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COMMUNICATION FROM THE OBSERVATORY AT LEIDEN.

Line absorption and Cepheid pulsation, by *J. Woltjer Jr.*

The coefficient of opacity is one of the fundamental quantities in the theory of the stellar interior. The dependence of this quantity on density and temperature is essential in two respects: firstly, it determines to a large extent the form of the mass-luminosity curve; secondly, as recently analysed by EDDINGTON *), it determines at least partly the algebraic sign of the dissipation of energy in Cepheid pulsation: if we do not have regard to changes of the liberation of sub-atomic energy, then a proportionality with $\rho/T^{\frac{7}{2}}$ makes the dissipation positive, while reduction of the exponent of T may reduce the dissipation to zero or change it to negative values.

The purpose of this note is to investigate the part played by line absorption in accounting for the total opacity; a summary of the results may be found at the end of the paper.

1. A theoretical determination of the intensity of emission in a quantised orbit of the series electron has recently become possible by the development of the new quantum mechanics.

The intensity of emission of the lines of the Lyman series has been computed by PAULI **); in the state of the H atom characterised by the total quantum number n the emission is proportional to:

$$\frac{(n-1)^{2n-1}}{n.(n+1)^{2n+1}}$$

Hence the intensity of emission of the Lyman lines out of the quantum states 2, 3, 4 ... is proportional to:

$$1 : 0.32 : 0.136;$$

these numbers show that the absorption of the Lyman lines by the normal state of the atom decreases rapidly with line number and contradicts my computation of the opacity by line absorption in a previous paper *), that proceeded by neglecting this decrease.

The absolute values of the intensities can also be computed. I have performed the necessary calculations according to the precepts of the new quantum mechanics and restate the formulas used by me to facilitate verification. **)

The energy levels of the atom are those values of E that allow a solution, fulfilling certain conditions of regularity, of the partial differential equation:

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} + \frac{8\pi^2 m_e}{h^2}\left(E + \frac{e^2}{r}\right)\psi = 0.$$

The constants have their usual meanings. These values of E appear to be

$$-2\pi^2 \frac{m_e e^4}{h^2} \frac{1}{n^2}$$

as demonstrated by SCHRÖDINGER ***); n is a whole number, the principal quantum number. The solution ψ is a function of r multiplied by a harmonic of order k; k is one of the numbers $0, \ldots, n-1$; this solution has been given in extenso by SCHRÖDINGER ****). I only use the solutions for $n = 1, 2, 3$; they are:

$$n=1 \;\; k=0: \;\; \psi = \frac{e^{-\frac{r}{A_1}}}{A_1^{\frac{3}{2}}\sqrt{\pi}} \qquad A_1 = \frac{h^2}{4\pi^2 m_e e^2}$$

$$n=2 \;\; k=1: \;\; \psi = \frac{r e^{-\frac{r}{A_2}}}{A_2^{\frac{5}{2}}\sqrt{\pi}} \sin\theta \sin\varphi \qquad A_2 = 2A_1$$

$$\psi = \langle\,.\,.\,\rangle \sin\theta\cos\varphi$$

$$\psi = \langle\,.\,.\,\rangle \cos\theta$$

*) A. S. EDDINGTON, *The internal constitution of the stars*, Cambridge, 1926, § 137.

**) Communicated by SCHRÖDINGER: *Annalen der Physik*. Band 80 (1926) p. 437 etc.

*) *B A.N.* 82.

**) Absolute values have been given by J. R. OPPENHEIMER, *Proc. of the Cambr. Phil. Soc.* 23. I have not been able to bring his numbers in accordance with PAULI's formula; hence I have performed the necessary computations myself, not pretending however to add anything to the development of the subject.

***) E. SCHRÖDINGER, *Annalen der Physik*, 79 (1926), p. 361 etc.

****) *l. c.*

$$n=3\ k=1: \psi = \frac{e^{-\frac{r}{A_3}}}{\sqrt{6\pi}A_3^{\frac{3}{2}}}\left(4\frac{r}{A_3}-2\frac{r^2}{A_3^2}\right)\sin\theta\sin\varphi;\ A_3=3A_1$$

$$\psi = \langle \ldots\ldots\ldots\ldots\ldots \rangle \sin\theta\cos\varphi$$

$$\psi = \langle \ldots\ldots\ldots\ldots\ldots \rangle \cos\theta$$

r, θ and φ are polar coordinates: $x = r\sin\theta\cos\varphi$ etc.; the functions ψ have been multiplied by the appropriate constants so as to make the volume integral of ψ^2 extended over the whole space unity; if $k \neq 0$ the solutions are multiple according to the arbitrary constants in the harmonics; they may be distinguished by a third number m.

According to the precepts of the new quantum mechanics the emission may be computed by the classical formula if we use the real frequencies and as "amplitudes" of the "vibrations" in x, y, z corresponding to the transition from n, k, m to n', k', m' the integral:

$$\int_0^\infty dr \int_0^\pi d\theta \int_0^{2\pi} d\varphi\, r^2 \sin\theta\, \psi_{n,k,m}\, \psi_{n',k',m'}\, x,$$

for the x coordinate, and analogous for y and z.

The properties of the harmonics make this integral zero if $k=2$, $k'=0$ and if $k=0$, $k'=0$. Hence the corresponding functions ψ have been omitted.

Performing the calculations I have found the following values of the "amplitude":

$n=2,\ k=1;\ n'=1,\ k'=0$	x coordinate	0	$\frac{64}{243}\sqrt{2}A_2$	0
	y "	$\frac{64}{243}\sqrt{2}A_2$	0	0
	z "	0	0	$\frac{64}{243}\sqrt{2}A_2$
$n=3,\ k=1;\ n'=1,\ k'=0$	x "	0	$\frac{27}{256}\sqrt{2}A_2$	0
	y "	$\frac{27}{256}\sqrt{2}A_2$	0	0
	z "	0	0	$\frac{27}{256}\sqrt{2}A_2$

Hence the total emission summed for all values of k and m of the transition $n=2$ to $n=1$ is equal to:

$$\frac{2^{11}}{3^6}\pi^8\frac{e^{14}m_e^2}{c^3h^8} = 0.00773 \text{ erg per sec. per atom;}$$

of the transition $n=3$ to $n=1$:

$$\frac{8}{9}\pi^8\frac{e^{14}m_e^2}{c^3h^8} = 0.00243 \text{ erg per sec. per atom;}$$

the ratio of these numbers conforms to PAULI's formula.

If we compute the emission from the acceleration according to the classical formula we get the value:

$$\frac{2}{3}\frac{e^2}{c^3}\frac{e^4}{m_e^2}\cdot(\text{radius of the orbit})^{-4}.$$

As the radius of the second orbit is equal to:

$$\frac{4h^2}{4\pi^2 m_e e^2},$$

the classical emission is equal to:

$$\frac{2}{3}\pi^8\frac{e^{14}m_e^2}{c^3h^8}.$$

Hence the ratio

intensity according to new quantum mechanics
to " " " the classical theory

is equal to: $\frac{2^{10}}{3^5} = 4.21.$

For a nucleus with charge Ze accompanied by a single electron both intensities are to be multiplied by Z^6, so their ratio remains the same.

2. We are now in a position to revise EDDINGTON's calculations of the maximum value of the line absorption coefficient *); this value is computed by equating the emission from a second quantum orbit (calculated from the acceleration in the classical way) to the absorption.

The absorption coefficient found in this way is the arithmetic mean with weights B_ν (intensity of black body radiation).

The computation must be revised in two respects.

1°. We have the lemma: the harmonic mean **) is equal or less than the arithmetic mean computed

*) EDDINGTON *l. c.* § 166.
**) *Cf.* E. A. MILNE *M. N.* 85, p. 979.

with the same weights *). If ε_ν is the emission per unit volume per unit solid angle then

$$\varepsilon_\nu = \varkappa_\nu \rho B_\nu$$

ρ being the density, $\varkappa_\nu$ the mass-absorption coefficient for frequency ν.
Hence:

$$\frac{\varepsilon_\nu}{B_\nu} \frac{\partial B_\nu}{\partial T} = \varkappa_\nu \rho \frac{\partial B_\nu}{\partial T},$$

T being the temperature. Integrating with respect to ν:

$$\frac{1}{\rho} \int_0^\infty \frac{\varepsilon_\nu}{B_\nu} \frac{\partial B_\nu}{\partial T} d\nu = \varkappa \int_0^\infty \frac{\partial B_\nu}{\partial T} d\nu$$

$\varkappa$ being the arithmetic mean value with weights $\frac{\partial B_\nu}{\partial T}$.

In the case considered by EDDINGTON the left hand integral reduces to a single element corresponding to the ν that belongs to the transition from $n = 2$ to $n = 1$. EDDINGTON uses the relation:

$$\frac{1}{\rho} \int_0^\infty \varepsilon_\nu \, d\nu = \varkappa \int_0^\infty B_\nu \, d\nu$$

$\varkappa$ being the arithmetic mean value with weights B_ν. Hence the change of weight factors is effected by multiplying EDDINGTON's maximum value by the factor:

$$\frac{1}{B_\nu} \frac{\partial B_\nu}{\partial T} \int_0^\infty B_\nu \, d\nu : \int_0^\infty \frac{\partial B_\nu}{\partial T} d\nu = \frac{T}{4} \frac{\partial \log B_\nu}{\partial T}.$$

If $\frac{h\nu}{kT}$ (notation of the constants as usual) is not too small:

$$\frac{T}{4} \frac{\partial \log B_\nu}{\partial T} = \frac{1}{4} \frac{h\nu}{kT}.$$

As EDDINGTON's maximum value is determined by the maximum of $y^4 e^{-y}$ $\left(y = \frac{h\nu}{kT}\right)$, so now the maximum is derived from the maximum of $\frac{1}{4} y^5 e^{-y}$; the ratio of the last quantity to the first is 1.12. The correction is of rather small importance.

2°. The rate of emission according to the new quantum mechanics must be substituted for the rate computed from the acceleration in the second quantum orbit as used by EDDINGTON. This correction amounts to the factor 4.21 as explained in the previous section.

Both corrections together raise EDDINGTON's upper limit of 14 *c.g.s.* to $14 \times 4.21 \times 1.12 = 66$ *c.g.s.*

However by fixing our attention too much on this upper limit we are apt to forget that in actual cases the contribution of line absorption to the value of the opacity inside a star may be much less; the chief factor accounting for this lessening is the deficiency of outspread of the lines for smaller frequencies in conjunction with the special properties of a harmonic mean.

3. The mass-luminosity relation points strongly to an opacity coefficient varying with density and temperature according to the law

$$\varkappa \propto \rho / T^{\frac{7}{2}}.$$

Hence with a value of $\varkappa$ of 49 *c.g.s.* at the centre of Capella corresponds a value of 2300 *c.g.s.* at the centre of Krüger 60 and of 29 *c.g.s.* at the centre of δ Cephei *).

The origin of this opacity is uncertain at present, as the amount computed from laboratory experiments on X-ray absorption does not fit in with these data **).

The discussion of the preceding section leads to the conclusion that line absorption may make it self felt for stars of high mass (low value of $\rho/T^{\frac{7}{2}}$); at the same time, as mentioned by EDDINGTON ***), its influence in stars of small mass must be quite insignificant.

Hence line absorption may affect the mass-luminosity curve in the direction of lowering it for very large masses, a result that seems not inconsistent with the empirical data.

4. The result of the preceding sections, that line absorption is ineffective for small masses but gets some influence for very large masses may have a decisive bearing on the problem of Cepheid variation.

The question how fast energy is dissipated in a pulsating star has been fully treated in EDDINGTON's recent book on the stellar interior. It appears that the law of variation of opacity with density and temperature has great importance in this respect. A law of variation according to the formula

$$\varkappa \propto \rho / T^{\frac{5}{2}}$$

may be able to allow the star a pulsation during an unlimited interval of time.

Now Cepheids are stars of large mass generally, hence of small opacity coefficient. So line absorption may take its share and change the law of variation with temperature and density.

5. EDDINGTON ****) has introduced a critical value of the ionisation potential, which may be considered as a mean value. The value of the corresponding quantity $h\nu_1/kT$ has been tabulated in his book on the stellar

*) As we wish an upper limit for ROSSELAND's absorption coefficient, we must compare with an arithmetic mean computed with ROSSELAND's weight factors: $\frac{\partial B_\nu}{\partial T}$.

*) EDDINGTON *l. c.* § 105.
**) EDDINGTON *l. c.* § 172.
***) EDDINCTON *l. c.* p. 240.
****) EDDINGTON. *M. N.* 84, p. 104.

interior, tables 31 *A* and 31 *B* (ψ_1/RT). Inspection of these tables shows at once that Cepheids are characterised by high values of $h\nu_1/kT$. A computation of this quantity for the centre of the Cepheids contained in EDDINGTON's table 25 confirms this result. (See table).

VALUES OF THE CRITICAL IONISATION POTENTIAL $\left(\frac{h\nu_1}{kT}\right)$ FOR CEPHEIDS.

Star	$h\nu_1 : kT$	Star	$h\nu_1 : kT$
l Car.	11.40	δ Cep.	8.71
Y Oph.	10.14	T Vul.	8.41
X Cyg.	9.70	SU Cyg.	7.91
ζ Gem.	9.77	RT Aur.	8.11
S Sge.	9.38	SZ Tau.	8.07
W Sgr.	8.92	SU Cas.	7.49
η Aql.	9.03	RR Lyr.	6.41
X Sgr.	9.13	Polaris	8.64
Y Sgr.	9.08	β Cep.	4.84

The corresponding value at the centre of Capella is 7.50 *).

Excepting RR Lyr. and β Cep. all values are large.

The value of $h\nu_1/kT$ will determine the amount of line absorption, as it (rather roughly) determines the part of the spectrum in which lines are preponderant. If $h\nu_1/kT$ becomes too large the line spectrum will be drawn out of the region able to contribute to ROSSELAND's opacity coefficient.

Hence *for these large values* of $h\nu_1/kT$ an increase diminishes line absorption and a decrease augments it. As $h\nu_1/kT$ is derived from the condition:

$$\frac{\rho}{T^{\frac{3}{2}}} \infty e^{-\frac{h\nu_1}{kT}}$$

the influence of line absorption (*in the Cepheids, but not generally*) changes in the same direction as $\rho/T^{\frac{3}{2}}$ (however, not necessarily proportional to the first power).

6. To have some definite example, suppose the total opacity in Cepheids in accordance with the developments of the preceding sections to obey the law:

$$x = \frac{A\rho}{T^{\frac{3}{2}}} + \frac{B\rho}{T^{\frac{7}{2}}}$$

and suppose in the non-oscillating star both terms (in a mean region) to have the same value, each $\frac{1}{2}$ x. The change δx effected by changes $\delta\rho$ and δT is:

$$\delta x = \frac{A\rho}{T^{\frac{3}{2}}}\left(\frac{\delta\rho}{\rho} - \frac{3}{2}\frac{\delta T}{T}\right) + \frac{B\rho}{T^{\frac{7}{2}}}\left(\frac{\delta\rho}{\rho} - \frac{7}{2}\frac{\delta T}{T}\right)$$

$$\frac{\delta x}{x} = \frac{\delta\rho}{\rho} - \frac{5}{2}\frac{\delta T}{T}$$

*) EDDINGTON, *M.N.* 84, p. 110.

hence formally equivalent to a change according to the law

$$x \infty \rho/T^{\frac{5}{2}}.$$

If we had supposed the part due to line absorption to vary according to a higher power of $\frac{\rho}{T^{\frac{3}{2}}}$, then a smaller amount would suffice to obtain the same formula for δx.

So we see that (*cf.* 4) the maintenance of the pulsation may be ensured by the influence of line absorption.

The conditions necessary to effect this condition of things are

1°. large mass, hence possibility for line absorption to become efficient;

2°. large values of $h\nu_1/kT$ to ensure a dependence on ρ and T as needed to maintain the pulsation.

Theoretically an exact formulation of these conditions must lead to the period-luminosity relation. I think, however, that our knowledge of the absorption processes inside a star is not yet accurate enough for this purpose.

7. *Summary.*

1°. A reconsideration of the part played by line absorption inside a star, partially made possible by results of the new quantum mechanics, has shown my former estimate of the line absorption coefficient to be much too large; on the other hand the necessity of multiplying EDDINGTON's upper limit by a factor of about 4. (2).

2°. The share of line absorption in the opacity of a star makes itself far more felt for large masses than for small ones, thus probably necessitating a lowering of the mass-luminosity curve for large masses, a result that seems rather more to be confirmed than to be contradicted by observation, though the evidence is very weak. (3).

3°. In stars that have *a*) large mass and *b*) a mean ionisation potential in the interior that is large compared with the mean kinetic energy of the particles, the law of change of opacity coefficient with temperature and density is probably able (as a consequence of the share played by line absorption) to allow the star a radial pulsation without dissipation of energy (4, 5, 6, *cf.* the reference to EDDINGTON's analysis in 4). Actually Cepheids appear to conform to these two conditions. Condition *a*) is needed to allow line absorption to become effective; condition *b*) ensures its operating in the right direction. Closer knowledge of the absorption processes inside a star may perhaps allow an exact formulation of these conditions and so lead to the period-luminosity relation.