Groups with finitely many non-normal subgroups

Lenstra, H.W.; Hekster, N.S.

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By

N S Hekster and H W Lenstra, Jr

1. Introduction. In this paper we describe all groups that have only finitely many non-normal subgroups Examples of such groups are, of course, groups that do not have any non-normal subgroups at all It is well-known that the latter groups can be completely described, as follows

Theorem 1. *Let G be a group Then all subgroups of G are normal if and only if G satisfies one of the following two conditions*

(i) *G is abelian,*
(ii) *there exist groups A, B such that*

(a) *$G \cong A \times Q \times B$, where Q denotes the quaternion group of order 8,*
(b) *A is an abelian group with the property that every $x \in A$ has finite odd order,*
(c) *B is an abelian group with $x^2 = 1$ for all $x \in B$*

For the proof, see [1, Theorem 12 5 4] A group G is called *hamiltonian* if it satisfies condition (ii) of Theorem 1

For a prime number p, denote by C_{p^∞} a multiplicatively written group that is isomorphic to the group of complex roots of unity of p-power order Our main result is as follows

Theorem 2. *Let G be a group Then the number of non-normal subgroups of G is finite if and only if G satisfies one of the following three conditions*

(i) *G is abelian or hamiltonian,*
(ii) *G is finite,*
(iii) *there exist a prime number p and groups A, B such that*

(a) *$G \cong A \times B$,*
(b) *A is a finite group of order not divisible by p, and it is abelian or hamiltonian,*
(c) *B has a normal subgroup C, contained in the centre of B, for which B/C is a finite abelian p-group and $C \cong C_{p^\infty}$*

For the proof we refer to Section 4

We can give a formula for the number of non-normal subgroups of the groups occurring in Theorem 2 (iii). Let G, p, A, B, C be as in Theorem 2 (iii). By $[,]$ we denote the map $B/C \times B/C \to C$ that is induced by the map $B \times B \to C$ sending (g, h) to $g^{-1}h^{-1}gh$. If J_1, J_2 are subgroups of B/C, then we denote by $[J_1, J_2]$ the subgroup of C generated by the image of $J_1 \times J_2$ under $[,]$. Finally, if D is a finite p-group, we write $l_p D$ for the number of factors p in the order of D, so $l_p D = (\log \# D)/\log p$.

Proposition 3. *Let the notation be as just defined, and let k denote the number of subgroups of A. Then the number of non-normal subgroups of G equals*

$$k \cdot \sum_J (l_p [B/C, J] - l_p [J, J]) \cdot \# J,$$

where J ranges over the set of subgroups of B/C.

The proof is given in Section 2.

It is easy to see from Proposition 3 that a group G as in Theorem 2 (iii) does not have non-normal subgroups at all if and only if B is abelian; this also follows from Theorem 1.

The sum appearing in Proposition 3 is clearly divisible by p. Assuming that B is not abelian one can, more precisely, show the following. If C equals the centre of B, then the sum is congruent to $p \bmod p^2$, and at least $p(p+1)$; and if C is properly contained in the centre of B, then the sum is congruent to $0 \bmod p^2$, and at least $p^2(p+2)$. In particular, any infinite group that has non-normal subgroups at all has at least 6 of them; equality occurs only for the unique non-abelian group containing C_{2^∞} as a central subgroup of index 4.

Another consequence is the following. If the number of non-normal subgroups of a group is a prime number, or the square or the cube of a prime number, then the group is finite.

Let G be a group and σ an automorphism of G. If H is a subgroup of G, we say that σ *fixes* H if $\sigma H = H$. The following result is needed in the proof of Theorem 2.

Proposition 4. *Let G be a group. Then the following two assertions are equivalent:*

(i) *G is an infinite abelian group, and it has an automorphism that fixes almost all but not all subgroups of G;*

(ii) *there exist a prime number p and groups A, D such that*

(a) $G \cong A \times C_{p^\infty} \times D$,

(b) *A is a finite abelian group of order not divisible by p,*

(c) *D is a non-trivial finite abelian p-group.*

The same is true if both in (i) *and in* (ii) (b) *"abelian" is replaced by "hamiltonian".*

The proof is given in Section 3.

2. Proof of Proposition 3. Let the notation be as in Proposition 3.

Any subgroup of G equals a subgroup of A times a subgroup of B, so the proof of Proposition 3 immediately reduces to the case that $G = B$, which we now assume. We write $\bar{G} = G/C$.

Let H be a subgroup of G, and $\bar{H}$ its image in $\bar{G}$ Clearly we have $[\bar{H}, \bar{H}] \subset H \cap C$, and H is normal in G if and only if $[\bar{G}, \bar{H}] \subset H \cap C$

It follows that the number of non-normal subgroups of G equals $\sum_{J\,D} n_{J\,D}$, where the sum ranges over all pairs of subgroups $J \subset \bar{G}$, $D \subset C$ for which $[J, J] \subset D$, $[\bar{G}, J] \not\subset D$, and where $n_{J\,D}$ is the number of subgroups H of G with $\bar{H} = J$ and $H \cap C = D$

For each J, the number of possible D equals $l_p[\bar{G}, J] - l_p[J, J]$ Now fix J and D Since D is characteristic in C it is normal in G Hence $n_{J\,D}$ equals the number of subgroups of G/D that map isomorphically to J under the natural map $G/D \to G/C$ Since D contains $[J, J]$, the inverse image of J in G/D is abelian From $C/D \cong C_{p^\infty}$ it follows that this inverse image is isomorphic to $C_{p^\infty} \times J$ Thus $n_{J\,D}$ is the number of subgroups of $C_{p^\infty} \times J$ mapping isomorphically to J, and this number equals $\# \operatorname{Hom}(J, C_{p^\infty}) = \# J$

We conclude that $\sum_{J\,D} n_{J\,D} = \sum_{J\,D} \# J = \sum_{J} (l_p[\bar{G}, J] - l_p[J, J]) \cdot \# J$, as required This proves Proposition 3

3. Proof of Proposition 4.

Lemma 5. *Let G be an infinite group, and suppose that G is written as the union of a finite set and a finite collection of subgroups Then the finite set can be omitted from this union*

Proof This is an immediate consequence of a lemma of B H Neumann, which asserts the following If a group is written as the union of finitely many cosets of subgroups, then the cosets occurring in that union belonging to subgroups of infinite index can be omitted For a proof of this lemma, see [2, (4 4), 3, Lemma 4 17] This proves Lemma 5

Lemma 6. *Let G be a group and σ an automorphism of G that fixes almost all subgroups of G Then σ fixes every infinite subgroup of G*

Proof Let $H \subset G$ be an infinite subgroup For every $x \in H - \sigma H$, the subgroup $\langle x \rangle$ generated by x clearly belongs to the finite collection of subgroups C of G with $\sigma C \neq C$ Since for every C there are only finitely many $x \in G$ with $C = \langle x \rangle$, it follows that $H - \sigma H$ is finite Lemma 5 now implies that $H \cap \sigma H = H$, so H is contained in σH (This can also be seen without Lemma 5) Likewise H is contained in $\sigma^{-1} H$, so $H = \sigma H$, as required This proves Lemma 6

Lemma 7. *Let G be an abelian group that has an element of infinite order, and σ an automorphism of G that fixes almost all subgroups of G Then σ fixes all subgroups of G*

Proof Let T be the subgroup of G consisting of all elements of finite order By Lemma 6, one has $\sigma x = x^{\pm 1}$ for every $x \in G - T$ Since G/T is not the union of two proper subgroups the sign is independent of x But $G - T$ generates G, so either σ is the identity on G or σ maps each $x \in G$ to x^{-1} This implies Lemma 7

Lemma 8. *Let G be an abelian group that contains a subgroup of the form $C_{p^\infty} \times C_{p^\infty}$, where p is a prime number Let σ be an automorphism of G that fixes almost all subgroups of G Then σ fixes all subgroups of G*

Proof. By Lemma 6, it suffices to prove that $\sigma H = H$ for every *finite* subgroup H of G. We may clearly assume that H is cyclic. Then it is easy to see that there exist subgroups C_1 and C_2 of $C_{p^\infty} \times C_{p^\infty}$, both isomorphic to C_{p^∞}, such that $C_1 \supset H \cap (C_{p^\infty} \times C_{p^\infty})$ and $C_1 \cap C_2 = \{1\}$. The infinite subgroups $C_1 H$, $C_2 H$ of G are fixed by σ, by Lemma 6, so the same is true for $(C_1 H) \cap (C_2 H) = H$. This proves Lemma 8.

We now prove Proposition 4. We treat the "abelian" and the "hamiltonian" case simultaneously.

To prove that (ii) implies (i), let $G = A \times C_{p^\infty} \times D$ as in Proposition 4(ii). It is clear that G is an infinite group that is abelian or hamiltonian, as the case may be. Let ϕ be any non-trivial homomorphism $D \to C_{p^\infty}$. We prove that the automorphism σ of G given by $\sigma(a, c, d) = (a, c\,\phi(d), d)$ fixes almost all but not all subgroups of G.

If $d \in D$ is such that $\phi(d) \neq 1$, then clearly the subgroup of G generated by $(1, 1, d)$ is not fixed by σ. It remains to prove that σ fixes almost all subgroups of G. Let p^n be the exponent of D. Since C_{p^∞} has only finitely many elements of order at most p^{2n}, almost any subgroup H of G has an element (a, c, d) with order $(c) > p^{2n}$. Taking the p^n-th power, we see that any such H also contains an element $(1, c', 1)$ with $c' \notin \phi D$. Then $\{1\} \times \phi D \times \{1\} \subset \langle (1, c', 1) \rangle \subset H$, and since σ acts modulo $\{1\} \times \phi D \times \{1\}$ as the identity this implies that σ fixes H. This proves that (ii) implies (i).

To prove that (i) implies (ii), let G be an infinite abelian or hamiltonian group, and let σ be an automorphism of G that fixes almost all but not all subgroups of G.

For a prime number l, let G_l be the subset of G consisting of all elements of finite l-power order. Since G is abelian or hamiltonian, each G_l is a subgroup of G, and it is clearly fixed by σ. Using Lemma 7 we see that G may be identified with the direct sum of all G_l. For any set π of primes, let G_π be the direct sum of all G_l with $l \in \pi$.

Let π be a set of primes, and π' its complement, so that $G = G_\pi \times G_{\pi'}$. Each subgroup of G is the direct sum of a subgroup of G_π and a subgroup of $G_{\pi'}$. It follows that at least one of G_π, $G_{\pi'}$ has a subgroup that is not fixed by σ, say this is $H \subset G_\pi$. Then $H \times G_{\pi'}$ is not fixed by σ, so Lemma 6 implies that $G_{\pi'}$ is finite.

This proves that, for any set π of primes, one of G_π, $G_{\pi'}$ has a subgroup not fixed by σ and the other one is finite.

If G_l is non-trivial for infinitely many l, then we can choose π such that both π and π' contain infinitely many such l, contradicting what we just proved. It follows that almost all G_l are trivial. Likewise we obtain a contradiction if G_l is infinite for two distinct primes l. Hence there exists a unique prime p such that G_p is infinite, and this G_p has a subgroup not fixed by σ. For this prime the group $A = G_{\{p\}'}$ is finite, it is either abelian or hamiltonian, and we have $G = A \times G_p$.

We now first prove that G_p is abelian. If this is not the case, then we have $p = 2$ and $G_p \cong Q \times B$, where B is an abelian group of exponent 2. In this group, two elements generate the same subgroup if and only if they are conjugate. Hence the hypothesis that σ fixes almost all subgroups implies that G_p is the union of a finite set F and $\bigcup_{\phi} \{x \in G_p : \sigma x = \phi x\}$, with ϕ ranging over the inner automorphisms of G_p. Since there are only finitely many ϕ's, Lemma 5 implies that the finite set F can be omitted from the union. Hence $\sigma \langle x \rangle = \langle x \rangle$ for every $x \in G_p$, so $\sigma H = H$ for every subgroup

H of G_p, which is a contradiction. This proves that G_p is abelian. In particular, A is abelian if and only if G is abelian, and A is hamiltonian if and only if G is hamiltonian.

For a non-negative integer m, denote by $G(m)$ the subgroup of all elements of G_p of order dividing p^m. We prove that each $G(m)$ is finite. Suppose that this is not the case. Then we can choose m so large that $G(m)$ is infinite and contains an element x with $\sigma\langle x\rangle \neq \langle x\rangle$. The hypothesis that σ fixes almost all subgroups of G implies that $G(m)$ is the union of a finite set F and $\bigcup_a \{x \in G(m) : \sigma x = x^a\}$, with a ranging over the integers mod p^m that are not divisible by p. Lemma 5 now implies that the finite set F can be omitted from the union. Hence $\sigma\langle x\rangle = \langle x\rangle$ for every $x \in G(m)$, contradicting the choice of m. This proves that all $G(m)$ are finite.

For each m, let $C(m)$ be the subgroup $\bigcap_{n \geqq 0} G(m+n)^{p^n}$ of $G(m)$. Since $G(m)$ is finite, we have $C(m) = G(m+n)^{p^n}$ for all n exceeding a bound depending on m. This implies that $C(m+1)^p = C(m)$ for all m, which readily yields that the set $C = \bigcup_m C(m)$ is a subgroup of G_p that is isomorphic to the direct sum of t copies of C_{p^∞}, for some non-negative integer t. Because C_{p^∞} is divisible we have $G_p \cong C \times D$ for some subgroup D of G_p. If n is such that $C(1) = G(1+n)^{p^n}$, then $D^{p^n} = \{1\}$, so $D \subset G(n)$ and therefore D is finite. But G_p is infinite, so we must have $t > 0$. By Lemma 8 we have $t < 2$. Therefore $t = 1$, and $C \cong C_{p^\infty}$.

Since G_p has a subgroup not fixed by σ, not every subgroup of G_p is characteristic. Hence $G_p \neq C$, and D is non-trivial.

This proves Proposition 4.

4. Proof of Theorem 2. The *if*-part of Theorem 2 is clear in the cases (i) and (ii), and in case (iii) it suffices to refer to Proposition 3.

Before we prove the *only if*-part we derive a series of auxiliary results.

Lemma 9. *Let G be a group with only finitely many non-normal subgroups. Then every infinite subgroup of G is normal.*

Proof. This follows from Lemma 6, applied to inner automorphisms of G.

Lemma 10. *Let G be a group that has only finitely many non-normal subgroups, and p a prime number. Suppose that G contains a normal subgroup C isomorphic to C_{p^∞} for which G/C is a finite cyclic group. Then G is abelian.*

Proof. For almost all x in a generating coset of G modulo C the subgroup $\langle x\rangle$ is normal in G. Choose such an x. Then the natural map $C \to G/\langle x\rangle$ is surjective, so $G/\langle x\rangle$ is abelian. Therefore the commutator subgroup G' of G is contained in $\langle x\rangle$, so G' is finite. But G' is the homomorphic image C^{x-1} of C, so it is divisible as well. Hence $G' = \{1\}$. This proves Lemma 10.

Lemma 11. *Let G be a group that has only finitely many non-normal subgroups, and p a prime number. Suppose that G contains a subgroup C of finite index that is isomorphic to C_{p^∞}. Then C is contained in the centre of G.*

Proof. By Lemma 9 the subgroup C is normal. Now apply Lemma 10 to subgroups generated by C and a single element of G. This proves Lemma 11.

Lemma 12. *Let G be a group that has only finitely many non-normal subgroups, and p a prime number. Suppose that G contains a normal subgroup C that is isomorphic to C_{p^∞} for which G/C is a finite p-group. Then G/C is abelian.*

Proof. By Lemma 9 we can choose a positive integer m such that every non-normal subgroup of G has order less than p^m. Denote by $C(m)$ the unique subgroup of C of order p^m. Then every subgroup of G containing $C(m)$ is normal, so $G/C(m)$ is abelian or hamiltonian. But the orders of the elements of $G/C(m)$ are exactly all powers of p, so $G/C(m)$ is not hamiltonian. Therefore $G/C(m)$ is abelian, and it follows that G/C is abelian as well. This proves Lemma 12.

Lemma 13. *Let G be an infinite group that has only finitely many non-normal subgroups, and that is neither abelian nor hamiltonian. Then G has a normal subgroup F of finite index that is abelian or hamiltonian, and that contains a subgroup H that is non-normal in G.*

Proof. We construct a sequence of non-normal subgroups $H_1, H_2, \ldots$, of G and a sequence of normal subgroups $F_1, F_2, \ldots$, of finite index in G with $H_i \subset F_i$ in the following way.

Let H_1 be any non-normal subgroup of G, and F_1 its normalizer in G. All conjugates of H_1 are non-normal in G, so they are finite in number. Hence F_1 is of finite index in G, and by Lemma 9 it is normal.

Suppose, inductively, that H_i, F_i have been constructed. If F_i is abelian or hamiltonian, then the construction stops, and $F = F_i$, $H = H_i$ satisfy the conclusion of the lemma. If F_i is not abelian or hamiltonian, we let H_{i+1} be any non-normal subgroup of F_i, and F_{i+1} its normalizer in F_i. Then F_{i+1} is normal of finite index in G.

To prove that the process stops, it suffices to show that $H_i \neq H_j$ for $j > i$. But H_i is normal in F_i, whereas H_j is not even normal in the subgroup F_{j-1} of F_i. This proves Lemma 13.

We now prove the *only if*-part of Theorem 2.

Let G be a group that has only finitely many non-normal subgroups, and that is not as in (i) or (ii) of Theorem 2, i.e., G is infinite, and neither abelian nor hamiltonian. Let F, H be chosen as in Lemma 13, and let ϕ be an inner automorphism of G with $\phi H \neq H$. Then F is an infinite group that is abelian or hamiltonian, and the restriction of ϕ to F is an automorphism of F that fixes almost all but not all subgroups of F. Applying Proposition 4 to F we see that F has a subgroup C of finite index that is isomorphic to C_{p^∞}. Then C is of finite index in G, so by Lemma 9 each subgroup of G containing C is normal. Therefore G/C is abelian or hamiltonian, and we can write $G/C = A \times D$, where D is a finite p-group and A is a finite group of order not divisible by p that is abelian or hamiltonian. Also, C is contained in the centre of G, by Lemma 11.

Let E be the unique subgroup of G for which $A = E/C$, and let a be the order of A. Since C is uniquely divisible by a, the cohomology group $H^2(A, C)$ vanishes, so E can be identified with $A \times C$. From $A = \{x \in E : x^a = 1\}$ we see that A is characteristic in E, and therefore normal in G The subgroup B of G with $D = B/C$ is also normal in G, and it follows that $G = A \times B$ Finally, applying Lemma 12 with B in the role of G we see that B/C is abelian

This completes the proof of Theorem 2

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Anschriften der Autoren

N S Hekster
IBM The Netherlands
Paalbergweg 9–11
1105 AG Amsterdam
The Netherlands

H W Lenstra, Jr
Department of Mathematics
University of California
Berkeley, California 94720
USA