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ON THE STABILITY OF A HELICAL MAGNETIC FIELD IN A SPIRAL ARM

G. SETTI*

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The hydromagnetic stability of a spiral arm in which the field lines are helices around the axis is considered. As a simplified model an infinite cylinder with a longitudinal magnetic field on the inside and an azimuthal one on the outside is investigated. Both the incompressible and the compressible cases are discussed. It is found that the gravitational field given by the distribution of the stars in a spiral arm may stabilize the configuration, even for a magnetic field intensity of the order of 3×10^{-5} gauss. A qualitative discussion is given on the stability of the actual toroidal configuration.

1. Introduction

CHANDRASEKHAR and FERMI (1953a) derived a model of a spiral arm with a general magnetic field of the order of 7×10^{-6} gauss running almost parallel to the axis. Discussing the polarization of starlight SHAJN (1956) has pointed out the presence of a local magnetic field in Sagittarius inclined to the galactic equator. Ireland has confirmed the Shajn results and suggested in a recent paper (IRELAND, 1961) that the data on interstellar polarization of starlight are favourable to a helical model of the magnetic field in the spiral arms. Now there are many arguments which indicate a magnetic field intensity B of about 3×10^{-5} gauss in the spiral arm near the Sun (WOLTJER, 1962a). It had been shown by WOLTJER (1962b) that with such a value of B only helical fields can satisfy the equilibrium condition in the z-direction where the gravitational attraction is small. Starting from these considerations we can visualize a real model in which the field lines are parallel near the axis of a spiral arm and gradually become more helical in the outer regions, so that the magnetic field is largely force-free in the z-direction. Then the problem arises whether or not such a configuration can be stable from the hydromagnetic standpoint. The azimuthal component of the magnetic field tends to cause "pinch"-type instabilities. In fact, if we outline a spiral arm as an infinite cylinder with a toroidal magnetic field on the outside, then it is well known that such a system is unstable to perturbations of a sufficient long wavelength. But the gravitational field given by the distribution of the stars in a spiral arm may improve the situation. The investigation of the stabilizing effect of such a gravitational field in the geometry of the infinite cylinder will be the object of this paper. Clearly the results could have a direct meaning only for perturbations transverse to the galactic plane, because the gravitational force in the galactic plane is not determined by the local distribution of matter. For radial perturbations one has to consider that actually a spiral arm is a ring in the galactic plane and therefore a torus constitutes the proper geometry of the problem. Then, if angular momentum is conserved, the gravitational force due to the main body of the Galaxy may provide the restoring force. However, the analysis of the toroidal configuration raises great mathematical difficulties. We shall return to this problem only in the concluding remarks.

We investigate as a simplified model the stability of an infinitely long cylinder (radius ϖ_0) of ideally conducting plasma subject to the action of an axisymmetric gravitational field of the form

$$\Phi = g\varpi^2 \ , \qquad (1)$$

where $g = \pi G\rho^+$, G is the gravitational constant and ρ^+ is the mean stellar density in a spiral arm. The equation (1) is derived on the assumption that ρ^+ is constant through the cylinder. The magnetic field lines are longitudinal inside the fluid and azimuthal in the

* Osservatorio Astronomico di Roma and Laboratorio di Astrofisica, Frascati, Roma, Italy.

surrounding empty space. The two regions are separated by a current sheet.

It may be noted that if one considers a uniform flow of the fluid, it is always possible in the geometry of an infinite cylinder to transform to an equivalent frame at rest with the fluid. We thus assume that the fluid is at rest in the equilibrium state.

If a mean magnetic field of the order of 10^{-5} gauss is considered, then the pressure associated with random cloud motions in a spiral arm turns out to be about three orders of magnitude less than the magnetic pressure, and therefore it may be considered as negligible. It will be shown that that is true only for the incompressible fluid, while in the compressible configuration the hydrostatic pressure, even if small, plays an essential role. As the density of interstellar matter is only a few per cent of the mean star density we shall neglect the self-gravitation of the fluid. It may be noted that the self-gravitation leads to the well known gravitational instability of the infinite cylinder (CHANDRASEKHAR and FERMI, 1953b), in which we are not interested in this paper. However, it may be anticipated that also in this case the gravitational field (1) increases the stability. This point will be discussed further on.

In section 2 we shall discuss the case of an incompressible fluid by means of a normal mode technique. It will be possible to find a complete solution.

The analysis of the compressible fluid encounters analytical difficulties. In section 3 we shall discuss an approximation method in which the gravitational field will be taken as weak. Comparing the results obtained through this perturbation method for both the incompressible and the compressible case, we shall get some information about the stabilizing effect of the gravitational field on the compressible fluid.

Section 4 will be devoted to the examination of two somewhat different configurations. The results clarify and complete the analysis developed in the previous sections. The linearized hydromagnetic equations which will be used later on are

$$\rho \frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} = \frac{1}{4\pi}\left[(\nabla \times \delta\mathbf{B})\times\mathbf{B}+(\nabla\times\mathbf{B})\times\delta\mathbf{B}\right]-\nabla\delta p-\delta\rho\nabla\Phi, \quad (2)$$

$$\nabla\cdot\delta\mathbf{B} = 0, \quad (3)$$

$$\delta\mathbf{B} = \nabla\times(\boldsymbol{\xi}\times\mathbf{B}), \quad (4)$$

$$\delta\rho = -\nabla\cdot(\rho\boldsymbol{\xi}), \quad (5)$$

$$\delta p = -\gamma p\nabla\cdot\boldsymbol{\xi}-\boldsymbol{\xi}\cdot\nabla p, \quad (6)$$

where $\boldsymbol{\xi}$ is a small displacement applied to the equilibrium configuration, $\mathbf{B}$ the magnetic field, p the hydrostatic pressure, ρ the fluid density, and γ the ratio of specific heats. For an incompressible fluid one has to add the equation

$$\nabla\cdot\boldsymbol{\xi} = 0. \quad (7)$$

As the equilibrium configuration has cylindrical symmetry we suppose that all perturbed quantities depend on t, φ and z through a factor exp $i(\sigma t + m\varphi + kz)$ only.

We derive from KRUSKAL and SCHWARZSCHILD (1954) the boundary conditions which apply at the interface between the fluid and the vacuum

$$[\Pi] = 0, \quad (8)$$

$$\mathbf{n}\cdot\mathbf{B}^{\text{in}} = \mathbf{n}\cdot\mathbf{B}^{\text{ex}} = 0, \quad (9a,\ b)$$

where $\Pi = p+|\mathbf{B}|^2/8\pi$ is the total pressure, $[\Pi]$ the discontinuity of Π, $\mathbf{B}^{\text{in}}$ and $\mathbf{B}^{\text{ex}}$ the magnetic fields inside and outside the fluid, and $\mathbf{n}$ the unit vector normal to the interface.

2. Incompressible fluid

Let us consider a fluid with uniform density ρ and let the magnetic fields be given by

$$\mathbf{B}^{\text{in}} = (0, 0, b), \qquad \mathbf{B}^{\text{ex}} = \left(0, h\frac{\varpi_0}{\varpi}, 0\right), \quad (10a,\ b)$$

where b and h are constants.

With equation (1) we have for the pressure in the static state

$$p = p_c - g\rho\varpi^2. \quad (11)$$

From equations (2), (4) and (7) it follows that

$$\rho(\sigma^2-k^2V_A^2)\boldsymbol{\xi} = \nabla\delta\Pi, \qquad \nabla^2\delta\Pi = 0, \quad (12a,\ b)$$

where V_A is the Alfvén velocity and $\delta\Pi$ the perturbed total pressure. As the fluid is incompressible and the gravitational field not perturbed no explicit gravitational term occurs in (2). It enters through the boundary conditions. The solution of equation (12b) which is finite at $\varpi = 0$ is

$$\delta\Pi = AI_m(x)\exp i(\sigma t + m\varphi + kz), \quad (13)$$

where $x = |k|\varpi$, $I_m(x)$ is the modified Bessel function of the first kind of order m, and A is a constant.

The perturbation equations for the vacuum field are

$$\nabla\times\delta\mathbf{B}^{ex} = 0, \qquad \nabla\cdot\delta\mathbf{B}^{ex} = 0, \quad (14a, b)$$

hence

$$\delta\mathbf{B}^{ex} = \nabla\psi, \qquad \nabla^2\psi = 0, \quad (15a, b)$$

and the desired solution which tends exponentially to zero as $\varpi \to \infty$ is

$$\delta\mathbf{B}^{ex} = C\nabla[K_m(x)\exp i(\sigma t + m\varphi + kz)], \quad (16)$$

where $K_m(x)$ is a modified Bessel function of the second kind of order m, and C is a constant.

Applying (8) and (9a, b) to the deformed boundary and developing everything to the first order we find

$$\left[\delta\Pi(\varpi_0) + \left(\frac{d\Pi}{d\varpi}\right)_{\varpi_0}\xi_\varpi(\varpi_0)\right] = 0, \quad (17)$$

$$\delta B_\varpi^{ex}(\varpi_0) - \frac{imh}{\varpi_0}\xi_\varpi(\varpi_0) = 0, \quad (18)$$

while equation (9a) is satisfied identically. Then making use of the equations (11), (12a), (13) and (16) one obtains a homogeneous system of equations in A and C which has a non-trivial solution only if

$$V_A^{-2}\varpi_0^2\sigma^2 = x_0^2 + x_0\left(2g\varpi_0^2V_A^{-2} - \frac{h^2}{b^2}\right)\frac{I_m'(x_0)}{I_m(x_0)} - m^2\frac{h^2 I_m'(x_0)K_m(x_0)}{b^2 I_m(x_0)K_m'(x_0)}, \quad (19)$$

where the prime means the derivation with respect to the total argument. For all positive values of x_0: $I_m'(x_0)/I_m(x_0) > 0$ and $K_m(x_0)/K_m'(x_0) < 0$. Therefore the gravitational field actually gives a positive contribution. The only term which contributes towards instability is $-x_0h^2b^{-2}I_m'(x_0)/I_m(x_0)$.

Clearly in the equilibrium state the magnetic fields must satisfy the condition $h \geq b$. Let us consider the case $h = b$ which is the force-free case. If $m = 0$, equation (19) becomes

$$V_A^{-2}\varpi_0^2\sigma^2 = x_0[x_0 - I_1(x_0)/I_0(x_0)] + 2gx_0\varpi_0^2V_A^{-2}I_1(x_0)/I_0(x_0). \quad (20)$$

From the properties of the Bessel functions it is easily seen that $I_1(x_0)/I_0(x_0) < \frac{1}{2}x_0$, so that the first term on the right-hand side of equation (20) is positive. Thus the system is stable to all axisymmetric perturbations even in the absence of the gravitational field.

Let us next consider the case $m = 1$. Let $\rho = 2 \times 10^{-24}$ g/cm^3, $\rho^+ = 2 \times 10^{-23}$ g/cm^3, $\varpi_0 = 150$ parsecs and $b = h = 3 \times 10^{-5}$ gauss be the physical quantities characteristic of a spiral arm. Then we find the results summarized in table 1, which show that the configuration is stable to all perturbation wavelengths. The stabilizing effect of the gravitational field is apparent.

Let us consider the modes $m \geq 2$. From the recurrence formulae of the Bessel functions (WATSON, 1958) we obtain

$$-\frac{I_m'(x_0)}{I_m(x_0)}\left(x_0 + m^2\frac{K_m(x_0)}{K_m'(x_0)}\right) > x_0\frac{I_m'(x_0)}{I_m(x_0)}\left(-x_0 + \frac{m^2}{x_0+m}\right) \geq 0 \quad (21a)$$

TABLE 1

The values of the various terms in equation (19) as function of x_0 ($m=1$)

x_0	x_0^2	x_0I_1'/I_1	$\frac{I_1' K_1}{I_1 K_1}$	$2g\varpi_0^2x_0V_A^{-2}\frac{I_1'}{I_1}$	$V_A^{-2}\varpi_0^2\sigma^2$
0.1	0.01	1.002	− 0.972	0.050	0.030
0.2	0.04	1.010	− 0.939	0.050	0.019
0.3	0.09	1.022	− 0.899	0.051	0.018
0.4	0.16	1.039	− 0.862	0.052	0.035
0.5	0.25	1.060	− 0.827	0.053	0.070

for all $x_0 \leq m$, and

$$x_0\left(x_0 - \frac{I'_m(x_0)}{I_m(x_0)}\right) > x_0(x_0-1)-m \geq 0 \quad (21b)$$

for all $x_0 \geq m$. Thus all modes $m \geq 2$ are stable even in the absence of the gravitational field.

3. Compressible fluid

3.1. *Solution of the compressible cylinder when $g = 0$*

We shall first discuss the case of an axisymmetric compressible fluid in the absence of a gravitational field, as we shall need this later on. The geometry of the problem is the same as before, so that in the static state our assumptions are: $\Phi = 0$, p and ρ are constant, and the magnetic fields are given by (10a, b).

From equations (2), (4) and (5) we obtain

$$\rho(\sigma^2 - k^2 V_A^2)\xi_\varpi = \frac{\partial}{\partial\varpi}\delta\Pi\,,$$

$$\rho(\sigma^2 - k^2 V_A^2)\xi_\varphi = \frac{im}{\varpi}\delta\Pi\,, \qquad (22)$$

$$\rho\sigma^2\xi_z = \rho V_A^2 \frac{ik}{\varpi}\left[\frac{\partial}{\partial\varpi}(\varpi\xi_\varpi) + im\xi_\varphi\right] + ik\delta\Pi\,.$$

Taking the divergence of equation (22) and making use of equations (4), (6) and of the third component of (22) we have

$$\frac{1}{\varpi}\frac{\partial}{\partial\varpi}\left(\varpi\frac{\partial}{\partial\varpi}\delta\Pi\right) - \left(\frac{m^2}{\varpi^2} + k^2F\right)\delta\Pi = 0\,, \quad (23)$$

where

$$F = 1 - \left(\frac{\sigma^2}{k^2}\right)^2\left[(V_S^2 + V_A^2)\frac{\sigma^2}{k^2} - V_S^2V_A^2\right]^{-1}, \quad (24)$$

with V_S the sound velocity.

The solution of equation (23) which is finite at the origin is

$$\delta\Pi = AJ_m(ixF^{\frac{1}{2}})\exp i(\sigma t + m\varphi + kz)\,, \qquad (25)$$

where $x = |k|\varpi$ and J_m is the Bessel function of order m, the argument of which can be real or imaginary according to the sign of F. $\delta\mathbf{B}^{ex}$ is again given by equation (16). Making use of the boundary conditions (8) and (9a, b) we obtain the following dispersion equation

$$\frac{J_m(ix_0F^{\frac{1}{2}})}{J'_m(ix_0F^{\frac{1}{2}})} + \frac{ih^2F^{\frac{1}{2}}}{4\pi\rho\varpi_0^2(\sigma^2 - k^2V_A^2)}\left[x_0 + m^2\frac{K_m(x_0)}{K'_m(x_0)}\right] = 0\,. \qquad (26)$$

The solution of equation (26) for any possible pair of the parameters k and m gives an infinite set of real eigenvalues $\sigma^2_{n,km}$. It is usually assumed that the corresponding set of eigenvectors $\boldsymbol{\xi}_{n,km}$ is complete. This property will be used in the development of the perturbation theory. Let us consider the occurrence of negative σ_n^2 (instability). From expression (24) it follows that $F > 0$, so that the argument of the functions J_m in (26) is imaginary. Using some properties of the Bessel functions it is possible to show that for every pair (m,k) there may be one and only one negative σ_n^2 which is a solution of equation (26). This eigenvalue has been calculated (for the mode $m = 1$) for two values of the fluid pressure: $p = 0$ and $p = 0.1\, b^2/8\pi$. All other quantities entering in (26) had the same values as in section 2. The results are given in figure 1, together with the curves for the corresponding incompressible cases obtained from equation (19) by putting $g = 0$.

It may be noted that the marginal states for the incompressible and compressible fluids coincide as for $\sigma^2 = 0$ and $g = 0$ equation (26) reduces to equation (19).

3.2. *Perturbation theory*

Formally we can represent the problem which arises from equations (2) to (6) together with the boundary conditions (8) and (9a, b) as

$$\begin{cases} \sigma^2\boldsymbol{\xi} = L(\boldsymbol{\xi}) \\ M(\boldsymbol{\xi}) = 0, \end{cases} \qquad (27)$$

where L is a self-adjoint second-order linear differential operator (HAIN *et al.*, 1957) and M applies to the interface between the fluid and the vacuum. Since L is self-adjoint the eigenvectors form an orthogonal system. It is also physically reasonable to assume that such a system is complete for every vector which satisfies the boundary conditions $M(\boldsymbol{\xi}) = 0$. If a weak gravitational field is introduced we have to the first order in g

$$\begin{cases} [\sigma_n^{(1)}]^2 \boldsymbol{\xi}_n + \sigma_n^2 \boldsymbol{\xi}_n^{(1)} = L(\boldsymbol{\xi}_n^{(1)}) + L^{(1)}(\boldsymbol{\xi}_n) \\ M(\boldsymbol{\xi}_n^{(1)}) + M^{(1)}(\boldsymbol{\xi}_n) = 0 , \end{cases} \tag{28}$$

where the index (1) marks first-order changes in the various quantities, due to the gravitational field.

We cannot directly develop the $\boldsymbol{\xi}^{(1)}$'s in terms of the unperturbed eigenvectors $\boldsymbol{\xi}_n$ as the boundary conditions in the systems (27) and (28) are different. Let us put $\boldsymbol{\xi}^{(1)} = \boldsymbol{\zeta} + \boldsymbol{\eta}$, where $\boldsymbol{\eta}$ satisfies the eigenvalue problem

$$\begin{cases} \mu^2 \boldsymbol{\eta} = L(\boldsymbol{\eta}) \\ M(\boldsymbol{\eta}) + M^{(1)}(\boldsymbol{\xi}_n) = 0 . \end{cases} \tag{29}$$

Substituting in equation (28) we obtain a system in which the $\boldsymbol{\zeta}$'s obey the boundary condition $M(\boldsymbol{\zeta}) = 0$, and thus they can be expanded in terms of the complete set of the unperturbed eigenvectors.

After multiplication by $\boldsymbol{\xi}_n^*$, the complex conjugates of $\boldsymbol{\xi}_n$, and integration over the whole configuration we obtain

$$[\sigma_n^{(1)}]^2 = [\sigma_{L,n}^{(1)}]^2 + [\sigma_{M,n}^{(1)}]^2 = \left(\int \boldsymbol{\xi}_n^* \boldsymbol{\xi}_n \, d\tau\right)^{-1} \int \boldsymbol{\xi}_n^* L^{(1)}(\boldsymbol{\xi}_n) \, d\tau + \left(\int \boldsymbol{\xi}_n^* \boldsymbol{\xi}_n \, d\tau\right)^{-1} (\mu_l^2 - \sigma_n^2) \int \boldsymbol{\xi}_n^* \boldsymbol{\eta}_l \, d\tau , \tag{30}$$

with $[\sigma_{L,n}^{(1)}]^2$ and $[\sigma_{M,n}^{(1)}]^2$ the changes in σ_n^2 due to the changes in the operator L and in the boundary conditions.

Writing the equation (30) for two different possible values of μ_l^2, μ_α^2 and μ_β^2, subtracting and developing the vector $\boldsymbol{\eta}_\alpha - \boldsymbol{\eta}_\beta$, which obeys the boundary condition $M(\boldsymbol{\eta}_\alpha - \boldsymbol{\eta}_\beta) = 0$, in terms of the $\boldsymbol{\xi}_n$'s, it is easily shown that $[\sigma_n^{(1)}]^2$ is unchanged if a different solution of the system (29) is chosen, as it should be.

3.3. *Determination of the gravitational perturbation to L*

We introduce a gravitational field of the form $\Phi = g\varpi^2$ as the perturbing quantity on the hydromagnetic system of section 3.1, where g is now a small parameter which represents a first-order perturbation. As we do not want to change the model of section 2, we assume that in the static state the perturbing gravitational field only modifies the fluid pressure, while ρ and $\mathbf{B}$ are left constant. The pressure at ϖ_0 will also be kept constant. Then we have for the change in the pressure due to the gravitational field

$$p' = g\rho(\varpi_0^2 - \varpi^2) . \tag{31}$$

So the magnetic terms in the operator L are not modified and we easily find for the change in L due to the g-term

$$L^{(1)}(\boldsymbol{\xi}) = g\{-\gamma(\varpi_0^2 - \varpi^2)\nabla\nabla \cdot \boldsymbol{\xi} + (\gamma - 1)\nabla(\varpi^2) + \nabla[\boldsymbol{\xi} \cdot \nabla(\varpi^2)]\} . \tag{32}$$

3.4. *Determination of the gravitational perturbation to M*

From the boundary conditions (8) and (9a, b), and the second component of (22) we find

$$\left(\frac{dp'}{d\varpi}\right)_{\varpi_0} \xi_\varpi(\varpi_0) + kE\xi_\varpi(\varpi_0) + \frac{\varpi_0}{im}(\sigma^2 - k^2 V_A^2)\xi_\varphi(\varpi_0) = 0, \tag{33}$$

where

$$E = \frac{h^2}{4\pi\rho x_0^2}\left[x_0 + m^2 \frac{K_m(x_0)}{K_m'(x_0)}\right] .$$

Hence

$$M(\boldsymbol{\xi}) = kE\xi_\varpi(\varpi_0) + (\sigma^2 - k^2 V_A^2)\frac{\varpi_0}{im}\xi_\varphi(\varpi_0) \tag{34}$$

and

$$M^{(1)}(\boldsymbol{\xi}) = -2g\varpi_0 \xi_\varpi(\varpi_0) . \tag{35}$$

3.5. *Computation of* $[\sigma_M^{(1)}]^2$

Here we shall derive a new expression for $[\sigma_M^{(1)}]^2$ somewhat different from that found in the end of section 3.2. First a more useful expression for the unperturbed eigenvectors $\boldsymbol{\xi}$ is derived from equations (22), (23) and (24):

$$\boldsymbol{\xi} = (\sigma^2 - k^2 V_A^2)^{-1}\left\{\frac{\partial u}{\partial \varpi}, \frac{im}{\varpi}u, \frac{ik}{\sigma^2}[k^2 V_A^2(F-1) + \sigma^2]u\right\}, \tag{36a}$$

where u is in the place of $\delta\Pi$ and the other symbols have the usual meaning. Moreover, the eigenvectors of system (29) have the same structure as those of (27), so that one can write

$$\boldsymbol{\eta} = (\mu^2 - k^2 V_A^2)^{-1} \left\{ \frac{\partial v}{\partial \varpi}, \frac{im}{\varpi} v, \frac{ik}{\mu^2} [k^2 V_A^2 (F_\mu - 1) + \mu^2] v \right\}, \tag{36b}$$

where F_μ is defined in the same way as F except for the fact that μ^2 replaces σ^2. The functions u and v obey the equations

$$\varpi u'' + u' + \left(k^2 F - \frac{m^2}{\varpi^2}\right) \varpi u = 0,$$

$$\varpi v'' + v' + \left(k^2 F_\mu - \frac{m^2}{\varpi^2}\right) \varpi v = 0. \tag{37a, b}$$

Now μ^2 is a continuous function of g, so that to the first order we have

$$[\sigma_M^{(1)}]^2 = g \left(\frac{d\mu^2}{dg}\right)_{g=0}. \tag{38}$$

Making use of equations (34), (35) and (37a, b) we can write down the boundary condition of the system (29) as a function of g

$$\Lambda(g, \mu^2) = M(\boldsymbol{\eta}) + M^{(1)}(\boldsymbol{\xi}) =$$

$$v + \frac{kE}{\mu^2 - k^2 V_A^2} \left(\frac{\partial v}{\partial \varpi}\right)_{\varpi_0} - \frac{2g\varpi_0}{\sigma^2 - k^2 V_A^2} \left(\frac{\partial u}{\partial \varpi}\right)_{\varpi_0} = 0. \tag{39}$$

The total derivative of the latter equation with respect to g gives an expression for $d\mu^2/dg$ to be substituted in equation (38)

$$\frac{d\mu^2}{dg} = -\left(\frac{\partial \Lambda}{\partial g} \bigg/ \frac{\partial \Lambda}{\partial \mu^2}\right)_{g=0}. \tag{40}$$

From equation (39) and (37a, b) it follows

$$\frac{\partial \Lambda}{\partial g} = -\frac{2igx_0 F^{\frac{1}{2}}}{\sigma^2 - k^2 V_A^2} J_m'(ix_0 F^{\frac{1}{2}}), \tag{41}$$

$$\left(\frac{\partial \Lambda}{\partial \mu^2}\right)_{g=0} =$$

$$\frac{1}{2} \frac{k^2 x_0 E}{\sigma^2 - k^2 V_A^2} \left(1 + \frac{m^2}{x_0^2 F}\right) \left(\frac{\partial F_\mu}{\partial \mu^2}\right)_{g=0} J_m(ix_0 F^{\frac{1}{2}}) + \tag{42}$$

$$\left[\frac{x_0}{2F} \left(\frac{\partial F_\mu}{\partial \mu^2}\right)_{g=0} - \frac{k^4 E}{(\sigma^2 - k^2 V_A^2)^2}\right] iF^{\frac{1}{2}} J_m'(ix_0 F^{\frac{1}{2}}).$$

But the unperturbed boundary condition $(\Lambda)_{g=0} = 0$ gives

$$J_m(ix_0 F^{\frac{1}{2}}) = -\frac{ik^2 F^{\frac{1}{2}} E}{\sigma^2 - k^2 V_A^2} J_m'(ix_0 F^{\frac{1}{2}}), \tag{43}$$

so that the expression (42) becomes

$$\left(\frac{\partial \Lambda}{\partial \mu^2}\right)_{g=0} = \left\{ \left[\frac{1}{F} - \frac{k^4 E^2}{(\sigma^2 - k^2 V_A^2)^2} \left(1 + \frac{m^2}{x_0^2 F}\right)\right] \left(\frac{\partial F_\mu}{\partial \mu^2}\right)_{g=0} - \frac{2k^4 E}{x_0 (\sigma^2 - k^2 V_A^2)^2} \right\} \tfrac{1}{2} i x_0 F^{\frac{1}{2}} J_m'(ix_0 F^{\frac{1}{2}}). \tag{44}$$

Therefore the expression of $[\sigma_M^{(1)}]^2$ is immediately obtained by substitution of (41) and (44) in equation (40).

3.6. *Computation of* $[\sigma_L^{(1)}]^2$

We begin by seeking an expression for $\int \boldsymbol{\xi}^* L^{(1)}(\boldsymbol{\xi})\, d\tau$. The equations (4), (6), (22) and (23) lead to

$$\nabla \cdot \boldsymbol{\xi} = \frac{k^2}{\sigma^2} (F - 1) u. \tag{45}$$

With this expression for $\nabla \cdot \boldsymbol{\xi}$ and integrating the equation (32) over a unit length of the cylinder, after multiplication by $\boldsymbol{\xi}^*$, we obtain

$$\int \boldsymbol{\xi}^* L^{(1)}(\boldsymbol{\xi})\, d\tau =$$

$$2\pi g \left\{ -\frac{\gamma k^2 (F-1)}{\sigma^2 (\sigma^2 - k^2 V_A^2)} \int_0^{\varpi_0} (\varpi_0^2 - \varpi^2) \left\{ u' u'^* + \frac{m^2}{\varpi^2} u u^* + \frac{k^2}{\sigma^2} [k^2 V_A^2 (F-1) + \sigma^2] u u^* \right\} \varpi\, d\varpi + \right.$$

$$2k^2 (F-1) \frac{\gamma - 1}{\sigma^2 (\sigma^2 - k^2 V_A^2)} \int_0^{\varpi_0} u u'^* \varpi^2\, d\varpi + \tag{46}$$

$$\frac{2}{(\sigma^2 - k^2 V_A^2)^2} \int_0^{\varpi_0} u' u'^* \varpi\, d\varpi +$$

$$\left. \frac{2}{(\sigma^2 - k^2 V_A^2)^2} \int_0^{\varpi_0} \left\{ u'' u'^* + \frac{m^2}{\varpi^2} u' u^* + \frac{k^2}{\sigma^2} [k^2 V_A^2 (F-1) + \sigma^2] u' u^* \right\} \varpi^2\, d\varpi \right\}.$$

The integrals are calculated in the Appendix. After straightforward but lengthy algebra the expression (46) becomes

$$\left(2\pi u u^* \Big|_0^{\varpi_0}\right)^{-1} \int \xi^* L^{(1)}(\xi)\, d\tau =$$
$$g\left\{\frac{2x_0^2}{k^4E^2} + \frac{2(F-1)(Q-x_0^2)}{\sigma^2(\sigma^2-k^2V_A^2)} + \right. \tag{47}$$
$$\gamma(F-1)^2\sigma^{-4}\left[Q\left(\tfrac{5}{12}x_0^2 + \frac{m^2-1}{F}\right) + \right.$$
$$\left.\left.\frac{x_0^2}{3F}\left(1 + \frac{x_0}{k^2E}(\sigma^2-k^2V_A^2)\right)\right]\right\},$$

where

$$Q = x_0^2 - F^{-1}\left[\frac{x_0^2(\sigma^2-k^2V_A^2)}{k^4E^2} - m^2\right].$$

By the same procedure we get

$$\left(2\pi u u^* \Big|_0^{\varpi_0}\right)^{-1} \int \xi^*\xi\, d\tau =$$
$$(\sigma^2-k^2V_A^2)^{-2}\left\{-\frac{x_0}{k^2E}(\sigma^2-k^2V_A^2) + \right. \tag{48}$$
$$\left.\tfrac{1}{2}Q\left[\left(\frac{k^2}{\sigma^2}V_A^2(F-1)+1\right)^2 - F\right]\right\}.$$

As derived in section 3.2 the ratio of (47) to (48) determines $[\sigma_L^{(1)}]^2$.

3.7. *Calculation of* $[\sigma^{(1)}]^2$

Let us look at the $m = 1$ mode which tends to be the most unstable in our problem. Let us first consider the case $p = 0$. From inspection of figure 1 (a) it is seen that the lowest eigenvalue is given by $\varpi_0^2\sigma^2 = -1.2 \times 10^{12}$ for $x_0 = 0.28$. Substituting this in the expressions determined in sections 3.5 and 3.6 and using the same values of section 2 for the physical parameters involved, we get: $[\sigma_M^{(1)}]^2 = 2.09\, g$ and $[\sigma_L^{(1)}]^2 = 1.96\, g$. Thus

$$\left|\frac{[\sigma^{(1)}]^2}{\varpi_0^2\sigma^2}\right|_{\text{comp.}} = 3.38 \times 10^{-12}\, g. \tag{49}$$

We want to compare this result with the one that is valid in the case of an incompressible fluid. It follows from equation (19) that the eigenvalues are shifted because of the gravitational field by

$$[\sigma^{(1)}]^2 = 2g x_0 I_m'(x_0)/I_m(x_0). \tag{50}$$

With the same physical conditions as in the compressible case, the lowest eigenvalue for $g = 0$ is at about $x_0 = 0.26$, where $\varpi_0^2\sigma^2 = -1.16 \times 10^{12}$. From equation (50) it follows that $[\sigma^{(1)}]^2 = 2.03\, g$ and therefore

$$\left|\frac{[\sigma^{(1)}]^2}{\varpi_0^2\sigma^2}\right|_{\text{inc.}} = 1.75 \times 10^{-12}\, g, \tag{51}$$

and so the relative stabilization is twice as large in the compressible case, at least for small g.

We have made a second calculation in which $p = 0.1\, b^2/8\pi$. In this case, as shown in figure 1 (b), the lowest eigenvalue for the compressible fluid is given by $\varpi_0^2\sigma^2 = -1.48 \times 10^{12}$ at $x_0 = 0.30$. We find $[\sigma_M^{(1)}]^2 = 2.08\, g$ and $[\sigma_L^{(1)}]^2 = 2.00\, g$, and thus

$$\left|\frac{[\sigma^{(1)}]^2}{\varpi_0^2\sigma^2}\right|_{\text{comp.}} = 2.77 \times 10^{-12}\, g. \tag{52}$$

Similarly for the incompressible fluid we have $\varpi_0^2\sigma^2 = -1.42 \times 10^{12}$ at $x_0 = 0.3$, $[\sigma^{(1)}]^2 = 2.04\, g$ and thus

$$\left|\frac{[\sigma^{(1)}]^2}{\varpi_0^2\sigma^2}\right|_{\text{inc.}} = 1.44 \times 10^{-12}\, g, \tag{53}$$

and the relative stabilization is therefore about the same as in the case $p = 0$. The results should not be interpreted to mean that the compressible fluid is "more stable" than the incompressible one, because for $g = 0$ the lowest eigenvalue in the compressible case is smaller than the corresponding one in the incompressible case.

4. Discussion of two other models

In this section we investigate two different cylindrical models: one in which the fluid is considered as isothermal and the other one in which a magnetic pressure gradient wholly balances the gravitational force given by (1). Also in this case the "normal modes" analysis encounters analytical difficulties. Here the stability of the two configurations will be treated following the procedures developed by BERNSTEIN *et al.* (1958) for the energy principle technique. Again it would be very difficult to find a complete solution, but it will be possible to find some general results without solving the whole problem of stability.

It may be noted that the application of the energy principle technique to the models of sections 2 and 3 does not lead to some simple criteria that could clarify the role of the compressibility of the fluid on the stability, because the pressure and the gravitational terms are coupled.

4.1. In the previous sections we have investigated a model in which the fluid density had been kept constant throughout the configuration, while a pressure gradient balances the gravitational force. Thus implicitly a temperature gradient has been assumed. Let us now consider an isothermal fluid with $p = \kappa\rho$. The gravitational and magnetic fields are again given by equations (1) and (10a, b). Then the equilibrium pressure is given by

$$p = p_c e^{-g\varpi^2/\kappa}, \tag{54}$$

where p_c is the pressure at the axis.

Again $\delta\mathbf{B}^{ex}$ is expressed by (16), so that, making use of the boundary conditions and of equation (6), one obtains the following relation which must be satisfied at $\varpi = \varpi_0$:

$$-\gamma p \nabla\cdot\boldsymbol{\xi} + \frac{1}{4\pi}\mathbf{B}^{in}\ \delta\mathbf{B}^{in} = -\frac{h^2}{4\pi\varpi_0 x_0}\left[x_0 + m^2\frac{K_m(x_0)}{K'_m(x_0)}\right]. \tag{55}$$

From equations (2), (5) and (6) and integrating over a unit length of the undeformed cylinder, after multiplication by $\boldsymbol{\xi}^*$, we obtain

$$\sigma^2\int\rho\boldsymbol{\xi}^*\boldsymbol{\xi}\,d\tau = -\frac{1}{4\pi}\int\boldsymbol{\xi}^*\cdot[(\nabla\times\delta\mathbf{B}^{in})\times\mathbf{B}^{in}]\,d\tau - \int\boldsymbol{\xi}^*\cdot\nabla(\gamma p\nabla\cdot\boldsymbol{\xi}+\boldsymbol{\xi}\cdot\nabla p)\,d\tau - \int\boldsymbol{\xi}^*\cdot\nabla\Phi(\boldsymbol{\xi}\cdot\nabla\rho+\rho\nabla\cdot\boldsymbol{\xi})\,d\tau. \tag{56}$$

The first integral on the right-hand side can be transformed by using well known vector identities. Then, introducing the real variables

$$\xi = \xi_\varpi,\qquad \eta = \nabla\cdot\boldsymbol{\xi} - \frac{1}{\varpi}\frac{\partial}{\partial\varpi}(\varpi\xi_\varpi) = \frac{im}{\varpi}\xi_\varphi + ik\xi_z,\qquad \zeta = i\xi_\varphi \tag{57}$$

and making use of equations (5), (55) and of the equilibrium conditions one can rewrite equation (56) as

$$\begin{aligned}\sigma^2\int_0^{\varpi_0}\rho\boldsymbol{\xi}^*\boldsymbol{\xi}\varpi\,d\varpi &= \frac{1}{4\pi}\int_0^{\varpi_0}(\delta\mathbf{B}^{in})^*\cdot\delta\mathbf{B}^{in}\varpi\,d\varpi + \\ &\gamma\int_0^{\varpi_0}p\left\{\left[\eta+\frac{1}{\varpi}\frac{\partial(\varpi\xi)}{\partial\varpi}\right]-\frac{\rho}{\gamma p}\frac{d\Phi}{d\varpi}\xi\right\}^2\varpi\,d\varpi + \\ &\left(1-\frac{1}{\gamma}\right)\int_0^{\varpi_0}\frac{\rho^2}{p}\left(\frac{d\Phi}{d\varpi}\right)^2\xi^2\varpi\,d\varpi + \\ &\left\{2g\rho\varpi_0^2 - \frac{h^2}{4\pi x_0}\left[x_0+m^2\frac{K_m(x_0)}{K'_m(x_0)}\right]\right\}\xi^2(\varpi_0),\end{aligned} \tag{58}$$

where

$$(\delta\mathbf{B}^{in})^*\cdot\delta\mathbf{B}^{in} = b^2\left\{k^2(\xi^2+\zeta^2)+\frac{1}{\varpi^2}\left[\frac{\partial(\varpi\xi)}{\partial\varpi}+m\zeta\right]^2\right\}.$$

As $\gamma \geq 1$ all integrals are positive definite and the only negative contribution comes from the surface term. The configuration is unstable if, and only if, it is possible to find a set of functions ξ, η, ζ subject to the boundary condition (55) that makes the right-hand side of equation (58) negative.

If we would have applied the same procedure to the incompressible model of section 2 we would have obtained

$$\sigma^2\int_0^{\varpi_0}\rho\boldsymbol{\xi}^*\boldsymbol{\xi}\varpi\,d\varpi = \frac{1}{4\pi}\int_0^{\varpi_0}(\delta\mathbf{B}^{in})^*\cdot\delta\mathbf{B}^{in}\varpi\,d\varpi + \left\{2g\rho\varpi_0^2 - \frac{h^2}{4\pi x_0}\left[x_0+m^2\frac{K_m(x_0)}{K'_m(x_0)}\right]\right\}\xi^2(\varpi_0). \tag{59}$$

It is easily shown that the minimization of the right-hand side of equation (59) with respect to ξ and ζ immediately leads to the result of section 2, as it should do.

For arbitrary functions ξ,ζ the right-hand part of (58) is minimized with respect to η by taking

$$\eta + \frac{1}{\varpi}\frac{\partial(\varpi\xi)}{\partial\varpi} = \frac{\rho}{\gamma p}\frac{d\Phi}{d\varpi}\xi. \tag{60}$$

Then if $\gamma = 1$ the problem (58) is reduced to the problem (59) and in other cases it differs from it only by an additional term which is positive definite in ξ. Thus if the incompressible model of section 2 is stable, then the isothermal compressible configuration, which has the same fluid density at $\varpi = \varpi_0$, and the same magnetic and gravitational fields, is stable too. Of course the two models are not equivalent because of the different mass and pressure distributions. Let us consider $\gamma = 1$, and require that the total amount of fluid is the same in both systems. Due to the exponential form of equation (54) the fluid density at ϖ_0 of the isothermal configuration is lower than that of the incompressible system of section 2, and therefore it may in general be unstable, the physical reason being that the gravitational force is proportional to ϖ while in the isothermal cylinder the fluid is concentrated near the axis. But, if $\gamma > 1$, the last integral in equation (58) (the perturbed thermal energy) increases the stability.

4.2. Here we consider a model in which the gravitational field is wholly balanced by a gradient in the magnetic pressure. Therefore in the equilibrium state we have: ρ = const., p = constant, $\mathbf{B}^{\text{in}} = [0,0,b(\varpi)]$, while $\mathbf{B}^{\text{ex}}$ and Φ are again given by (10b) and (1). Then in the equilibrium state

$$b^2/8\pi = \rho g(\varpi_0^2-\varpi^2)-p+h^2/8\pi. \tag{61}$$

Making use of the boundary conditions and of equations (6), (16) and (61) one obtains the following relation that must be satisfied on the unperturbed interface

$$\gamma p \nabla\cdot\boldsymbol{\xi} + \frac{b^2}{4\pi\varpi_0}\left\{\left[\frac{\partial(\varpi\xi)}{\partial\varpi}\right]_{\varpi_0} + m\zeta\right\} = \frac{h^2}{4\pi x_0\varpi_0}\left[x_0+m^2\frac{K_m(x_0)}{K_m'(x_0)}\right]\xi(\varpi_0). \tag{62}$$

By the same procedure as followed in the preceding subsection we obtain

$$\sigma^2\rho\int_0^{\varpi_0}\boldsymbol{\xi}^*\boldsymbol{\xi}\varpi\,\mathrm{d}\varpi = \frac{1}{4\pi}\int_0^{\varpi_0}\left\{k^2(\xi^2+\zeta^2)+\frac{1}{\varpi^2}\left[\frac{\partial(\varpi\xi)}{\partial\varpi}+m\zeta\right]^2\right\}b^2\varpi\,\mathrm{d}\varpi +$$

$$\gamma p\int_0^{\varpi_0}\left[\eta+\frac{1}{\varpi}\frac{\partial(\varpi\xi)}{\partial\varpi}\right]^2\varpi\,\mathrm{d}\varpi - \tag{63}$$

$$2\rho\int_0^{\varpi_0}\left[\eta+\frac{1}{\varpi}\frac{\partial(\varpi\xi)}{\partial\varpi}\right]\xi\frac{\mathrm{d}\Phi}{\mathrm{d}\varpi}\varpi\,\mathrm{d}\varpi +$$

$$\left\{2g\rho\varpi_0^2 - \frac{h^2}{4\pi x_0}\left[x_0+m^2\frac{K_m(x_0)}{K_m'(x_0)}\right]\right\}\xi^2(\varpi_0).$$

The two integrals which contain η clearly represent the compressibility terms. It may be noted by comparing equation (63) for

$$\eta + \frac{1}{\varpi}\frac{\partial(\varpi\xi)}{\partial\varpi} = 0$$

with equation (55) that the incompressible configuration of the model here investigated is more stable than that of section 2, because of the greater magnetic energy accumulated in the longitudinal field. To find a complete solution of the compressible configuration would again be very difficult. Moreover the analysis has been complicated by the presence of interchange instabilities in which we are not interested. It is, however, worth while to consider the case $p = 0$. Then η is no longer restricted by the boundary condition (62) and it may be chosen in such a way that the last integral in equation (63) becomes arbitrarily large and negative. Thus the configuration is unstable. It may be noted that all modes come out to be unstable. Looking at the minimization problem involved in equation (63) and recalling the definition of η, it is also clear that the instability is associated with the longitudinal component of $\boldsymbol{\xi}$. Therefore it seems physically reasonable to explain the origin of the instability as due to the fact that, when the fluid pressure vanishes, there are no restoring forces against displacements parallel to the field lines, which are induced by the tangential component of the gravitational field (1).

5. Concluding remarks

For the incompressible models of sections 2 and 4 we have shown that, under the conditions prevailing in a spiral arm, the gravitational attraction supplied by the stars can effectively suppress instabilities connected with the helical component of the magnetic field.

But, obviously, the matter of which a spiral arm is made is not incompressible, and therefore the investigation of a compressible fluid deserves particular attention. Due to the mathematical difficulties we have no complete analytical solutions for a compressible model, but the results of the perturbation theory developed in section 3 strongly suggest that the stability of the incompressible model of section 2 is not substantially modified by compressibility effects. The main effect which could be relevant to our problem can be clearly understood by looking at the kink-deformed cylinder. Then the fluid particles may fall down along the perturbed magnetic lines towards regions of lower gravitational potential, and thus the stabilization due to the gravitational field (1) may be strongly reduced. As a matter of fact we have shown in section 4.2 that a configuration with a zero-pressure fluid on the inside is always unstable. This may be considered as a proof of the existence of the above mentioned effect. In section 4.1, on the other hand, it has been explicitly demonstrated that under suitable conditions a compressible cylinder may be stable. It appears that, though the compressibility contributes towards the instability, the destabilization itself depends essentially on the particular model at hand. However, let us consider in more detail the qualitative aspects of the problem. Generally, a gradient of the perturbed pressure along the field lines, that might prevent the fall of the fluid, could take place if the equilibrium pressure did not vanish. But this is possible only if the pressure is greater than a minimum value p_{m} that we can evaluate as an order of magnitude in the following manner. Let us consider in the kink-deformed cylinder a perturbed field line on the $\varphi = 0$ plane. Evaluating the first-order component of the gravitational field (1) tangent to the line and integrating over a half wavelength of the perturbation we obtain

$$p_{\mathrm{m}} \approx 4g\rho\varpi\xi_{\varpi}(\varpi). \tag{64}$$

It may noted that p_{m} does not depend on the wave number k in agreement with the remark made at the end of section 4.2. Let us give some numerical estimates. For instance, assuming $\varpi = 100$ parsecs, $\xi_{\varpi}(\varpi) = 10^{-2}\,\varpi$, and for g and ρ, the above used values, we obtain $p_{\mathrm{m}} \approx 2 \times 10^{-14}$ dyne/cm^2, which incidentally turns out to be of the same order of magnitude as the pressure associated with the random clouds motion in a spiral arm. This shows that the fluid pressure, even if small compared to the magnetic pressure, is very important in the investigation of the stability of a compressible model. Though the foregoing discussion is very crude and not conclusive it supports the idea that the compressibility of the fluid does not substantially modify the results of the incompressible models if appropriate values of the pressure in a spiral arm are considered. Which fact confirms the result obtained in section 3.

Throughout the whole analysis the self-gravity of the fluid has been neglected. Introducing the self-gravitation term in the model of section 2, and putting there $b = h = 0$, one easily finds the dispersion equation

$$\frac{\sigma^2}{4\pi g\rho} = x_0\,\frac{I'_m(x_0)}{I_m(x_0)}\left[\tfrac{1}{2}\left(1+\frac{\rho^+}{\rho}\right) - K_m(x_0)I_m(x_0)\right]. \tag{65}$$

Thus, even in this case, the gravitational attraction due to the stars increases the stability, but, like in the classical paper of CHANDRASEKHAR and FERMI (1953b), it cannot cancel the gravitational instability of an infinite incompressible cylinder. However, using the same quantities of section 2, we find that the mode of maximum instability occurs at $x_0 = 2.3 \times 10^{-3}$, with a characteristic time for break-up of the order of 3.4×10^{10} years, which is longer than the life time of the Galaxy. When compressibility is considered the problem appears to be more difficult, and in fact there is no clear understanding of the self-gravitation effects in such a case (SIMON, 1963). Anyhow it may be noted that the persistence of the gravitational instability for sufficiently long waves is strictly connected with the geometry of an infinite cylinder, but the wavelengths may be so long as to be meaningless in the case of a spiral arm.

As remarked in the Introduction, from the point of view of the stability of a spiral arm the analysis developed in the present paper is strictly valid only for perturbations perpendicular to the galactic plane. But in other cases, e.g. for perturbations on the galactic plane, one has to consider the actual shape of a spiral arm in the galactic disk. Then a restoring force could arise due to the conservation of the energy and of the angular momentum. A rough estimate on the efficiency of this mechanism may be given here. Let us assume that a spiral arm is a ring in the galactic plane in equilibrium under the action of centrifugal and gravitational forces. Now let us suppose that, due to perturbations, an element of fluid is displaced by the quantity Δ in radial direction. Then, by imposing the conservation of the angular momentum and eliminating the radial gravitational force in terms of the circular velocity Θ, we obtain a restoring force of the order of $2\rho\Theta^2\Delta/R^2$ where R is the distance from the galactic centre. It should be compared now with the gravitational attraction in the z-direction. For instance, from Oort's data (Oort, 1960) for the gravitational acceleration K_z near the Sun and for a displacement Δ = 100 pc, we find that the two forces are of the same order of magnitude. Thus it may reasonably be argued that a spiral arm could be stable to all kinds of perturbations discussed above. But this clearly is only tentative. In a refined theory the stability of a toroidal configuration with magnetic fields has to be fully investigated.

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Appendix

The reduction of the expression (46) in section 3.6 involves the following integrals

$$\int_0^{\varpi_0} uu^*\varpi\,d\varpi = \tfrac{1}{2}\left[\frac{\varpi^2}{k^2F}u'u'^* + \left(\varpi^2 - \frac{m^2}{k^2F}\right)uu^*\right]\Bigg|_0^{\varpi_0},$$

$$\int_0^{\varpi_0}\left(u'u'^* + \frac{m^2}{\varpi^2}uu^*\right)\varpi\,d\varpi = \varpi u'u^*\Big|_0^{\varpi_0} + k^2F\int_0^{\varpi_0} uu^*\varpi\,d\varpi,$$

$$\int_0^{\varpi_0}\left(u'u'^* + \frac{m^2}{\varpi^2}uu^*\right)\varpi^3\,d\varpi = \varpi^3u'u^*\Big|_0^{\varpi_0} - \varpi^2uu^*\Big|_0^{\varpi_0} + 2\int_0^{\varpi_0} uu^*\varpi\,d\varpi - k^2F\int_0^{\varpi_0}\varpi^3uu^*\,d\varpi,$$

$$\int_0^{\varpi_0} uu'^*\varpi^2\,d\varpi = \tfrac{1}{2}\varpi^2uu^*\Big|_0^{\varpi_0} - \int_0^{\varpi_0} uu^*\varpi\,d\varpi,$$

$$\int_0^{\varpi_0}\left(u''u'^*\varpi + u'u'^* + \frac{m^2}{\varpi}u'u^*\right)\varpi\,d\varpi = m^2uu^*\Big|_0^{\varpi_0} - k^2F\int_0^{\varpi_0} uu'^*\varpi^2\,d\varpi,$$

$$\int_0^{\varpi_0} uu^*\varpi^3\,d\varpi = \frac{\varpi^2}{6k^2F}[(k^2F\varpi^2 - m^2 + 2)uu^* - 2\varpi u^*u' + \varpi^2u'u'^*]\Big|_0^{\varpi_0} + \tfrac{2}{3}\frac{m^2-1}{k^2F}\int_0^{\varpi_0} uu^*\varpi\,d\varpi.$$

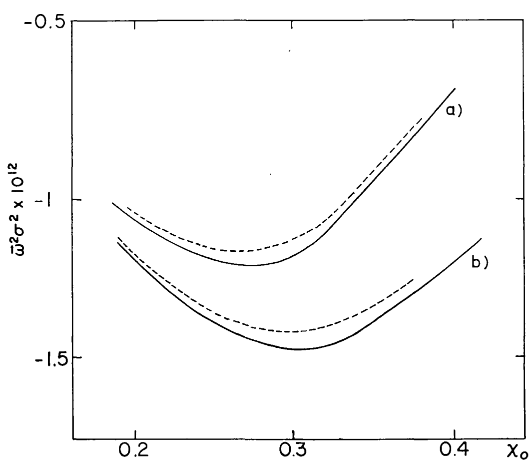

Figure 1. The dispersion relations for the unstable mode $m = 1$ and for $g = 0$: a) $p = 0$; b) $p = 0.1\ b^2/8\pi$.
– – – – – – incompressible fluid
———— compressible fluid

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