Note on the circular vortex in the theory of sunspots

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Note on the circular vortex in the theory of sunspots, by *J. Woltjer Jr.*

Theoretical consideration on a supposed vertical motion in sunspots starts from a particular solution of the hydrodynamical equations in which each particle uniformly describes a circle centered on a common axis. An extensive investigation of this type of motion has been made by BJERKNES [1]).

ROSSELAND [2]) has extended this method to the case of two interacting coaxial vortices relating to the motion of the electrons and the positive ions in the same sunspot with a view of obtaining a relative motion of the particles of opposite charge, apt to generate a magnetic field of the type observed in sunspots.

Purpose of this note is to show that generally two interacting circular coaxial vortices do not represent a particular solution of the hydrodynamical equations.

1. Suppose the axis of motion to be the axis of z; the angular velocity of rotation is ω, a function of z and $\Delta = \sqrt{(x^2 + y^2)}$; V_x, V_y, V_z are the components of linear velocity. Hence

$$(1.1) \qquad V_x = -\omega y \ , \ V_y = \omega x \ , \ V_z = 0.$$

The equation of continuity is satisfied. The three equations that express the change of momentum parallel to the three axes of coordinates are satisfied if

$$(1.2) \qquad \omega^2 = \frac{1}{\rho}\frac{1}{\Delta}\frac{\partial p}{\partial \Delta} \ , \ \frac{\partial p}{\partial z} = -g\rho\,;$$

p is the pressure, ρ the density, g the acceleration of gravity supposed to have the direction of $-z$. If ω is an arbitrarily chosen function of Δ and z, we are able to derive the values of p and ρ from these equations in such a way that $\lim_{\Delta \to \infty} p$ equals the undisturbed value of p outside the spot region. The development of these equations has been extensively treated in the memoir by BJERKNES already referred to.

2. Consider the case of two circular coaxial vortices, that interact by means of internal friction and possibly also by electric and magnetic fields. However, as the electromagnetic field is irrelevant to the problem of this note, I do not consider the complication arising from its action.

[1]) V. BJERKNES Solar Hydrodynamics, *Aph. J.* 64 p. 93 etc.

[2]) S. ROSSELAND: On the structure and origin of solar magnetic fields, *Astroph. J.* **62** p. 387, etc.

As the equations of continuity are satisfied we only have to examine the six hydrodynamical equations:

$$(2.1) \qquad \frac{dV_x}{dt} = -\frac{1}{\rho}\left(\frac{\partial p_{xx}}{\partial x} + \frac{\partial p_{xy}}{\partial y} + \frac{\partial p_{xz}}{\partial z}\right) + \gamma\,(V_x' - V_x); \quad \text{etc.}$$

As div $V = 0$ the tensor p_{xx} etc. has the components:

$$(2.2) \qquad \begin{aligned} p_{xx} &= p - 2\theta\frac{\partial V_x}{\partial x} \\ p_{xy} &= -\theta\left(\frac{\partial V_x}{\partial y} + \frac{\partial V_y}{\partial x}\right) \\ p_{xz} &= -\theta\left(\frac{\partial V_x}{\partial z} + \frac{\partial V_z}{\partial x}\right), \end{aligned}$$

θ being a factor depending on the physical state of the gas; the coefficient γ measures the resistance experienced by the particle on account of the motion relative to the particles of the other vortex [1]).

Substitution of the vortex equation gives the equations between ω, p, ρ and between ω', p', ρ' as in the previous section; two new equations of condition arise from the internal friction terms:

$$(2.3) \qquad \begin{aligned} 4\frac{\theta}{\Delta}\frac{\partial\omega}{\partial\Delta} + \Delta\frac{\partial}{\partial\Delta}\left(\frac{\theta}{\Delta}\frac{\partial\omega}{\partial\Delta}\right) + \frac{\partial}{\partial z}\left(\theta\frac{\partial\omega}{\partial z}\right) &= -\rho\gamma\,(\omega' - \omega) \\ 4\frac{\theta'}{\Delta}\frac{\partial\omega'}{\partial\Delta} + \Delta\frac{\partial}{\partial\Delta}\left(\frac{\theta'}{\Delta}\frac{\partial\omega'}{\partial\Delta}\right) + \frac{\partial}{\partial z}\left(\theta'\frac{\partial\omega'}{\partial z}\right) &= -\rho'\gamma'\,(\omega - \omega'). \end{aligned}$$

The solution of these four equations must be possible if two coaxial circular vortices form a particular solution of the hydrodynamical equations.

3. The possibility of the solution of these equations of condition may be investigated in this way: choose ω and ω' arbitrarily; then the equations (1.2) determine p, ρ and p', ρ'; however as p, ρ and p', ρ' must correspond to the same temperature, a relation is established between ω and ω'; so only one vortex-motion may be assumed arbitrarily. As the equations (2.3) impose two new restrictions it is generally impossible to satisfy these equations. Hence generally two interacting coaxial circular vortices do not form a solution of the hydrodynamical equations.

These considerations decide the problem apart from the possibility that equations (2.3) considered as a system of partial differential equations, being homogeneous, do not allow a solution, not identically zero, satisfying the necessary boundary conditions.

[1]) Cf. ROSSELAND *l.c.* p. 446.