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# Decay of the Loschmidt Echo for Quantum States with Sub-Planck-Scale Structures

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Quantum states extended over a large volume in phase space have oscillations from quantum interferences in their Wigner distribution on scales smaller than $\hbar$ [W. H. Zurek, Nature (London) **412**, 712 (2001)]. We investigate the influence of those sub-Planck-scale structures on the sensitivity to an external perturbation of the state's time evolution. While we do find an accelerated decay of the Loschmidt Echo for an extended state in comparison to a localized wave packet, the acceleration is described entirely by the classical Lyapunov exponent and hence cannot originate from quantum interference.

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One common interpretation of the Heisenberg uncertainty principle is that phase-space structures on scales smaller than $\hbar$ have no observable consequence. The recent assertion of Zurek [1] that sub-Planck-scale structures in the Wigner function enhance the sensitivity of a quantum state to an external perturbation, therefore, came out as particularly intriguing [2] and even controversial [3]. His argument can be summarized as follows. The overlap (squared amplitude of the scalar product) of two quantum states $\psi$ and $\psi'$ is given by the phase-space integral of the product of their Wigner functions,

$$I_{\psi,\psi'} \equiv |\langle\psi|\psi'\rangle|^2 = (2\pi\hbar)^d \int d\mathbf{r} d\mathbf{p} W_{\psi} W_{\psi'}. \quad (1)$$

For an extended quantum state covering a large volume $A \gg \hbar^d$ of $2d$-dimensional phase space, the Wigner function $W_\psi$ exhibits oscillations from quantum interferences on a scale corresponding to an action $\delta S \simeq \hbar^2/A^{1/d} \ll \hbar$. These sub-Planck-scale oscillations are brought out of phase by a shift $\delta p$, $\delta x$ with $\delta p \delta x \simeq \delta S \ll \hbar$. The shifted state $\psi'$ is then nearly orthogonal to $\psi$ since $I_{\psi,\psi'} \approx 0$. Zurek concludes that sub-Planck structures substantially enhance the sensitivity of a quantum state to an external perturbation.

A measure of this sensitivity is provided by the *Loschmidt Echo* [4,5]

$$M(t) = |\langle\psi| \exp(iHt) \exp(-iH_0 t)|\psi\rangle|^2, \quad (2)$$

giving the decaying overlap of two wave functions that start out identically and evolve under the action of two slightly different Hamiltonians $H_0$ and $H = H_0 + H_1$. (We set $\hbar = 1$.) One can interpret $M(t)$ as the fidelity with which a quantum state can be reconstructed by inverting the dynamics with a perturbed Hamiltonian. In the context of environment-induced dephasing, $M(t)$ measures the decay of quantum interferences in a system with few degrees of freedom interacting with an environment (with many more degrees of freedom) [6]. In this case $\psi$ represents the state of the *environment*, which in general extends over a large volume of phase space. This motivated Karkuszewski, Jarzynski, and Zurek [7] to investigate the dependence of $M(t)$ on short-scale structures.

In this paper we study the same problem as in Ref. [7], but arrive at opposite conclusions. Finer and finer structures naturally develop in phase space when an initially narrow wave packet $\psi_0$ evolves in time under the influence of a chaotic Hamiltonian $H_0$ [7,8]. As in Ref. [7], we observe numerically a more rapid decay of $M(t)$ for $\psi = \exp(-iH_0 T)\psi_0$ as the preparation time $T$ is made larger and larger, with a saturation at the Ehrenfest time. However, we demonstrate that this enhanced decay is described entirely by the classical Lyapunov exponent and hence is insensitive to the quantum interference that leads to the sub-Planck-scale structures in the Wigner function.

In the case of a narrow initial wave packet, $M(t)$ has been calculated semiclassically by Jalabert and Pastawski [5]. Before discussing extended states with short-scale structures, we recapitulate their calculation. The time evolution of a wave packet centered at $\mathbf{r}_0$ is approximated by

$$\psi(\mathbf{r}, t) = \int d\mathbf{r}_0 \sum_s K_s^H(\mathbf{r}, \mathbf{r}_0, t) \psi_0(\mathbf{r}_0), \quad (3)$$

$$K_s^H(\mathbf{r}, \mathbf{r}_0, t) = C_s^{1/2} \exp[i S_s^H(\mathbf{r}, \mathbf{r}_0, t) - i\pi\mu_s/2]. \quad (4)$$

The semiclassical propagator is a sum over classical trajectories (labeled $s$) that connect $\mathbf{r}$ and $\mathbf{r}_0$ in the time $t$. For each $s$, the partial propagator is expressed in terms of the action integral $S_s^H(\mathbf{r}, \mathbf{r}_0, t)$ along $s$, a Maslov index $\mu_s$ (which will drop out), and the determinant $C_s$ of the monodromy matrix. After a stationary phase approximation, one gets

$$M(t) \simeq \left| \int d\mathbf{r} \sum_s K_s^H(\mathbf{r}, \mathbf{r}_0, t)^* K_s^{H_0}(\mathbf{r}, \mathbf{r}_0, t) \right|^2 \quad (5)$$

Squaring the amplitude in Eq. (5) leads to a double sum

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over classical paths $s$, $s'$ and a double integration over final coordinates $\mathbf{r}$, $\mathbf{r}'$. Accordingly, $M(t)$ splits into diagonal ($s = s'$, $\mathbf{r} = \mathbf{r}'$) and nondiagonal ($s \neq s'$ or $\mathbf{r} \neq \mathbf{r}'$) contributions. Since quantum phases entirely drop out of the diagonal contribution, its decay is solely determined by the classical quantity $C_s \propto \exp(-\lambda t)$. Here $\lambda$ is the Lyapunov exponent of the classical chaotic dynamics, which we assume is the same for $H$ and $H_0$. The nondiagonal contribution also leads to an exponential decay, which however originates from the phase difference accumulated when traveling along a classical path with two different Hamiltonians [5]. The slope $\Gamma$ of this decay is the golden rule spreading the width of an eigenstate of $H_0$ over the eigenbasis of $H$ [9,10]. Since $M(t)$ is given by the sum of these two exponentials, the Lyapunov decay will prevail for $\Gamma > \lambda$.

The Lyapunov decay sensitively depends on the choice of an initial narrow wave packet $\psi_0$ [11]. The faster decay of $M(t)$ resulting from the increased complexity of the initial state can be quantitatively investigated by considering prepared states $\psi = \exp(-iH_0T)\psi_0$, i.e., narrow wave packets that propagate during a time $T$ with the Hamiltonian $H_0$ [12], thereby developing finer and finer structures in phase space as $T$ increases [7,8]. The stationary phase approximation to the fidelity then reads

$$M_T(t) = \left| \int d\mathbf{r} \sum_s K_s^{H_\tau}(\mathbf{r}, \mathbf{r}_0; t+T)^* K_s^{H_0}(\mathbf{r}, \mathbf{r}_0; t+T) \right|^2 \tag{6}$$

with the time-dependent Hamiltonian $H_\tau = H_0$ for $\tau < T$ and $H_\tau = H$ for $\tau > T$.

We can apply the same analysis as in Ref. [5] to the time-dependent Hamiltonian. Only the time interval $(T, t+T)$ of length $t$ leads to a phase difference between $K_s^{H_\tau}$ and $K_s^{H_0}$, because $H_\tau = H_0$ for $\tau < T$. Hence the nondiagonal contribution to $M_T(t)$, which is entirely due to this phase difference, still decays $\propto \exp(-\Gamma t)$, independent of the preparation time $T$. We will see below that this is in agreement with a fully quantum mechanical approach according to which the golden rule decay is independent of the complexity of the initial state.

The preparation does however have an effect on the diagonal contribution $M_T^{(d)}(t)$ to the fidelity. It decays $\propto \exp[-\lambda(t+T)]$ instead of $\propto \exp(-\lambda t)$, provided $t, T \gg \lambda^{-1}$. This is most easily seen from the expression

$$M_T^{(d)}(t) = \int d\mathbf{r} \sum_s |K_s^{H_\tau}(\mathbf{r}, \mathbf{r}_0; t+T)|^2 |K_s^{H_0}(\mathbf{r}, \mathbf{r}_0; t+T)|^2, \tag{7}$$

by following a path from its endpoint $\mathbf{r}$ to an intermediate point $\mathbf{r}_t$ reached after a time $t$. The time evolution from $\mathbf{r}$ to $\mathbf{r}_t$ leads to an exponential decrease $\propto \exp(-\lambda t)$ as in Ref. [5]. Because of the classical chaoticity of $H_0$, the subsequent evolution from $\mathbf{r}_t$ to $\mathbf{r}_0$ in a time $T$ brings in an additional prefactor $\exp(-\lambda T)$.

The combination of diagonal and nondiagonal contributions results in the biexponential decay (valid for $\Gamma t$, $\lambda t$, $\lambda T \gg 1$)

$$M_T(t) = A(t)\exp(-\Gamma t) + B(t)\exp[-\lambda(t+T)], \tag{8}$$

with prefactors $A$ and $B$ that depend algebraically on time. The Lyapunov decay prevails if $\Gamma > \lambda$ and $t > \lambda T/(\Gamma - \lambda)$, while the golden rule decay dominates if either $\Gamma < \lambda$ or $t < \lambda T/(\Gamma - \lambda)$. In both regimes the decay saturates when $M_T$ has reached its minimal value $1/I$, where $I$ is the total accessible volume of phase space in units of $\hbar^d$. In the Lyapunov regime, this saturation occurs at $t = t_E - T$, where $t_E = \lambda^{-1}\ln I$ is the Ehrenfest time. When the preparation time $T \to t_E$, we have a complete decay within a time $\lambda^{-1}$ of the fidelity down to its minimal value.

We now present numerical checks of these analytical results for the Hamiltonian

$$H_0 = (\pi/2\tau_0)S_y + (K/2S)S_z^2 \sum_n \delta(t - n\tau_0). \tag{9}$$

This kicked top model [13] describes a vector spin of magnitude $S$ undergoing a free precession around the $y$ axis and being periodically perturbed (period $\tau_0$) by sudden rotations around the $z$ axis over an angle proportional to $S_z$. The time evolution after $n$ periods is given by the $n$th power of the Floquet operator

$$F_0 = \exp[-i(K/2S)S_z^2]\exp[-i(\pi/2)S_y]. \tag{10}$$

Depending on the kicking strength $K$, the classical dynamics is regular, partially chaotic, or fully chaotic. We perturb the reversed time evolution by a periodic rotation of constant angle around the $x$ axis, slightly delayed with respect to the kicks in $H_0$,

$$H_1 = \phi S_x \sum_n \delta(t - n\tau_0 - \epsilon). \tag{11}$$

The corresponding Floquet operator is $F = \exp(-i\phi S_x)F_0$. We set $\tau_0 = 1$ for ease of notation. We took $S = 500$ (both $H$ and $H_0$ conserve the spin magnitude, the corresponding phase space being the sphere of radius $S$) and calculated the averaged decay $\overline{M}_T$ of $M_T(t = n) = |\langle\psi|(F^\dagger)^n F_0^n|\psi\rangle|^2$ taken over 100 initial states.

We choose $\psi_0$ as a Gaussian wave packet (coherent state) centered on a point $(\theta, \varphi)$ in spherical coordinates. The state is then prepared as $\psi = \exp(-iH_0T)\psi_0$. We can reach the Lyapunov regime by selecting initial wave packets centered in the chaotic region of the mixed phase space for the Hamiltonian (9) with kicking strength $K = 3.9$ [9]. Figure 1 gives a clear confirmation of the predicted decay $\propto \exp[-\lambda(t+T)]$ in the Lyapunov regime. The additional decay induced by the preparation time $T$

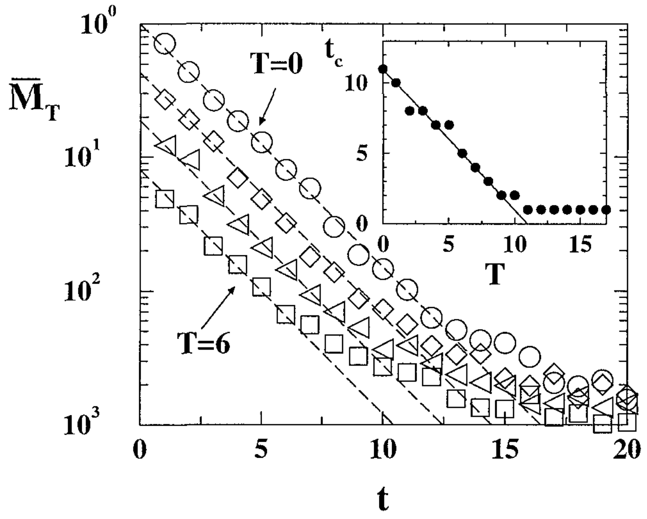


FIG 1 Decay of the average fidelity $\overline{M}_T$ for the kicked top with parameters $\phi = 1.2 \times 10^{-3}$, $K = 3.9$ and for preparation times $T = 0$ (circles), 2 (diamonds), 4 (triangles), and 6 (squares) In each case, the dashed lines give the analytical decay $\overline{M}_T = \exp[-\lambda(t + T)]$, in the Lyapunov regime with $\lambda = 0.42$ Inset threshold time at which $\overline{M}_T(t_c) = M_c = 10^{-2}$ The solid line gives the analytical behavior $t_c = -\lambda^{-1} \ln M_c - T$

can be quantified via the time $t_c$ it takes for $\overline{M}_T$ to reach a given threshold $M_c$ [7] We expect

$$t_c = -\lambda^{-1} \ln M_c - T, \tag{12}$$

provided $M_c > 1/I = 1/2S$ and $T < -\lambda^{-1} \ln M_c$ In the inset of Fig 1 we confirm this formula for $M_c = 10^{-2}$ As expected, $t_c$ saturates at the first kick ($t_c = 1$) when $T \simeq -\lambda^{-1} \ln M_c < t_E$ Numerical results qualitatively similar to those shown in the inset to Fig 1 [14] were obtained in Ref [7] and interpreted there as the accelerated decay resulting from sub-Planck-scale structures The fact that our numerical data are described so well by Eq (12) points to a classical rather than a quantum origin of the decay acceleration Indeed, Eq (12) contains only the classical Lyapunov exponent as a system dependent parameter, so that it cannot be sensitive to any fine structure in phase space resulting from quantum interference

We next illustrate the independence of $M_T(t)$ on the preparation time $T$ in the golden rule regime, i e , at larger kicking strength $K$ when $\lambda > \Gamma$ [9] As shown in Fig 2, the decay of $M_T(t)$ is the same for the four different preparation times $T = 0$, 5, 10, and 20 We estimate the Ehrenfest time as $t_E \approx 7$, so that increasing $T$ further does not increase the complexity of the initial state

These numerical data give a clear confirmation of the semiclassical result (8) Previous investigations have established the existence of five different regimes for the decay of $M(t)$ [4,5,9,10,15], and since only two of them are captured by the semiclassical approach used above, we now show that short-scale structures do not affect the remaining three The five regimes correspond to different decays (i) Parabolic decay, $M(t) = 1 - \sigma^2 t^2$, with $\sigma^2 \equiv \langle\psi_0|H_1^2|\psi_0\rangle - \langle\psi_0|H_1|\psi_0\rangle^2$, which exists for any perturbation strength at short enough times (ii) Gaussian decay, $M(t) \propto \exp(-\sigma^2 t^2)$, valid if $\sigma$ is much smaller than the level spacing $\Delta$ (iii) Golden rule decay, $M(t) \propto \exp(-\Gamma t)$, with $\Gamma \simeq \sigma^2/\Delta$, if $\Delta < \Gamma < \lambda$ (iv) Lyapunov decay, $M(t) \propto \exp(-\lambda t)$, if $\lambda < \Gamma$ (v) Gaussian decay, $M(t) \propto \exp(-B^2 t^2)$, if $H_1$ is so large that $\Gamma$ is larger than the energy bandwidth $B$ of $H$

All these regimes except regime (iv) can be dealt with quantum mechanically under the sole assumption that both $H_0$ and $H$ are classically chaotic, using random matrix theory (RMT) [16] Both sets of eigenstates $|\alpha\rangle$ of $H$ (with $N$ eigenvalues $\epsilon_\alpha$) and $|\alpha_0\rangle$ of $H_0$ (with $N$ eigenvalues $\epsilon_\alpha^0$) are then rotationally invariant [17] Expanding $\psi = \sum_\alpha \psi_\alpha |\alpha\rangle$ and assuming unbroken time-reversal symmetry, the fidelity (2) can be rewritten as

$$M(t) = \sum_{\alpha\beta\gamma\delta} \psi_\alpha \psi_\beta \psi_\gamma \psi_\delta \langle\alpha| \exp(-iH_0 t)|\beta\rangle\langle\gamma| \exp(iH_0 t)|\delta\rangle \exp[i(\epsilon_\alpha - \epsilon_\delta)t] \tag{13}$$

RMT implies the $\psi$-independent average $\overline{\psi_\alpha\psi_\beta\psi_\gamma\psi_\delta} = (\delta_{\alpha\beta}\delta_{\gamma\delta} + \delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma})/N^2$ The third contraction gives a contribution $N^{-1}$ representing the saturation of $M(t)$ for $t \to \infty$ The other two give the time dependence

$$\overline{M}(t) = N^{-1} + 2N^{-2}\Big|\sum_{\alpha\beta_0} |\langle\alpha|\beta_0\rangle|^2 \exp[i(\epsilon_\alpha - \epsilon_\beta^0)t]\Big|^2 \tag{14}$$

For perturbatively weak $H_1$ one has $\epsilon_\alpha = \epsilon_\alpha^0 + \langle\alpha|H_1|\alpha\rangle$ and $\langle\alpha|\beta_0\rangle = \delta_{\alpha\,\beta_0}$ According to RMT the matrix elements $\langle\alpha|H_1|\alpha\rangle$ are independent random numbers, and for large $N$ the central limit theorem leads to the Gaussian decay (ii) [or the parabolic decay (i) for short times] At larger perturbation strength, $|\langle\alpha|\beta_0\rangle|^2$ becomes Lorentzian,

$$|\langle\alpha|\beta_0\rangle|^2 = \frac{\Gamma/2\pi}{(\epsilon_\alpha - \epsilon_\beta^0)^2 + \Gamma^2/4}, \tag{15}$$

with a width $\Gamma \simeq \overline{|\langle\alpha_0|H_1|\beta\rangle|^2}/\Delta$ given by the golden rule This leads to regime (iii) Increasing $H_1$ further, one obtains an ergodic distribution $\overline{|\langle\alpha|\beta_0\rangle|^2} = N^{-1}$ and a straightforward calculation produces regime (v) This establishes that, under the sole assumption that $H_0$ and $H$ are classically chaotic, the decay of the fidelity in the three quantum regimes (ii), (iii), and (v) does not depend on the choice of the initial state $\psi$

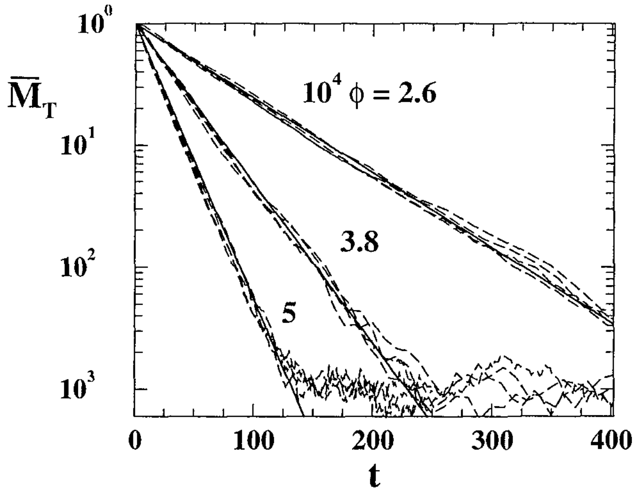


FIG. 2. Decay of $\overline{M}_T$ in the golden rule regime for $\phi = 2.6 \times 10^{-4}$, $3.8 \times 10^{-4}$, $5 \times 10^{-4}$, $K = 13.1$, and for preparation times $T = 0$, 5, 10, and 20 (nearly indistinguishable dashed lines). The solid lines give the corresponding golden rule decay with $\Gamma = 0.84\phi^2 S^2$ [9].

In summary, we have investigated the decay of the Loschmidt Echo, Eq. (2), for quantum states $\psi = \exp(-iH_0T)\psi_0$ that have spread over phase space for a time $T$. As in Ref. [7], we found a faster decay of $M_T(t)$ than for a localized wave packet, but only in the regime where the decay rate is set by the classical Lyapunov exponent $\lambda$. Since quantum interferences play no role in this regime, we conclude that sub-Planck-scale structures in the Wigner representation of $\psi$ do not influence the decay of the Loschmidt Echo.

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[14] The similarity between the data in our Fig. 1 and in Ref. [7] is only qualitative, mainly because of the much larger value $M_c = 0.9$ chosen in Ref. [7]. For values of $M_c$ close to 1, we expect that we can do perturbation theory in $t$ which gives $M_T(t) = 1 - \exp(\lambda T)\sigma^2 t^2$, and hence $t_c = \sqrt{1 - M_c}\exp(-\lambda T/2)/\sigma$. Analyzing the data presented in Fig. 2 of Ref. [7] gives the values $\sigma \approx 0.042$ and $\lambda \approx 0.247$.
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