Z' = V'^T$ ,  $\mathbf{T} = \mathbf{T}'$ .

The Fokker-Planck equation describes the evolution of slow variables after the elimination of fast variables. In our problem fast variables vary on the scale of the wavelength  $\lambda$ , while slow variables vary on the scale of the mean free path  $l$  or the amplification length  $|\sigma|^{-1}$ . We assume that both  $l$  and  $|\sigma|^{-1}$  are much greater than  $\lambda$  (This requires  $|\epsilon''| \ll 1$ ). The slow variables include  $\{\mathcal{R}_n\}$ ,  $\{\mathcal{T}_n\}$  and elements of  $Q = V^\dagger V'$ . We denote this set of slow variables collectively by  $\{\Phi_n\}$ . Each  $\Phi_i$  is incremented by  $\delta\Phi_i$  if a thin slice of length  $\delta L$  ( $\lambda \ll \delta L \ll l$ ) is added to the waveguide of length  $L$ . The increments are of order  $(\delta L/l)^{1/2}$  and can be calculated perturbatively. We specify an appropriate statistical ensemble for the scattering matrix  $\delta S$  of the thin slice and compute moments of  $\delta\Phi_i$ . The first two moments are of order  $\delta L/l$ ,

$$\langle \delta\Phi_i \rangle = a_i \delta L/l + \mathcal{O}(\delta L/l)^{3/2}, \tag{3.2a}$$

$$\langle \delta\Phi_i \delta\Phi_j \rangle = a_{ij} \delta L/l + \mathcal{O}(\delta L/l)^{3/2} \tag{3.2b}$$

Higher moments have no term of order  $\delta L/l$ . According to the general theory of Brownian motion [20], the Fokker-Planck equation for the joint probability distribution  $P(\{\Phi_n\}, L)$  reads

$$l \frac{\partial P}{\partial L} = - \sum_i \frac{\partial}{\partial \Phi_i} a_i P + \frac{1}{2} \sum_{ij} \frac{\partial^2}{\partial \Phi_i \partial \Phi_j} a_{ij} P \tag{3.3}$$

The average  $\langle \ \rangle$  in Eq (3.2) is defined by the statistics of  $\delta S$ . We specify this statistics using simplifying features of the waveguide geometry (length  $\gg$  width), which justify the equivalent channel or isotropy approximation [15,21]. We assume that amplification or absorption in the thin slice is independent of the scattering channel. This entails the relation

$$\delta S \delta S^\dagger = 1 + \bar{\sigma} \delta L, \tag{3.4}$$

where  $\bar{\sigma}$  is a modal and spatial average of the inverse amplification length  $\sigma$ . If  $\epsilon''$  is spatially constant, one has

$$\bar{\sigma} = -\frac{2k}{N} \sum_{n=1}^N \text{Im} (1 - \omega_n^2/\omega^2 + i\epsilon'')^{1/2}, \tag{3.5}$$

where  $\omega_n$  is the cutoff frequency of mode  $n$ . For  $N \rightarrow \infty$ , the sum over modes can be replaced by an integral. The result depends on the dimensionality of the waveguide,

$$\bar{\sigma} = -2k\epsilon'' \quad \text{for a 3D waveguide,} \tag{3.6a}$$

$$\bar{\sigma} = -(\pi/2)k\epsilon'' \quad \text{for a 2D waveguide,} \tag{3.6b}$$

where we have used that  $|\epsilon''| \ll 1$ .

Eq (3.4) ensures the existence of a polar decomposition for  $\delta S$ ,

$$\delta S = \begin{pmatrix} U_0 \sqrt{\delta \mathbf{R}} W_0 & U_0 \sqrt{\delta \mathbf{T}} Z_0 \\ V_0 \sqrt{\delta \mathbf{T}} W_0 & -V_0 \sqrt{\delta \mathbf{R}} Z_0 \end{pmatrix}, \tag{3.7}$$

with  $\delta \mathbf{T} + \delta \mathbf{R} = 1 + \bar{\sigma} \delta L$ . Note that a polar decomposition for  $\delta S$  does not imply a polar decomposition for  $S$ , because the special block structure of Eq (3.7) is lost upon composition of scattering matrices. We make the isotropy assumption that the matrices  $U_0, V_0, W_0, Z_0$  are uniformly distributed in the unitary group. In the presence of time-reversal symmetry one has  $W_0 = U_0^T$  and  $Z_0 = V_0^T$ . In the absence of time-reversal symmetry all four unitary matrices are independent. The diagonal matrices  $\delta \mathbf{R}$  and  $\delta \mathbf{T}$  may have arbitrary distributions. We specify the first moments,

$$\langle \text{Tr } \delta \mathbf{R} \rangle = N \delta L / l, \quad \langle \text{Tr } \delta \mathbf{T} \rangle = N + N(\gamma - 1) \delta L / l, \tag{3.8}$$

where we have defined  $\gamma = \bar{\sigma} l$ . The mean free path  $l$  in Eq (3.8) is related to the mean free path  $l_{tr}$  of radiative transfer theory by [18]

$$l = (4/3) l_u \quad \text{for a 3D waveguide,} \tag{3.9a}$$

$$l = (\pi/2) l_u \quad \text{for a 2D waveguide} \tag{3.9b}$$

This completes the specification of the statistical ensemble for  $\delta S$ .

We need the increments  $\Delta \mathcal{R}_n, \Delta \mathcal{T}_n$  of reflection and transmission eigenvalues to first order in  $\delta L / l$ ,

$$\Delta \mathcal{R}_n = \Delta R_{nm}^{(1)} + \Delta R_{mm}^{(2)} + \sum_{m \neq n} \frac{\Delta R_{nm}^{(1)} \Delta R_{mn}^{(1)}}{\mathcal{R}_n - \mathcal{R}_m}, \tag{3.10a}$$

$$\Delta \mathcal{T}_n = \Delta T_{nm}^{(1)} + \Delta T_{mm}^{(2)} + \sum_{m \neq n} \frac{\Delta T_{nm}^{(1)} \Delta T_{mn}^{(1)}}{\mathcal{T}_n - \mathcal{T}_m} \tag{3.10b}$$

The matrices of perturbation  $\Delta R^{(1)}, \Delta R^{(2)}, \Delta T^{(1)}, \Delta T^{(2)}$  are expressed through unitary matrices  $Q = V^\dagger V'$ ,  $\tilde{U} = Z' U_0$ ,  $\tilde{W} = W_0 V'$  and diagonal matrices  $\mathbf{T}, \mathbf{R}, \delta \mathbf{T}, \delta \mathbf{R}$ ,

$$\Delta R^{(1)} = \left[ \sqrt{\mathbf{R}} \tilde{U} \sqrt{\delta \mathbf{R}} \tilde{W} (1 - \mathbf{R}) + \text{H.c.} \right], \tag{3.11a}$$

$$\begin{aligned} \Delta R^{(2)} = & -\sqrt{\mathbf{R}} \tilde{U} (1 - \delta \mathbf{T}) \tilde{U}^\dagger \sqrt{\mathbf{R}} + \tilde{W}^\dagger \delta \mathbf{R} \tilde{W} + \sqrt{\mathbf{R}} \tilde{U} \sqrt{\delta \mathbf{R}} \tilde{W} \mathbf{R} \tilde{W}^\dagger \sqrt{\delta \mathbf{R}} \tilde{U}^\dagger \sqrt{\mathbf{R}} \\ & - \left[ \frac{1}{2} \tilde{W}^\dagger (1 - \delta \mathbf{T}) \tilde{W} \mathbf{R} + \sqrt{\mathbf{R}} \tilde{U} \sqrt{\delta \mathbf{R}} \tilde{W} \sqrt{\mathbf{R}} \tilde{U} \sqrt{\delta \mathbf{R}} \tilde{W} (1 - \mathbf{R}) + \text{H.c.} \right], \end{aligned} \tag{3.11b}$$

$$\Delta T^{(1)} = - \left[ Q \sqrt{\mathbf{R}} \tilde{U} \sqrt{\delta \mathbf{R}} \tilde{W} Q^\dagger \mathbf{T} + \text{H.c.} \right], \tag{3.11c}$$

$$\begin{aligned} \Delta T^{(2)} = & Q \sqrt{\mathbf{R}} \tilde{U} \sqrt{\delta \mathbf{R}} \tilde{W} Q^\dagger \mathbf{T} Q \tilde{W}^\dagger \sqrt{\delta \mathbf{R}} \tilde{U}^\dagger \sqrt{\mathbf{R}} Q^\dagger \\ & - \left[ \frac{1}{2} Q \tilde{W}^\dagger (1 - \delta \mathbf{T}) \tilde{W} Q^\dagger \mathbf{T} - Q \sqrt{\mathbf{R}} \tilde{U} \sqrt{\delta \mathbf{R}} \tilde{W} \sqrt{\mathbf{R}} \tilde{U} \sqrt{\delta \mathbf{R}} \tilde{W} Q^\dagger \mathbf{T} + \text{H.c.} \right] \end{aligned} \tag{3.11d}$$

(The abbreviation H.c. stands for Hermitian conjugate.) The moments (3.2) are computed by first averaging over the unitary matrices  $U_0, W_0$  and then averaging over  $\delta\mathbf{R}$  and  $\delta\mathbf{T}$  using Eq. (3.8). Averages over unitary matrices follow from

$$\langle U_{nl} U_{ml}^* \rangle = \frac{1}{N} \delta_{nm} \delta_{kl}, \tag{3.12a}$$

$$\langle U_{nk} U_{mk} U_{pl}^* U_{ql}^* \rangle = \frac{1}{N(N+1)} \delta_{kl} (\delta_{np} \delta_{mq} + \delta_{nq} \delta_{mp}) \tag{3.12b}$$

Without time-reversal symmetry averages over  $U_0$  and  $W_0$  are independent. With time-reversal symmetry we have  $W_0 = U_0^T$  so that only a single average remains. The results are as follows

*With time-reversal symmetry*

$$\begin{aligned} (1/\delta L) \langle \delta \mathcal{R}_n \rangle &= 1 + 2(\gamma - 1) \mathcal{R}_n + \frac{\mathcal{R}_n}{N+1} \left( \mathcal{R}_n + \sum_m \mathcal{R}_m \right) \\ &+ \frac{1}{N+1} \sum_{m \neq n} \frac{\mathcal{R}_n(1 - \mathcal{R}_m)^2 + \mathcal{R}_m(1 - \mathcal{R}_n)^2}{\mathcal{R}_n - \mathcal{R}_m}, \end{aligned} \tag{3.13a}$$

$$(1/\delta L) \langle \delta \mathcal{R}_n \delta \mathcal{R}_m \rangle = \frac{4\delta_{nm}}{N+1} \mathcal{R}_n(1 - \mathcal{R}_n)^2, \tag{3.13b}$$

$$\begin{aligned} (1/\delta L) \langle \delta \mathcal{T}_n \rangle &= \mathcal{T}_n(\gamma - 1) + \frac{\mathcal{T}_n}{N+1} \\ &\times \left( A_{nn} + \sum_{m \neq n} \frac{\mathcal{T}_m A_{nn} + \mathcal{T}_n A_{mm}}{\mathcal{T}_n - \mathcal{T}_m} + F_{nn} + \sum_{m \neq n} F_{nm} \frac{\mathcal{T}_n + \mathcal{T}_m}{\mathcal{T}_n - \mathcal{T}_m} \right), \end{aligned} \tag{3.13c}$$

$$(1/\delta L) \langle \delta \mathcal{T}_n \delta \mathcal{T}_m \rangle = \frac{2}{N+1} (\delta_{nm} \mathcal{T}_n^2 A_{nn} + \mathcal{T}_n \mathcal{T}_m F_{nm}), \tag{3.13d}$$

$$(1/\delta L) \langle \delta \mathcal{T}_n \delta \mathcal{R}_m \rangle = -\frac{4}{N+1} \mathcal{T}_n \mathcal{R}_m (1 - \mathcal{R}_m) |Q_{nm}|^2 \tag{3.13e}$$

*Without time-reversal symmetry*

$$\begin{aligned} (1/\delta L) \langle \delta \mathcal{R}_n \rangle &= 1 + 2(\gamma - 1) \mathcal{R}_n + \frac{\mathcal{R}_n}{N} \sum_m \mathcal{R}_m \\ &+ \frac{1}{N} \sum_{m \neq n} \frac{\mathcal{R}_n(1 - \mathcal{R}_m)^2 + \mathcal{R}_m(1 - \mathcal{R}_n)^2}{\mathcal{R}_n - \mathcal{R}_m}, \end{aligned} \tag{3.14a}$$

$$(1/\delta L) \langle \delta \mathcal{R}_n \delta \mathcal{R}_m \rangle = \frac{2\delta_{nm}}{N} \mathcal{R}_n(1 - \mathcal{R}_n)^2, \tag{3.14b}$$

$$(1/\delta L)\langle\delta\mathcal{F}_n\rangle = \mathcal{F}_n(\gamma - 1) + \frac{\mathcal{F}_n}{N} \left( A_{nn} + \sum_{m \neq n} \frac{\mathcal{F}_m A_{nm} + \mathcal{F}_n A_{mn}}{\mathcal{F}_n - \mathcal{F}_m} \right), \quad (3.14c)$$

$$(1/\delta L)\langle\delta\mathcal{F}_n\delta\mathcal{F}_m\rangle = \frac{2\delta_{nm}}{N} \mathcal{F}_n^2 A_{nm}, \quad (3.14d)$$

$$(1/\delta L)\langle\delta\mathcal{F}_n\delta\mathcal{R}_m\rangle = -\frac{2}{N} \mathcal{F}_n \mathcal{R}_m (1 - \mathcal{R}_m) |Q_{nm}|^2 \quad (3.14e)$$

We have abbreviated  $A_{nm} = (QRQ^\dagger)_{nm}$  and  $F_{nm} = |(Q\sqrt{\mathbf{R}}Q^\dagger)_{nm}|^2$

The moments of  $\delta\mathcal{R}_n$  contain only the set of reflection eigenvalues  $\{\mathcal{R}_n\}$ , so that from Eq (3.3) we can immediately write down a Fokker–Planck equation for the distribution of the  $\mathcal{R}_n$ 's. In terms of variables  $\mu_n = 1/(\mathcal{R}_n - 1) \in (-\infty, -1) \cup (0, \infty)$  it reads [16]

$$l \frac{\partial}{\partial L} P(\{\mu_n\}, L) = \frac{2}{\beta N + 2 - \beta} \sum_{n=1}^N \frac{\partial}{\partial \mu_n} \mu_n (1 + \mu_n) \times \left[ \frac{\partial P}{\partial \mu_n} + \beta P \sum_{m \neq n} \frac{1}{\mu_m - \mu_n} + \gamma(\beta N + 2 - \beta) P \right], \quad (3.15)$$

where the symmetry index  $\beta = 1(2)$  corresponds to the case of unbroken (broken) time-reversal symmetry. The evolution of the reflection eigenvalues is independent of the transmission eigenvalues – but not vice versa. The evolution of the  $\mathcal{F}_n$ 's depends on the  $\mathcal{R}_n$ 's, and in addition on the slow variables contained in the unitary matrix  $Q$ . To obtain a closed Fokker–Planck equation we also need to compute increments and moments of  $Q$ . The resulting expressions are lengthy and will not be written down here.

In the single-mode case ( $N = 1$ ) this complication does not arise, because  $Q = e^{i\phi}$  drops out of the scalars  $A$  and  $F$ . The single transmission and reflection eigenvalues  $\mathcal{T}, \mathcal{R}$  coincide with the transmittance and reflectance  $T, R$  defined by Eq (2.2). The resulting Fokker–Planck equation is [9]

$$l \frac{\partial P}{\partial L} = -\frac{\partial}{\partial R} [(1 - R)^2 + 2\gamma R] P + \frac{\partial^2}{\partial R^2} R(1 - R)^2 P - \frac{\partial}{\partial T} T(\gamma - 1 + R) P + \frac{\partial^2}{\partial T^2} T^2 R P - 2 \frac{\partial^2}{\partial T \partial R} T R (1 - R) P \quad (3.16)$$

In the case of absorption ( $\gamma < 0$ ), Eq (3.16) is equivalent to the moments equations [5]

#### 4. Localization length

The limit  $L \rightarrow \infty$  of the distribution of the reflection eigenvalues follows directly from Eq (3.15), by equating the left-hand-side to zero. The resulting distribution  $P_\infty$

is that of the Laguerre ensemble of random matrix theory [16],

$$P(\{\mu_n\}) \propto \prod_{i < j} |\mu_j - \mu_i|^\beta \prod_k \exp[-\gamma(\beta N + 2 - \beta)\mu_k] \tag{4.1}$$

The distribution looks the same for both signs of  $\gamma$ , but the support (and the normalization constant) is different  $\mu_n > 0$  for  $\gamma > 0$ , and  $\mu_n < -1$  for  $\gamma < 0$ . To determine the localization length we need the distribution of the transmission eigenvalues in the large- $L$  limit. We consider the cases  $N = 1$  and  $N \gg 1$ .

#### 4.1 Single-mode waveguide

We compute the distribution  $P(T, L)$  of the transmission probability through a single-mode waveguide in the limit  $L \rightarrow \infty$ . In the case of absorption ( $\gamma < 0$ ) this calculation was done by Rammal and Doucot [4], and by Freilikher et al. [5]. We generalize their results to the case of amplification ( $\gamma > 0$ ). The two cases are essentially different because, while the mean value of  $R$  is finite in the case of absorption,

$$\langle R \rangle_\infty = 1 - 2\gamma e^{-2\gamma} \text{Ei}(2\gamma) \quad \text{for } \gamma < 0, \tag{4.2}$$

it diverges in the case of amplification. The mean value of  $\ln R$  is finite in both cases,

$$\langle \ln R \rangle_\infty = \begin{cases} C + \ln 2\gamma - e^{2\gamma} \text{Ei}(-2\gamma) & \text{for } \gamma > 0, \\ -C - \ln(-2\gamma) + e^{-2\gamma} \text{Ei}(2\gamma) & \text{for } \gamma < 0 \end{cases} \tag{4.3}$$

Here  $C$  is Euler’s constant and  $\text{Ei}(x) = \int_\infty^x dt e^t/t$  is the exponential integral. The relation

$$\langle \ln R(\gamma) \rangle_\infty = -\langle \ln R(-\gamma) \rangle_\infty \tag{4.4}$$

holds, in accordance with the duality relation (2.5).

We now show that the asymptotic  $L \rightarrow \infty$  distribution of  $T$  is log-normal, with mean and variance of  $\ln T$  given by

$$\langle \ln T \rangle = -(1 + |\gamma|)L/l + 2c(\gamma) + \mathcal{O}(l/L), \tag{4.5a}$$

$$c(\gamma) = \begin{cases} 0 & \text{for } \gamma < 0, \\ C + \ln 2\gamma - e^{2\gamma} \text{Ei}(-2\gamma) & \text{for } \gamma > 0, \end{cases} \tag{4.5b}$$

$$\text{var } \ln T = [2 + 4|\gamma|e^{2|\gamma|} \text{Ei}(-2|\gamma|)]L/l + \mathcal{O}(1) \tag{4.6}$$

The constant  $c(\gamma) \approx -2\gamma \ln \gamma$  if  $0 < \gamma \ll 1$ . Note that  $\text{var } \ln T \ll \langle \ln T \rangle^2$  for  $L/l \gg 1$ . The localization length  $\xi = l(1 + |\gamma|)^{-1}$  is independent of the sign of  $\gamma$ , in accordance with the duality relation (1.2).

These results are easy to establish for the case of absorption, when Eq. (3.16) implies the evolution equations [4,5]

$$l \frac{\partial}{\partial L} \langle \ln T \rangle = -1 + \gamma, \quad l \frac{\partial}{\partial L} \text{var } \ln T = 2\langle R \rangle, \quad \text{for } \gamma < 0 \tag{4.7}$$

Making use of the initial condition  $T \rightarrow 1$  for  $L \rightarrow 0$  and the asymptotic value (4.2) of  $\langle R \rangle$ , one readily obtains Eqs (4.5) and (4.6) for  $\gamma < 0$ .

In the case of amplification, the evolution equations (4.7) hold only for lengths  $L$  smaller than  $L_c \simeq l c(\gamma)/|\gamma|$ . For  $L \lesssim L_c$  stimulated emission *enhances* transmission through the waveguide. On larger length scales stimulated emission *reduces* transmission. Technically, the evolution equations (4.7) break down for  $L \rightarrow \infty$  because the integration by parts of the Fokker–Planck equation produces a non-zero boundary term if  $L > L_c$ . To extend Eqs (4.5) and (4.6) to the case  $\gamma > 0$  we use the duality relation (2.4). It implies that for  $N = 1$  the distribution of the ratio  $T/R$  is an even function of  $\gamma$ . Eq (4.5) for  $\gamma > 0$  follows directly from the equality

$$\langle \ln T(\gamma)/R(\gamma) \rangle = \langle \ln T(-\gamma)/R(-\gamma) \rangle, \quad (4.8)$$

which holds for all  $L$ , plus Eq (4.4), which holds for  $L \rightarrow \infty$ . The constant  $c(\gamma)$  for  $\gamma > 0$  equals  $\langle \ln R(\gamma) \rangle_\infty$  and is substituted from Eq (4.3). The duality of  $T(\gamma)/R(\gamma)$  also implies Eq (4.6) for the variance, provided the covariance  $\langle (\ln T \ln R) \rangle = \langle \ln T \ln R \rangle - \langle \ln T \rangle \langle \ln R \rangle$  remains finite as  $L \rightarrow \infty$ . We have checked this directly from the Fokker–Planck equation (3.16), and found the finite large- $L$  limit

$$\begin{aligned} \langle (\ln T \ln R) \rangle_\infty &= -2e^{2\gamma} \text{Ei}(-2\gamma)c(\gamma) - c(\gamma)^2 \\ &\quad - 2\gamma \int_0^\infty d\mu e^{-2\gamma\mu} [\ln^2(1+\mu) - \ln^2\mu] \quad \text{for } \gamma > 0 \end{aligned} \quad (4.9)$$

#### 4.2 Multi-mode waveguide

We next consider a waveguide with  $N \gg 1$  modes. We compute the localization length  $\xi = -\lim_{L \rightarrow \infty} L^{-1} \langle \ln T \rangle$  in the case of absorption, and include the case of amplification invoking duality. For absorption the average reflectance  $\langle R \rangle = N^{-1} \langle \sum_k (1 + 1/\mu_k) \rangle$  remains finite as  $L \rightarrow \infty$ . The large- $L$  limit  $\langle R \rangle_\infty$  follows from the distribution (4.1), using known formulas for the eigenvalue density in the Laguerre ensemble [22]. For  $|\gamma|N^2 \gg 1$  the result is

$$\langle R \rangle_\infty = 1 + |\gamma| - \sqrt{|\gamma|(2 + |\gamma|)} + \mathcal{O}(1/N), \quad \gamma < 0 \quad (4.10)$$

The evolution of transmission eigenvalues is governed by the Fokker–Planck equation (3.3), with coefficients given by (3.2), (3.13) and (3.14). Each  $\mathcal{T}_n$  has its own localization length  $\xi_n = -\lim_{L \rightarrow \infty} L^{-1} \ln \mathcal{T}_n$ . We order the  $\xi_n$ 's from large to small,  $\xi_1 > \xi_2 > \dots > \xi_\Lambda$ . This implies that for  $L \rightarrow \infty$  the separation of the  $\mathcal{T}_n$ 's becomes exponentially large,  $\mathcal{T}_1 \gg \mathcal{T}_2 \gg \dots \gg \mathcal{T}_N$ . Hence we may approximate

$$\frac{\mathcal{T}_n + \mathcal{T}_m}{\mathcal{T}_n - \mathcal{T}_m} \approx \begin{cases} -1 & \text{for } n > m, \\ 1 & \text{for } n < m, \end{cases} \quad (4.11a)$$

$$\frac{\mathcal{T}_n A_{mm} + \mathcal{T}_m A_{nn}}{\mathcal{T}_n - \mathcal{T}_m} \approx \begin{cases} -A_{nn} & \text{for } n > m, \\ A_{mm} & \text{for } n < m \end{cases} \quad (4.11b)$$

The Fokker–Planck equation (3.3) simplifies considerably and leads to the following equation for the largest transmission eigenvalue

$$l \frac{\partial}{\partial L} \langle \ln \mathcal{F}_1 \rangle = \begin{cases} -1 - |\gamma| + \langle R \rangle - \frac{1}{N+1} \langle A_{11} + F_{11} \rangle & \text{for } \beta = 1, \\ -1 - |\gamma| + \langle R \rangle - \frac{1}{N} \langle A_{11} \rangle & \text{for } \beta = 2 \end{cases} \quad (4.12)$$

For  $|\gamma|N^2 \gg 1$  we may substitute Eq (4.10) for  $\langle R \rangle$  and omit the terms with  $\langle A_{11} \rangle$  and  $\langle F_{11} \rangle$ . The resulting localization length is given by

$$l/\xi = \sqrt{|\gamma|(2 + |\gamma|)} + \mathcal{O}(1/N) \quad (4.13)$$

Because of duality, Eq (4.13) holds regardless of the sign of  $\gamma$ . It agrees with radiative transfer theory for  $\gamma < 0$ , but not for  $\gamma > 0$ . Indeed, the exponential decay of the transmitted intensity in the case of amplification is an interference effect, which is not contained in the theory of radiative transfer.

Eq (4.13) is asymptotically exact for  $|\gamma| \gg 1/N^2$ . For smaller  $|\gamma|$  we cannot compute  $\xi$  rigorously because the distribution of the matrices  $\mathbf{A}$  and  $\mathbf{F}$  is not known. An interpolative formula for all  $\gamma$  can be obtained by substituting for  $\langle A_{11} \rangle$  and  $\langle F_{11} \rangle$  in Eq (4.12) their  $L \rightarrow \infty$  limits when  $\gamma = 0$ , which are  $\langle A_{11} \rangle = \langle F_{11} \rangle = 1$ . In this way, we arrive at the localization length

$$\xi = l \left[ \frac{2}{\beta N + 2 - \beta} + \sqrt{|\gamma|(2 + |\gamma|)} \right]^{-1}, \quad (4.14)$$

which interpolates between the known [13,14,23] value of  $\xi$  for  $\gamma = 0$  and Eq (4.13) for  $|\gamma| \gg 1/N^2$ .

The localization length  $\xi$  is the largest of the eigenvalue-dependent localization lengths  $\xi_n$ . What about the other  $\xi_n$ 's? For  $\gamma = 0$  it is known [14,15,23] that the inverse localization lengths are equally spaced, and satisfy the sum rule  $\sum_n 1/\xi_n = N/l$ . We have not succeeded in deriving the spacings for  $\gamma \neq 0$ , but we have been able to derive the sum rule from the Fokker–Planck equation (by computing the  $L$ -dependence of  $\langle \sum_n \ln \mathcal{F}_n \rangle$ ). The result is exact and reads

$$l \sum_{n=1}^N \xi_n^{-1} = (1 + |\gamma|)N \quad (4.15)$$

### 5. Numerical results

To test the analytical predictions on a model system, we have numerically solved a discretized version of the Helmholtz equation,

$$[\nabla^2 + k^2 \varepsilon(\mathbf{r})]E(\mathbf{r}) = 0, \quad (5.1)$$

on a two-dimensional square lattice (lattice constant  $d$ , length  $L$ , width  $W$ ). The real part  $\varepsilon'$  of the dielectric constant was chosen randomly from site to site with a uniform

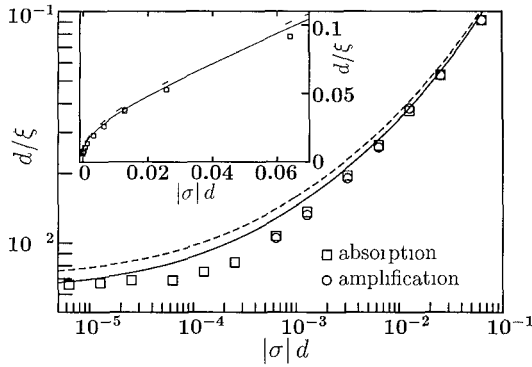


Fig. 1. Localization length  $\xi = -\lim_{L \rightarrow \infty} L^{-1} \langle \ln T \rangle$  of a disordered waveguide ( $N = 10$ ) versus the modal average  $\bar{\sigma}$  of the inverse absorption or amplification length. Data points are numerical solutions of the discretized (lattice constant  $d$ ) two-dimensional Helmholtz equation for the case of absorption (squares) and amplification (circles). The curves are the analytical prediction (4.14) for the case  $\beta = 1$  (unbroken time-reversal symmetry) for  $l = 29.6d$  [solid curve, determined from Eq. (5.3)] and for  $l = 26.1d$  [dashed curve, determined from Eq. (5.4)]. The inset shows the same data on a linear, rather than logarithmic, scale.

distribution between  $1 \pm \delta\epsilon$ . The imaginary part  $\epsilon''$  was the same at all sites. The scattering matrix was computed using the recursive Green function technique [24].

The parameter  $\bar{\sigma}$  is obtained from the analytical solution of the discretized Helmholtz equation in the absence of disorder ( $\delta\epsilon = 0$ ). The complex longitudinal wavenumber  $k_n$  of transverse mode  $n$  then satisfies the dispersion relation

$$\cos(k_n d) + \cos(n\pi d/W) = 2 - \frac{1}{2}(kd)^2(1 + i\epsilon''), \tag{5.2}$$

which determines  $\bar{\sigma}$  according to  $\bar{\sigma} = -2N^{-1} \text{Im} \sum_n k_n$ . Simulations with  $\epsilon'' = 0$  were used to obtain  $l$ , either from the large- $L$  relation [14]

$$-\lim_{L \rightarrow \infty} L^{-1} \langle \ln T \rangle = \left[ \frac{1}{2}(N+1)l \right]^{-1}, \tag{5.3}$$

or from the large- $N$  relation [25]

$$\lim_{N \rightarrow \infty} \langle T \rangle = (1 + L/l)^{-1} \tag{5.4}$$

The parameters chosen were  $W = 25d$ ,  $k = 1.22d^{-1}$ , corresponding to  $N = 10$ ,  $l = 29.6d$  from Eq. (5.3) and  $l = 26.1d$  from Eq. (5.4). The localization length was computed as a function of  $\sigma$  from the  $L$ -dependence of  $\ln T$  up to  $40l$ , averaged over 150 realizations of the disorder. Results are shown in Fig. 1. The localization length is the same for absorption and amplification, within the numerical accuracy. Comparison with the analytical result (4.14) for  $\beta = 1$  is plotted for the two values of the mean free path. The agreement is quite reasonable, given the approximate nature of Eq. (4.14) in the regime  $|\gamma|N^2 \simeq 1$  (corresponding to  $|\bar{\sigma}|d \simeq 10^{-4}$ ).

In conclusion, we have shown how absorption or amplification can be incorporated into the Dorokhov–Mello–Pereyra–Kumar equation for transmission through a multi-mode waveguide. The technical difficulty of the multi-mode case is that the

Fokker–Planck equation for the transmission eigenvalues  $\mathcal{T}_n$  depends not just on the transmission and reflection eigenvalues  $\mathcal{T}_n$ ,  $\mathcal{R}_n$ , but also on the eigenvectors of the matrices  $tt^\dagger$  and  $rr^\dagger$ . We could compute the localization length in the two regimes,  $|\gamma| \gg 1/N^2$  and  $|\gamma| \ll 1/N^2$ , and have given an interpolation formula for the intermediate regime. An exact solution for all  $\gamma$  remains an unsolved problem.

## Acknowledgements

We acknowledge useful discussions with P W Brouwer and K M Frahm. This research was supported in part by the “Nederlandse organisatie voor Wetenschappelijk Onderzoek” (NWO) and by the “Stichting voor Fundamenteel Onderzoek der Materie” (FOM).

## References

- [1] P Eidos and R C Herndon, *Adv Phys* 31 (1982) 65
- [2] J B Pendry, *Adv Phys* 43 (1994) 461
- [3] M E Gertsenshtein and V B Vasil’ev, *Teor Veroyatn Pimen* 4 (1959) 424–5 (1960) 3(E) [*Theor Probab Appl* 4 (1959) 391, 5 (1960) 340(E)]
- [4] R Rammal and B Douçot, *J Phys (France)* 48 (1987) 509
- [5] V Freilikher, M Pustilnik and I Yurkevich, *Phys Rev Lett* 73 (1994) 810
- [6] P Pishdian and N Kumar, *Phys Rev B* 50 (1994) 9644
- [7] A K Gupta and A M Jayannavar, *Phys Rev B* 52 (1995) 4156
- [8] Z-Q Zhang, *Phys Rev B* 52 (1995) 7960
- [9] J C J Paasschens, T Sh Misirpashaev and C W J Beenakker, *Phys Rev B* 54 (1996) 11887
- [10] V Freilikher, M Pustilnik, and I Yurkevich, preprint (cond-mat/9605090)
- [11] A K Sen, *Mod Phys Lett B* 10 (1996) 125
- [12] D J Thouless, *Phys Rev Lett* 39 (1977) 1167
- [13] K B Efetov and A I Larkin, *Zh Eksp Teor Fiz* 85 (1983) 764 [*Sov Phys JETP* 58 (1983) 444]
- [14] O N Dorokhov, *Pisma Zh Eksp Teor Fiz* 36 (1982) 259 [*JETP Lett* 36 (1982) 318], *Zh Eksp Teor Fiz* 85 (1983) 1040 [*Sov Phys JETP* 58 (1983) 606]
- [15] P A Mello, P Pereyra, and N Kumar, *Ann Phys (N Y)* 181 (1988) 290
- [16] C W J Beenakker, J C J Paasschens and P W Brouwer, *Phys Rev Lett* 76 (1996) 1368
- [17] N A Bruce and J T Chalker, *J Phys A* 29 (1996) 3761
- [18] For a review of the theory of the DMPK equation, see C W J Beenakker, *Rev Mod Phys* (to be published)
- [19] G L J A Rikken and B A van Tiggelen, *Nature* 381 (1996) 54
- [20] N G van Kampen, *Stochastic Processes in Physics and Chemistry* (North Holland, Amsterdam, 1981)
- [21] P A Mello and S Tomsovic, *Phys Rev B* 46 (1992) 15963
- [22] T Nagao and K Slevin, *J Math Phys* 34 (1993) 2075, 34 (1993) 2317
- [23] A D Stone, P A Mello, K A Muttalib and J-L Pichard, in *Mesoscopic Phenomena in Solids*, eds B L Al’tshuler, P A Lee and R A Webb (North-Holland, Amsterdam, 1991)
- [24] H U Baranger, D P DiVincenzo, R A Jalabert and A D Stone, *Phys Rev B* 44 (1991) 10637
- [25] P A Mello and A D Stone, *Phys Rev B* 44 (1991) 3559