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On the motion and the mutual perturbations of material particles in an expanding universe,

by *W. de Sitter.*

1. In an expanding universe, in which the material density and the pressure are distributed entirely homogeneously and isotropically, the line-element being

$$(1) \qquad ds^2 = R_1^2 \, (-y^2 \, d\sigma^2 + d\tau^2)\,,$$

the track of a material particle of negligible mass is a geodesic in the three-dimensional space of which the line-element is $d\sigma$, the position of the particle on its track being determined by the differential equation (for the derivation of which the reader may be referred to my Hitchcock lectures [1]), p. 187)

$$(2) \qquad \frac{d\sigma}{dy} = \frac{\eta}{\sqrt{y^2 + \eta^2}} \cdot \frac{1}{\sqrt{Q}}\,,$$

where η is a constant of integration belonging to the individual particle considered, and Q determines the value of y in the particular universe chosen, by

$$(3) \qquad \left(\frac{dy}{d\tau}\right)^2 = \frac{Q}{y^2}.$$

We have

$$(4) \qquad Q = \beta + \sqrt{y^2 + \eta_0^2} - ky^2 + \gamma y^4\,,$$

where β is the pressure of radiation, η_0 is an average of the values of η for all the particles in the universe, and represents the kinematical pressure, and $\gamma = \frac{1}{3} R_1^2 \lambda$, λ being the "cosmical constant". The equation (3) follows at once from LEMAÎTRE's equation [2]).

$$(5) \qquad \frac{1}{y^2}\left(\frac{dy}{d\tau}\right)^2 + \frac{k}{y^2} = \frac{1}{3}\,(\lambda + \varkappa\rho)$$

by means of the energy equation

$$(6) \qquad \frac{d\rho}{d\tau} + \frac{3}{y}\frac{dy}{d\tau}\,(\rho + p) = 0.$$

The three-dimensional space of line-element $d\sigma$ has constant unit curvature, $d\sigma$ being given by:

$$(7) \qquad d\sigma^2 = d\chi^2 + \chi^2\, \mathbf{s}^2(\chi)\,(d\psi^2 + \sin^2\psi\, d\theta^2)\,,$$

where we have put for convenience

$$\mathbf{s}(x) = \frac{\sin(x\sqrt{k})}{x\sqrt{k}}, \quad \mathbf{c}(x) = \cos(x\sqrt{k}), \quad k = +1, 0, -1.$$

We have thus

$$\frac{d}{dx}[x\,\mathbf{s}(x)] = \mathbf{c}(x)\,, \quad \frac{d}{dx}\,\mathbf{c}(x) = -k\,x\,\mathbf{s}(x)\,,$$

$$\mathbf{c}^2(x) + kx^2\,\mathbf{s}^2(x) = 1, \quad \mathbf{s}(2x) = \mathbf{c}(x)\,\mathbf{s}(x), \quad \text{etc.}$$

These and similar formulae enable us to use the same standard form (7) for positive, negative or zero curvature ($k = +1$, -1 or 0). For $k = 0$ we have, of course, $\mathbf{c} = \mathbf{s} = 1$.

The space of line-element $d\sigma$, with reference to which the material particles, if their mutual gravitation is neglected, have no systematic motion, may be called the "cosmical space". An observer in interpreting his observations does not refer them to this "cosmical" space, but uses his own local galilean coordinates, of which the radius vector and time are, in the neighbourhood of $\chi = 0$, $\tau = 0$, given by

$$dr = R_1 y\, d\chi\,, \qquad dt = R_1 c\, d\tau.$$

We are thus led to introduce another three-dimensional space, of which the line-element is $R_1 y\, d\sigma$, which me may call the "observer's space". Its curvature is also constant (in space), and it is

$$\varepsilon = \frac{k}{R_1^2\, y^2}\,. \qquad (k + 1, 0, -1)$$

The constant R_1, which arises in the integration of the energy equation (6), is the natural unit of

1) The astronomical aspect of the theory of relativity, by W. DE SITTER, *University of California Publications in Mathematics*, Volume 2, No. 8, pp. 143—196, Berkeley, University of California Press, 1933. See also *B. A. N.*, Vol. V, No. 193, pp. 217—218, 1930.

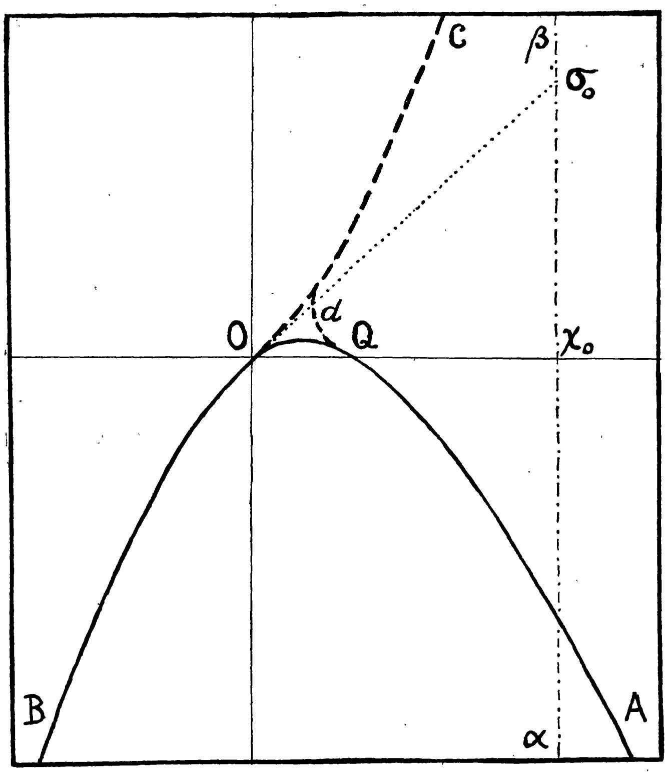
2) See e. g. *Hitchcock lectures*, p. 166. or *B. A. N.* Vol. V, Nrs. 193 and 223 (1930 and 1931).

length and time. It is probably of the order of 10^{27} cm, or 10^9 years. We will in the present article use this unit, i.e. we will put $R_1 = 1$.

The linear and areal velocities of the particle referred to the cosmical space are given by the first integrals of the differential equations for the geodesic, which are [1]).

$$(8) \qquad \varphi = \frac{d\sigma}{ds} = \frac{\eta}{y_2}, \qquad \Gamma = \chi^2 \, \mathbf{s}^2(\chi) \frac{d\theta}{ds} = \frac{\varpi}{y^2},$$

where η and $\varpi = \eta \chi_0 \, \mathbf{s}(\chi_0)$ are constants and χ_0 is the minimum radius vector of the track.



The track in the cosmical space ($d\sigma$) is thus (if we take $k = 0$, or consider such a small portion of space that the effect of the curvature is negligible) a straight line like $\alpha\beta$ in the diagram. If we take the plane of (χ, θ) through this track, we have $\psi = 90°$, $d\psi / ds = 0$, and, counting σ along the track from the point of minimum distance χ_0, we have

$$(9) \qquad \sigma = \chi_0 \tan \theta, \qquad \chi = \chi_0 \sec \theta.$$

The value of σ, i.e. of $\tan \theta$, is determined by the integration of the equation (2).

The track in the observer's space ($y \, d\sigma$) is derived from this by leaving θ unaltered and multiplying χ by y, the radius vector thus becoming $r = y\chi$. It thus becomes a curve like $A\ O\ C$, having a sharp cusp at the origin O corresponding to the point σ_0 on the track $\alpha\beta$ where the particle was at the time $\tau = 0$ for which y becomes zero. Since the sign of y, or of χ, is undetermined, the branch $O\ C$ is indistinguishable from $O\ B$, which is derived from it by changing the sign of r, or, which comes to the same thing, by adding 180° to θ, while leaving r unaltered. We then get the parabola-like curve $A\ O\ B$. The velocity dr/dt at the point O is equal to the velocity of light c, independently of the value of η, i.e. of the velocity on the "cosmical" track $\alpha\beta$ [1]).

¹) See *Hitchcock lectures*, p. 186, or *B. A. N.* 193, p. 217.

If we suppose that in the actual universe the exact value $y = 0$ is not reached, y having a finite minimum, then the real track will not be $A\ O\ B$ or $A\ O\ C$, but presumably something like $A\ d\ C$.

In the Hitchcock lectures (art. 33, pp. 186—190, fig. 4) some examples have been computed for different universes, i.e. different values of η_0, k and γ (β was taken zero throughout), and different values of η, χ_0 and of the value of y corresponding to $\sigma = 0$, $\chi = \chi_0$. The curve $A\ O\ C$ of the diagram in the present paper is actually the curve II of the Hitchcock lectures, but the track $\alpha\beta$ has for the sake of clearness been drawn much too near the origin. Its correct position is at a distance from O equal to 25 times the distance from O to the right hand edge of the diagram.

2. By assuming the line-element (1) we neglect the mutual gravitational interaction of the different individual material particles constituting the universe, and replace their combined action on any one of them by that of the whole universe filled homogeneously with matter of a certain density. It has been pointed out [2]) that this procedure, which is a good approximation if the mutual distances between the material bodies are large as compared with their dimensions, as they are at the present time in the actual universe, ceases to be an approximation when these mutual distances become very small, i.e. for very small values of y. It thus becomes important to determine at least the order of magnitude of the deviations of the actual tracks of material particles, subject to their mutual gravitational interaction, from the idealized simple tracks described in the preceding article. A rigorous, or even an approximate, determination of the field of a number of massive particles, and the motion of each of them in this field, is evidently a problem of enormous complication, surpassing the power of our

¹) See *Hitchcock lectures*, p. 186, or *B. A. N.* 193, p. 218.
²) *Hitchcock lectures*, p. 190; "On the expanding universe", *Proceedings Acad. of Sciences Amsterdam*, XXXV, p. 596—607 (May 1932); *The Observatory*, June 1933, pp. 182—185; *M. N.* **93**, pp. 628—634, June 1933.

mathematical resources. We must perforce introduce simplifying assumptions.

I will assume that the line-element is of the form

$$(10) \qquad ds^2 = -y^2 e^{2\mu} d\sigma^2 + e^{2\nu} d\tau^2,$$

where y is still a function of τ alone, while $d\sigma^2$ retains the form (7). This involves spherical symmetry, i.e. the active mass is restricted to a sphere, which may be of finite radius, round the origin, the gravitational action of the rest of the universe being still replaced by that of a homogeneous and isotropic distribution of density, as in (1). With regard to μ and ν I will assume that they are functions of the distance-variable r alone, r being defined by

$$\mathrm{r} = y . \omega$$

where ω is a function of χ alone. It would be simpler to take $\omega = \chi$, and therefore $\mathrm{r} = r$, but it appears that this is only possible for $k = 0$. Thus, although in the numerical application I will restrict myself to this simple case, I will retain the more general definition of r in the general formulas.

We denote differential quotients with respect to τ by dots, and with respect to r by accents. Further we put

$$\frac{d\omega}{d\chi} = \omega', \qquad \frac{d^2\omega}{d\chi^2} = \omega''.$$

Then we have

$$\frac{\partial\mu}{\partial\chi} = y\,\omega'\,\mu', \quad \frac{\partial\mu}{\partial\tau} = \dot{y}\,\omega\,\mu', \quad \frac{\partial^2\mu}{\partial\chi^2} = y^2\,\omega'^2\,\mu'' + y\,\omega''\,\mu',$$

and similarly for ν.

The Christoffel symbols become

$$\left\{{11 \atop 1}\right\} = y\,\omega'\,\mu', \quad \left\{{12 \atop 2}\right\} = \left\{{13 \atop 3}\right\} y\,\omega'\,\mu' + \frac{\mathsf{c}}{\chi \mathsf{s}},$$

$$\left\{{22 \atop 1}\right\} = -\chi\,\mathsf{cs} - \chi^2\,\mathsf{s}^2\,y\,\omega'\,\mu', \quad \left\{{33 \atop 1}\right\} = \left\{{22 \atop 1}\right\} \sin^2\psi,$$

$$\left\{{23 \atop 3}\right\} = \cot\psi, \quad \left\{{33 \atop 2}\right\} = -\sin\psi\cos\psi,$$

$$\left\{{11 \atop 4}\right\} = e^{2(\mu-\nu)} (1 + \mathrm{r}\,\mu')\,y\,\dot{y},$$

$$\left\{{22 \atop 4}\right\} = \left\{{11 \atop 4}\right\} \chi^2\,\mathsf{s}^2, \quad \left\{{33 \atop 4}\right\} = \left\{{22 \atop 4}\right\} \sin^2\psi,$$

$$\left\{{14 \atop 1}\right\} = \left\{{24 \atop 2}\right\} = \left\{{34 \atop 3}\right\} = \frac{\dot{y}}{y} (1 + \mathrm{r}\,\mu'), \quad \left\{{14 \atop 4}\right\} = y\,\omega'\,\nu',$$

$$\left\{{44 \atop 1}\right\} = e^{2(\nu-\mu)} \frac{\omega'\,\nu'}{y}, \quad \left\{{44 \atop 4}\right\} = \dot{y}\,\omega\,\nu',$$

where c and s stand for $\mathsf{c}\,(\chi)$ and $\mathsf{s}\,(\chi)$ respectively.

If we take ψ = constant = $\frac{1}{2}\pi$, the equations of the geodesic become

$$\frac{d^2\chi}{ds^2} + y\,\omega'\,\mu' \left(\frac{d\chi}{ds}\right)^2 - (\chi\,\mathsf{sc} + \chi^2\,\mathsf{s}^2\,y\,\omega'\,\mu') \left(\frac{d\theta}{ds}\right)^2 +$$
$$+ 2\frac{\dot{y}}{y} (1 + \mathrm{r}\,\mu') \frac{d\chi}{ds}\frac{d\tau}{ds} + e^{2(\nu-\mu)} \frac{\omega'\,\nu'}{y} \left(\frac{d\tau}{ds}\right)^2 = 0,$$

$$\frac{d^2\theta}{ds^2} + 2\left(\frac{\mathsf{c}}{\chi\mathsf{s}} + y\,\omega'\,\mu'\right) \frac{d\chi}{ds}\frac{d\theta}{ds} + 2\frac{\dot{y}}{y} (1 + \mathrm{r}\,\mu') \frac{d\theta}{ds}\frac{d\tau}{ds} = 0.$$

The value of $d\tau/ds$ is found at once from (10). It is

$$\left(\frac{d\tau}{ds}\right)^2 = e^{-2\nu} + y^2\,e^{2(\mu-\nu)}\,\varphi^2.$$

For $\varphi = d\sigma/ds$ and $\Gamma = \chi^2\,\mathsf{s}^2\,d\theta/ds$ we find then instead of (8)

$$(11) \qquad y^4\,\varphi^2 = \eta^2\,(1 + \Delta),$$

where Δ is determined by

$$(12) \quad \Delta = -4\int (1 + \Delta) \left[\mu'\,\omega\,dy + \frac{\mu' + \nu'}{2} y\,d\omega\right] -$$
$$- 2 \int e^{-2\mu}\,y^3\,\nu' \frac{d\omega}{\eta^2},$$

and

$$(13) \qquad y^2\,\Gamma\,e^{2\mu} = \varpi.$$

For $\chi = \chi_0$ we have $\Gamma_0 = \chi_0\,\mathsf{s}\,(\chi_0)\,\varphi_0$. Therefore

$$\varpi = \eta\,\chi_0\,\mathsf{s}\,(\chi_0)\,e^{2\mu_0} \sqrt{1 + \Delta_0} \quad .$$

For the materal energy tensor I will continue to use the values

$$(14) \quad T_{ab} = -g_{ab}\,p, \quad T_{44} = g_{44}\,\rho, \quad T_{a4} = T_{4a} = 0.$$

The field-equations then become

$$\frac{G_{11}}{g_{11}} = \frac{G_{22}}{g_{22}} = \frac{G_{33}}{g_{33}} = \lambda + \frac{1}{2}\kappa(\rho - p), \quad \frac{G_{44}}{g_{44}} = \lambda - \frac{1}{2}\kappa(\rho + 3p),$$
$$G_{14} = 0.$$

The conditions $G_{14} = 0$ and $G_{11}/g_{11} = G_{22}/g_{22} = G_{33}/g_{33}$ give:

$$(15) \qquad 2\,\dot{y}\,\omega'\,[\mu' - \nu' + \mathrm{r}\,(\mu'' - \mu'\,\nu')] = 0,$$

from which we find at once (if we exclude the possibility $\dot{y} = 0$);

$$(16) \qquad 1 + \mathrm{r}\,\mu' = e^{\nu},$$

and

$$(17) \quad e^{-2\mu}\Big[\omega'^2\,(\mu'' + \nu'') + \frac{\omega''}{y} (\mu' + \nu') +$$
$$+ \omega'^2\,(\nu'^2 - \mu'^2 - 2\,\mu'\,\nu') - \frac{\mathsf{c}}{\chi\mathsf{s}} \frac{\omega'\,(\mu' + \nu')}{y}\Big] = 0.$$

LEMAÎTRE's equation is

$$\frac{1}{2}\left(\frac{G_{aa}}{g_{aa}} - \frac{G_{44}}{g_{44}}\right) = \lambda + \kappa\rho,$$

where the term G_{aa}/g_{aa} is to be summed over a from 1 to 3. This gives instead of (5):

$$(18)\quad \frac{\dot{y}^2}{y^2}e^{-2\nu}(1+\mathrm{r}\mu')^2+\frac{k}{y^2}e^{-2\mu}-\tfrac{1}{3}e^{-2\mu}\Big[\omega'^2(2\mu''+ +\mu'^2)+\frac{\mu'}{y}\Big(2\omega''+4\frac{\mathbf{C}\omega'}{\chi\mathbf{S}}\Big)\Big]=\tfrac{1}{3}(\lambda+\varkappa\rho).$$

The energy equations $\operatorname{div} T^{\alpha}_{\mu}=0$ give:

$$(19)\qquad \frac{\partial p}{\partial\chi}+y\omega'\nu'(\rho+p)=0$$

$$\frac{\partial p}{\partial\psi}=\frac{\partial p}{\partial\theta}=0,$$

$$(20)\qquad \dot{\rho}+3\,\frac{\dot{y}}{y}(1+\mathrm{r}\mu')(\rho+p)=0.$$

If instead of the variables χ and τ we introduce r and y, these become

$$(19')\qquad \frac{\partial p}{\partial\mathrm{r}}+\nu'(\rho+p)=0$$

$$(20')\qquad \mathrm{r}\frac{\partial\rho}{\partial\mathrm{r}}+y\frac{\partial\rho}{\partial y}+3(1+\mathrm{r}\mu')(\rho+p)=0.$$

The equations (15), (17), (18), (19), (20) or (16), (17), (18), (19′), (20′) serve to determine y, p, ρ, μ and ν. If we neglect squares and products of μ and ν, (17) is satisfied by

$$(21)\qquad \mu+\nu=0.$$

Then we find from (16)

$$(22)\qquad \mu=\frac{m}{\mathrm{r}},$$

where m is a constant [1]), and (18) becomes

$$(23)\qquad \frac{\dot{y}^2}{y^2}+\frac{k}{y^2}(1-\tfrac{5}{2}\mu)=\tfrac{1}{3}(\lambda+\varkappa\rho).$$

The potential (22) becomes infinite for $\mathrm{r}=0$. It corresponds to a dimensionless particle with mass $M=8\pi m/\varkappa$ at the origin. If we wish to replace this by a sphere of finite radius, we cannot use the particular solution (21) of (17), but must replace it by the general solution. Since we need only an approximate solution, we continue to neglect the second order, and we also neglect the curvature, i. e. we take $k=0$ and $\omega=\chi$ (and consequently $\omega'=1$, $\omega''=0$). The difference between $\mathrm{r}=y\omega$ and $r=y\chi$ thus disappears, and we will write r for r. We have then from (17),

$$(24)\qquad \mu'+\nu'=ar,$$

and from (15) or (16) we find then

$$(25)\qquad \mu=\frac{1}{6}ar^2+b\,,\quad \nu=\frac{1}{3}ar^2+c.$$

If we take

$$a=-\frac{3m}{r_1^{\,3}}\,,\quad b=\frac{3m}{2r_1}\,,\quad c=0\,,$$

r_1 being the radius of the sphere, then μ, ν and μ' are continuous at the boundary. It is evidently impossible to make also ν' continuous. Substituting the solution (24) in LEMAÎTRE's equation (18) we get

$$(26)\qquad \frac{\dot{y}^2}{y^2}-\frac{2}{3}a=\frac{1}{3}[\lambda+\varkappa(\rho+\rho_1)]\,,$$

where we have written $\rho+\rho_1$ for ρ, ρ_1 being the addition to the normal density of the universe corresponding to the presence of the material sphere within the radius r_1. If we could suppose that (23), which for $k=0$ does not contain μ or ν, remains true inside the sphere, we would find by subtraction

$$-2a=\varkappa\rho_1\,,$$

or, since $a=-3m/r_1^{\,3}$, $M=8\pi m/\varkappa$,

$$M=\frac{4}{3}\pi\rho_1 r_1^{\,3}.$$

3. We suppose the universe to be of the expanding type of the first kind, i.e, y decreases from infinity to zero and then increases again to infinity. The velocity of decrease and increase in the unperturbed case is given by

$$dy^2=\frac{Q}{y^2}d\tau^2.$$

Instead of τ I will use as independent variable z, defined by $dz=\sqrt{Q}\,d\tau/y$, or $dz=\mp dy$, the upper sign being used before and the lower sign after the minimum $y=0$. We have thus $z=\mp y$, y being the unperturbed value. Since we are only interested in the events very near the origin of time and space, we can as a good approximation neglect the curvature

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[1]) In a remarkable and valuable paper: "The mass-particle in an expanding universe", read in the April meeting of the Royal Astronomical Society (*M. N.* **93**, pp. 325—339, 1933), Dr. MC VITTIE gives a rigorous solution of the equations (15) and (17), to which his equations (10) and (11) are equivalent. The correspondence of the notations is:

MCVITTIE:	r	R^2	β	ζ	ν	μ	$\frac{\partial}{\partial r}$
present paper:	$\frac{\chi\mathbf{s}(\frac{1}{2}\chi)}{\mathbf{c}(\frac{1}{2}\chi)}$	$kR_1^2=k$	$2\lg y$	2ν	$2\lg y+2\mu+4\lg\mathbf{c}(\frac{1}{2}\chi)$	$\frac{m}{y}$	$\mathbf{c}^2(\frac{1}{2}\chi)\frac{\partial}{\partial\chi}$

MC VITTIE's solution is

$$\omega=\chi\mathbf{s}(\tfrac{1}{2}\chi)\,,\ \omega'=\mathbf{c}(\tfrac{1}{2}\chi)\,,\ \omega''=-\tfrac{1}{4}k\omega\,,$$
$$\mu=2\lg(1+\gamma)\,,\ \nu=\lg(1-\gamma)-\lg(1+\gamma)\,,$$

where $\gamma=m/2\mathrm{r}$. It will be seen that this agrees with the solution given above to the first order of γ. The equation (18) becomes

$$\frac{\dot{y}^2}{y^2}+\frac{k}{y^2}(1+\gamma)^{-5}=\frac{1}{3}(\lambda+\varkappa\rho),$$

which is the same as (23) to the first order.

of space and the influence of the cosmical constant. Also the influence of the pressure of radiation is considered as negligible. We take thus $\beta = 0$, $k = 0$, $\gamma = 0$. Then, if we put

$$\tan\psi = \frac{y}{\eta_0}, \qquad \tan\upsilon = \frac{y}{\eta},$$

we have in the "unperturbed" motion, by (2)

$$\sigma = \chi_0 \tan\theta, \qquad \chi = \chi_0 \sec\theta,$$

$$(27) \qquad d\sigma = S\,dz, \quad d\chi = S\sin\theta\,dz, \quad d\theta = \frac{S\cos^2\theta}{\chi_0}\,dz,$$

where

$$(28) \qquad S = \frac{\cos\upsilon\cos^{\frac{1}{2}}\psi}{\sqrt{\eta_0}}.$$

It has been found convenient to continue to denote the "unperturbed" angle θ by the symbol θ. The angle θ is thus defined by the formulas (27). The "perturbed" value then is $\theta + \delta\theta$. For χ, on the contrary, we denote the "perturbed" value by χ, so that $\chi = \chi_0 \sec\theta + \delta\chi$.

In the "perturbed" motion $d\sigma$ and $d\theta$ are given by (11), (12) and (13) and $d\chi$ must be derived from these. We will neglect second orders of μ, ν and of the "perturbations" $\delta\chi$, $\delta\theta$. From (12) we have then at once

$$(29) \quad d\Delta = \pm\left[4\chi_0\mu'\sec\theta + 2(\mu' + \nu'\sec^2\upsilon)\,Sz\sin\theta\right]dz.$$

If we change the sign of sin θ and cos θ after passing through the minimum $y = 0$, i.e. if we choose for the "unperturbed orbit" AOB in the diagram, instead of AOC, the sign $\pm$ in (29) must be dropped.

From

$$ds^2 = \frac{e^{2\nu}\,d\tau^2}{1 + y^2\,e^{2\mu}\varphi^2}$$

we find, replacing $d\tau$ by its value in terms of dz,

$$(30) \qquad \frac{\eta\,ds}{y^2} = \left[1 + \nu - (\mu + \tfrac{1}{2}\Delta)\cdot\cos^2\upsilon\right] S\,dz.$$

Since

$$d\theta = \frac{\Gamma ds}{\chi^2} = \frac{\Gamma\cos^2\theta\,ds}{\chi_0^2}\left(1 - 2\frac{\delta\chi}{\chi}\right),$$

we find from (13), using the value (30) of $\eta\,ds/y^2$,

$$d\theta = \frac{S\cos^2\theta\,dz}{\chi_0}\left[1 - 2(\mu-\mu_0) - \mu\cos^2\upsilon + \nu - \right.$$
$$\left. - \frac{1}{2}(\Delta\cos^2\upsilon - \Delta_0) - 2\frac{\delta\chi}{\chi}\right],$$

where μ_0 and Δ_0 are the values of μ and Δ for $\chi = \chi_0$, $\theta = 0$.

Subtracting the unperturbed value $d\theta = S\cos^2\theta\,dz/\chi_0$, we have

$$(31) \quad d\delta\theta = -\frac{S\cos^2\theta}{\chi_0}\left[2(\mu-\mu_0) + \mu\cos^2\upsilon - \nu + \right.$$
$$\left. + \frac{1}{2}(\Delta\cos^2\upsilon - \Delta_0) + 2\frac{\delta\chi}{\chi}\right]dz.$$

Further, since $d\chi^2 = \varphi^2 ds^2 - \chi^2 d\theta^2$ and $\chi^2 d\theta^2 = \Gamma^2/\chi^2$, we have from (11) and (13) by subtraction

$$d\chi^2 = \frac{\eta^2 ds^2}{y^4}\left[\sin^2\theta + 4(\mu-\mu_0)\cos^2\theta + \Delta - \Delta_0\cos^2\theta + \right.$$
$$\left. + 2\frac{\delta\chi}{\chi}\cos^2\theta\right],$$

or, taking the square root and introducing (30)

$$d\chi = S\sin\theta\,dz\left[1 + B + \frac{\delta\chi}{\chi}\cot^2\theta\right],$$

where we have written for abbreviation

$$(32) \quad \begin{aligned} B = \nu + \mu(2\cot^2\theta - \cos^2\upsilon) - 2\mu_0\cot^2\theta + \\ + \tfrac{1}{2}\Delta(\mathrm{cosec}^2\theta - \cos^2\upsilon) - \tfrac{1}{2}\Delta_0\cot^2\theta. \end{aligned}$$

Subtracting the unperturbed value $d\chi = S\sin\theta\,dz$, and remembering that $S\,dz = \chi_0\sec^2\theta\,d\theta$ by (27), we have

$$\frac{d\delta\chi}{\chi_0} = \frac{B}{\chi_0}S\sin\theta\,dz + \frac{\delta\chi}{\chi}\,\mathrm{cosec}\,\theta\,d\theta.$$

Substituting in the last term $\chi = \chi_0\sec\theta$, multiplying by cosec θ, and putting

$$\xi = \frac{\delta\chi}{\chi_0}\,\mathrm{cosec}\,\theta,$$

we find

$$(33) \qquad d\xi = \frac{B}{\chi_0}S\,\cancel{\sin\theta}\,dz,$$

where B is given by (32). After the integration we take

$$(34) \qquad \frac{\delta\chi}{\chi} = \xi\sin\theta\cos\theta.$$

Recapitulating we determine first Δ by (29), then $\delta\chi/\chi$ by (33), (34) and finally $\delta\theta$ by (31). In accordance with (21), (22) and (24), (25), the values of μ, ν, μ', ν' are as follows, r_1 being the radius of the material sphere round the origin:

for $r = y\chi_0\sec\theta > r_1$:

$$\mu = \frac{m}{r}\,, \quad \mu' = -\frac{m}{r^2}\,, \quad \nu = -\mu\,, \quad \nu' = -\mu'\,;$$

for $r < r_1$:

$$\mu = \frac{3m}{2r_1} - \frac{m}{2r_1^3}r^2\,, \quad \mu' = -\frac{m}{r_1^3}r\,,$$

$$\nu = -\frac{m}{r_1^3}r^2\,, \qquad \nu' = -\frac{2m}{r_1^3}r\,.$$

The values of $\delta\theta$ and $\delta\chi$ thus determined are the "perturbations" of the coordinates of the moving particle in the "cosmical space". For those of the coordinates θ and r in the "observer's space" we then have

$$(35)\qquad \delta r = r\frac{\delta\chi}{\chi} + \chi_0 \sec\theta\, \delta y,$$

whilst $\delta\theta$ remains the same. For the determination of δy we must consider LEMAÎTRE's equation, which in our simplified case becomes

$$(36)\qquad \frac{\dot y^2}{y^2} = \frac{1}{3}\varkappa\rho,$$

where ρ must be found from the energy equations (19) and (20) or (19′) and (20′), to which must be added an "equation of state" connecting ρ and p. If our original assumption, that the line-element is of the form (10), is admissible, it should be possible to choose this equation of state in such a way that the resulting value of y becomes independent of χ, whilst ρ and p are functions of both χ and τ, or of r and y. I have not attempted to investigate this possibility. The following analysis is, of course, not rigorous. It may be interpreted as giving, not a "radius" y, which would be valid for the whole universe, but a multiplyer to reduce the radius vector of the particular particle considered from the "cosmical" to the "observer's" space by the formula $r = y\chi$.

In the "unperturbed" case the equation of state to be used is [1])

$$3p = \frac{\eta_0^2}{y^2}\rho_0.$$

By the same reasoning which leads to this equation we are led in the perturbed case to adopt

$$(37)\qquad 3p = e^{2\mu}\frac{\eta_0^2}{y^2}\rho_0,$$

or, since $\rho = \rho_0 + 3p$,

$$\rho = \left(1 + e^{2\mu}\frac{\eta_0^2}{y^2}\right)\rho_0.$$

Then, neglecting higher orders of μ and ν, we find from (19′)

$$\frac{\partial\rho}{\partial r} + \left[7\nu' + 2\mu' + \left(4\frac{\eta_0^2}{y_0} + 3\frac{y^2}{\eta_0^2}\right)\nu'\right]\rho_0 = 0,$$

and (20′) becomes

$$(38)\qquad \frac{1}{\rho}\frac{\partial\rho}{\partial y} = \frac{a y^4 - b y^2 - c}{y^3 + e y},$$

where

$$a = \frac{3r\nu'}{\eta_0^2},\quad c = 4\eta_0^2(1 + 2\mu - r\mu' - r\nu'),$$
$$b = 3 + r\mu' - 7r\nu',\quad e = \eta_0^2(1 + 2\mu).$$

[1]) See LEMAÎTRE *B. A. N.* Vol. V, No. 200, p. 273 (1930) or *Hitchcock lectures*, p. 162.

Treating r, and therefore also μ, ν, μ', ν', as constants we find from (38)

$$\lg\rho = \text{const.} + \tfrac{1}{2}ay^2 - \frac{c}{e}\lg y + \left(\frac{c}{2e} - \frac{ae + b}{2}\right)\lg(y^2 + e),$$

or

$$(39)\qquad \rho = C\frac{\sqrt{y^2 + \eta_0^2}}{y^4}(1 + 2\alpha),$$

where

$$\alpha = \tfrac{1}{2}\mu\frac{\eta_0^2}{y^2 + \eta_0^2} + \tfrac{3}{4}r\nu'\frac{y^2}{\eta_0^2} + 2r(\mu' + \nu')\lg y - \tfrac{5}{4}r\mu'\lg(y^2 + \eta_0^2).$$

If for C we write $3R_1/\varkappa$ as in the unperturbed case, and take again $R_1 = 1$, we have by substitution in (36)

$$\delta\dot y = \alpha\dot y,$$

or

$$(40)\qquad d\,\delta y = \mp\alpha\, dz$$

The corresponding perturbation in the time is found by

$$(41)\qquad d\,\delta\tau = -\frac{y}{\sqrt{Q}}\,d\,\delta y = \pm\frac{\sin\psi\cos\psi}{S}\alpha\, dz.$$

For the greater part of the track the value of α is negative, consequently the value of $\dot y$ in the perturbed motion is numerically smaller than in the unperturbed case: y decreases less rapidly, and the value $y = 0$ is not reached.

The velocity of the particle in its track in the observer's space near the origin is

$$V = y\frac{d\sigma}{d\tau} = \frac{\eta\, e^{\nu}\sqrt{1 + \Delta}}{[y^2 + e^{2\mu}\eta^2(1 + \Delta)]^{\frac{1}{2}}},$$

of which the limit for $y = 0$ is $1 + \nu - \mu$. This is less than unity, so the velocity never reaches the velocity of light, as it does in the unperturbed case.

4. Some examples have been computed in order to get an idea of the order of magnitude of the perturbations. As unit of length and time we use R_1, for which we take 10^{27} cm. For the unit of mass we take $10^{11}\odot = 2.10^{44}$ gr. In these units the value of $\varkappa$ is very nearly

$$\varkappa = 4.10^{-10}.$$

For m I take as a convenient value

$$m = \tfrac{1}{4}.10^{-10},$$

corresponding to $M = 8\pi m/\varkappa = 1{\cdot}57.10^{11}\odot$, a rather high estimate, which has been chosen on purpose to exaggerate the perturbations. For the radius of the material sphere near the origin I take

$$r_1 = 10^{-5},$$

or roughly 3000 parsecs. For η_0 I take

$$\eta_0 = 0{\cdot}01,$$

which corresponds to an average peculiar velocity of the extragalactic nebulae at the present moment of about 300 km/sec. For the minimum distance χ_0 I take

$$\chi_0 = 0{\cdot}001,$$

which is about the average distance in the cosmical space between neighbouring galaxies.

For the velocity of the perturbed galaxy two values were used, viz:

(I) $\eta = 0{\cdot}01,$

equal to the average velocity η_0, and a relatively small velocity

(II) $\eta = 0{\cdot}001.$

For the value of τ, or of z, at which the minimum distance χ_0 is reached, also two different cases were considered, viz:

(A) $\sigma = 0$ for $z = 0$

(B) $\sigma = 0$ for $z = -0{\cdot}01.$

The case (B) is as drawn in the diagram. In the case (A) σ_0 coincides with χ_0 and Q with O. The track is then symmetrical with respect to O.

The integrations were started at $z = -{\cdot}020$, but the resulting perturbations only become appreciable much nearer the origin. The values of τ corresponding to those of z are

z	τ
± ·020	± ·001563
16	1061
12	635
8	299
4	78
0	0

The unit of τ is roughly 10^9 years.

The results of the computations are given in the following tables. The values of θ given (in degrees) are those for the track AOC. The values of $\delta\theta$ are, of course, in radians.

IA. $z_1 = \mp{\cdot}000997$

z	σ	θ	$10^5 r$	$10^7 \Delta$	$10^7 \xi$	$10^7 \frac{\delta\chi}{\chi}$	$10^7 \delta\theta$	$10^{10} \delta y$	$10^{11} \delta r$	$10^{11} \delta\tau$
∓ ·0200	∓ ·12552	∓ 89°·54	251·0	0	0	0	0	0	0	0
160	11194	89·49	179·1	— ·02	— 2·1	+ 0·02	0	+ 4	+ 4	— 4
120	9414	89·39	113·0	·10	6·6	·07	0	9	10	10
80	7026	89·18	56·22	·45	19	·27	+ 0·04	23	18	20
40	3852	88·52	15·41	2·7	72	1·9	+ ·20	59	25	38
20	1980	87·11	3·97	11·9	192	9·7	+ ·13	126	29	57
16	1590	86·40	2·55	18·8	253	15·8	— ·25	160	29	76
12	1196	85·22	1·44	33·9	356	29·5	1·6	206	30	78
8	799	82·86	0·644	86	529	65	7·4	160	17	67
4	400	75·95	·165	110	687	162	39	99	7	64
2	200	63·42	·045	111	762	305	162	94	4	63
1	100	45·0	·014	112	800	400	419	92	2	63
0	0	0	0	112	837	0	—817	90	1	63

IB. $z_1 = \overline{\mp}{\cdot}000122, \quad +{\cdot}000119$

z	σ	θ	$10^5 r$	$10^8 \Delta$	$10^6 \xi$	$10^7 \frac{\delta\chi}{\chi}$	$10^5 \delta\theta$	$10^{10} \delta y$	$10^{10} \delta r$	$10^{11} \delta\tau$
— ·0200	— ·04243	— 88°·65	84·9	0	0	0	0	0	0	0
180	3608	88·41	71·5	+ 2	— 0·1	0	+ 5	+ 2	+ 0·1	— 6
160	2885	88·01	46·2	6	·4	+ 0·1	10	11	·4	14
140	2058	87·22	28·8	13	·8	0·4	16	22	·5	27
120	1105	84·83	13·31	31	1·7	1·5	22	42	·7	46
112	682	81·66	7·72	58	2·4	3·5	24	59	·7	61
108	462	77·78	5·10	88	3·0	6·2	25	73	·6	75
104	234	66·89	2·65	173	4·3	15·5	26	100	·6	95
102	118	49·72	1·48	296	7·2	35	24	123	·6	120
101	— 59	— 30·63	1·17	397	9·7	+ 42	19	141	·6	137
100	0	0	1·00	457	13·2	0	15	163	+ 0·2	154

IB. $z_1 = -\cdot 000122,\ +\cdot 000119$ *(continued)*

z	σ	θ	$10^5 r$	$10^8 \Delta$	$10^6 \xi$	$10^7 \frac{\delta\chi}{\chi}$	$10^5 \delta\vartheta$	$10^{10} \delta y$	$10^{10} \delta r$	$10^{11} \delta\tau$
− ·0100	0	0°	1·00	+ 457	− 13·2	0	+ 15	+ 163	+ 0·2	− 154
99	+ ·00060	+ 30·80	1·15	377	16·9	− 74	18	186	− 0·6	171
98	120	50·15	1·53	263	19·5	96	29	205	1·2	188
96	241	67·50	2·51	127	22·8	81	53	229	1·5	213
92	490	78·47	4·60	+ 28	26·3	52	98	257	1·1	234
88	747	82·37	6·63	− 18	28·3	37	135	274	− 0·4	247
80	1283	85·54	10·29	53	32·4	25	201	304	+ 1·3	266
60	2766	87·93	16·61	103	40	14·5	359	343	7·1	291
40	4457	88·71	17·83	140	46	10·2	522	373	14·8	305
20	6329	89·09	12·66	196	53	8·4	695	409	25	317
10	7312	89·22	7·31	268	58	7·9	783	438	31	321
2	8109	89·29	1·62	783	112	13.7	853	495	40	323
− 1	8209	89·30	0·82	1339	119	14·4	859	486	40	323
0	8309	89·31	0	1678	119	14·3	859	443	37	323
+ 1	8409	89·32	0·84	1326	119	14·2	859	400	34	323
2	8509	89·33	1·70	774	120	14·2	860	446	38	323
10	9306	89·38	9·31	324	129	13·9	868	496	48	323
20	·10289	89·44	20·58	272	134	12·9	878	514	56	323
40	·12161	89·53	48·6	254	138	11·3	898	532	70	328
60	·13852	89·59	83·1	248	143	10·3	916	540	83	331
80	·15335	89·63	122·7	246	150	9·7	930	545	96	335
+ ·0100	+ ·16618	89·65	166·2	− 244	156	9·4	941	548	+ 107	337

IIA. $z_1 = \mp\cdot 00107$

z	σ	θ	$10^5 r$	$10^7 \Delta$	$10^7 \xi$	$10^7 \frac{\delta\chi}{\chi}$	$10^6 \delta\vartheta$	$10^{10} \delta y$	$10^{11} \delta r$	$10^{11} \delta\tau$
∓ ·0200	∓ ·03426	∓ 88°33	68·6	0	0	0	0	0	0	0
160	3271	88·25	52·4	+ 12	+ 4	− 0·1	+ ·009	+ 12	+ 4	− 15
120	3052	88·12	36·8	26	24	·8	·024	31	12	35
80	2712	87·89	21·71	45	81	3·0	·054	65	24	64
40	2078	87·55	8·32	71	248	11·9	·144	142	40	104
20	1440	86·03	2·89	86	434	30·0	·34	249	45	135
10	881	83·52	0·886	70	471	52·8	·74	353	36	156
6	569	80·03	·346	+ 4·5	394	67·2	1·40	228	15	152
4	390	75·62	·161	− 1·8	339	81·6	2·33	213	10	152
2	199	63·29	·044	3·3	273	110	5·7	208	4	152
1	100	44·94	·014	3·4	237	118	12	206	3	152
0	0	0	0	− 3·5	200	0	19	204	2	152

IIB. $z_1 = -\cdot 00040,\ +\cdot 00031$

z	σ	θ	$10^5 r$	$10^6 \Delta$	$10^5 \xi$	$10^6 \frac{\delta\chi}{\chi}$	$10^6 \delta\vartheta$	$10^9 \delta y$	$10^{10} \delta r$	$10^{10} \delta\tau$
− ·020	− ·00524	− 79°19	10·67	0	0	0	0	0	0	0
18	452	77·58	8·37	+ 27	− 0·3	+ 0·6	+ 7·4	+ 3·6	+ 0·7	− 5
16	368	74·81	6·11	65	+ 0·4	− 0·9	19·6	8·8	− 0·2	11
14	269	69·61	4·02	129	2·9	9·5	43·7	17·1	3·3	20
12	149	56·13	2·15	263	9·7	45	116	32·0	9·1	35
11	− 79	− 38·24	1·40	393	16·3	− 79	235	45·4	− 10·6	50
10	0	0	1·00	506	26·4	0	477	66·3	+ 0·7	67
9	+ 89	+ 41·70	1·20	387	39·2	+ 195	504	89·2	24·7	86
8	191	62·37	1·72	265	58·4	240	438	107	43·7	104
6	449	77·44	2·76	175	111	235	373	130	70·9	114
4	825	83·09	3·33	149	167	198	355	147	78·5	122
2	1463	86·09	2·93	139	246	168	349	164	73·1	127
− 1	2022	87·17	2·02	135	294	145	347	175	65	128
0	2902	88·03	0	121	312	108	347	199	58	129
+ 1	3783	88·49	3·79	132	321	85	347	218	115	130
2	4342	88·68	8·69	133	326	75	347	223	162	131
4	4980	88·85	19·93	133	338	68	348	227	248	132
6	5356	88·93	32·1	133	350	65	349	229	332	133
8	5614	88·98	44·9	133	363	65	350	230	429	134
+ ·010	5805	89·01	58·1	133	377	+ 65	350	231	+ 511	135

It will be seen that the "perturbations" δy are always positive; δr, though in some cases negative in the beginning, always becomes positive before the origin of coordinates is reached. The minimum distance is in the case B of the order of a few lightyears (4 in I B and 6 in II B). In the cases I A and II A, where a considerable part of the track lies inside the material sphere (galaxy) at the origin of coordinates, the minimum distance is much smaller, of the order of a few hundredths of a lightyear, but still essentially positive.

These are the perturbations of a material particle (galaxy) by *one* galaxy at the origin of coordinates. It would be rash to conclude from these to the perturbations in the actual universe, which contains uncounted millions of galaxies, but the following reasoning may be of some help.

The perturbations of the minimum distance are always positive, and therefore additive [1]). The perturbations in the time are somewhat smaller than those in δy, the ratio being of the order of one tenth in the cases considered in this paper. The deviation from simultaneity of the passing of the different galaxies near the origin will thus be less than the spreading in space. Suppose we consider a million galaxies. If these occupied the same space that is occupied by the one in our example, the perturbations would be a million times as large. They do, however, occupy a much larger space, say ten times the diameter at least, and consequently the case A, which gives much smaller perturbations, will be more frequent, and the total perturbations will be considerably smaller than a million times that by one galaxy. If a still larger number is considered this same reasoning applies with still more force, and the total perturbation will fall still further below the simple sum.

[1]) The estimate made *M. N.* **93**, p. 634, that the total perturbation will be proportional to the square root of the number of galaxies, is therefore not correct. It will be much larger.