TRIAxIAL GALAXY MODELS WITH THIN TUBE ORBITS

C. HUNTER

Department of Mathematics, Florida State University, Tallahassee, FL 32306-3027

AND

P. T. DE ZEEUW

Institute for Advanced Study, Princeton, NJ 08540; and Sterrewacht, Huygens Laboratorium, Postbus 9513,
 2300 RA Leiden, The Netherlands¹

Received 1991 May 16; accepted 1991 October 18

ABSTRACT

We show how to construct self-consistent distribution functions for triaxial galaxy models with Stäckel potentials. The Stäckel potentials cause the dynamics to be simple, with all the stellar orbits regular and belonging to one of four families. By also restricting each of the three families of tube orbits to be infinitesimally thin and with no radial epicyclic motion, we are able to obtain explicit expressions for their phase-space distribution functions, and then their densities and velocity moments throughout space. We also derive and discuss the properties of the integral equation of Volterra type that then has to be solved to obtain the distribution function for the box orbits, although no explicit solution of it is obtained.

Our models are not unique. Our method of construction begins with a choice of how to allocate, between the two kinds of long-axis tubes, the density on a part of the principal plane perpendicular to the intermediate axis. But once this choice has been made, no other arbitrariness remains. A number of general properties of our models, such as the fact that orbits of all four families must be populated, and the way in which these populations vary with geometry, can be deduced analytically. Numerical methods are needed to evaluate the quadratures in many of our formulae. They are also needed for the volume integrations that must be done to evaluate the masses in the separate families, and the angular momenta of their mean streaming motions. These properties have been calculated for four different continuous sets of perfect ellipsoids which span the triangle $0 \leq c/a \leq b/a \leq 1$ of geometrically possible axis ratios, as well as for the discrete set of 21 ellipsoids studied earlier by Statler.

Subject headings: celestial mechanics, stellar dynamics — galaxies: kinematics and dynamics

1. INTRODUCTION

Schwarzschild's (1979) classic paper established the existence of self-consistent triaxial galaxy models with strongly inhomogeneous density profiles and nonrotating figures. He did this numerically, using a combination of orbits of two different families. In a subsequent study he showed that models with slow figure rotation can be constructed in the same manner (Schwarzschild 1982). Here he also used orbits of a third family. Although some additional nonrotating and scale-free triaxial models were built by Levison & Richstone (1987), it was Statler (1987) who used Schwarzschild's methods to make the first major exploration of the solution space. He constructed a large variety of equilibrium solutions for a set of 21 triaxial ellipsoids, all nonrotating and with the same density profile, but with different axis ratios. He showed that the presence of four major orbit families, each of which can contribute density at any point, guarantees the existence of many different self-consistent phase-space distribution functions for the same mass model. This nonuniqueness is reflected in the large range of kinematic properties displayed by the resulting dynamical models.

Statler was able to calculate so many phase-space distribution functions because his mass models were chosen to have potentials of Stäckel form (Kuzmin 1973; de Zeeuw 1985a, hereafter Paper I). Orbital densities are available in analytic form for such potentials, so that the numerical integration of orbits, needed in Schwarzschild's earlier work, is avoided. Still, Statler was forced to do the actual reproduction of the model density by a numerical combination of orbital densities on a truncated grid of rather modest resolution, due to computational limitations. As a result, the precise division of mass between the orbit families, and the detailed structure of the models, particularly that near the focal curves of the ellipsoidal coordinate system in which the motion separates, cannot be deduced with confidence from Statler's models. It is important to address these issues, because these models are expected to describe the dynamics of slowly rotating giant elliptical galaxies, and accurate measurements of the two-dimensional velocity fields of these systems generally show interesting structure (e.g., Wilkinson et al. 1986; Davies & Birkinshaw 1988; Wagner 1990). We have chosen to do so using analytical methods as far as possible for a more detailed investigation of the internal structure of triaxial dynamical models with a Stäckel potential. This is not an easy task, and we have therefore approached this problem in steps. We first analyzed the two-dimensional problem of the construction of self-consistent elliptic disks (de Zeeuw, Hunter, & Schwarzschild 1987, hereafter Paper II). This analysis showed clearly that the nonuniqueness that Schwarzschild (1979) encountered in the original application of his numerical method is real, and is not just an artifact of the use of a finite grid. It also showed that the special equilibrium model which has just thin tubes, but the full variety of box orbits, is useful for an understanding of other

¹ Postal address.

solutions employing all available tubes. We subsequently built oblate (de Zeeuw & Hunter 1990, hereafter Paper III) and prolate models (Hunter et al. 1990, hereafter Paper IV) exclusively with infinitesimally thin tube orbits, i.e., tube orbits around the symmetry axis without radial epicyclic motion. After this preparatory work we are—finally—in a position to address the triaxial models, which contain all three families of tube orbits as well as box orbits. We use only infinitesimally thin tube orbits of the three families, but the full variety of box orbits. As in Papers III and IV, the use of infinitesimally thin tubes allows us to obtain explicit expressions for their distribution functions from the solution of one-dimensional integral equations of Abel type. The box orbit distribution function can then be found by an extension of the method that was first developed in Paper II for the (flat) box orbits of the elliptic disk.

Oblate dynamical models built exclusively with infinitesimally thin short-axis tubes (hereafter thin S-tubes) are unique (Paper III; see also Bishop 1987). Prolate models built with thin tubes are not unique (Paper IV), because there is a level of freedom to exchange inner and outer long-axis tubes (hereafter I- and O-tubes). We shall see that there is a similar freedom of exchange between I- and O-tubes in the triaxial case too, so that our models are again not unique. Once the division between I- and O-tubes has been chosen, however, the distribution of the thin S-tubes and the box orbits is fixed.

In § 2 we derive the basic integral equations for the phase-space distribution functions of the different orbit families in an arbitrary triaxial Stäckel model. The calculation of the three thin tube–orbit distribution functions is discussed in § 3. Then in § 4 we consider what part of the density in a given triaxial model must be assigned to the three families of thin tubes. The distribution function needed for the box orbits to reproduce the remaining density is discussed in § 5. We derive explicit expressions for the mean streaming velocities in § 6; similar expressions for the velocity dispersions are given in Appendix B. In § 7 we obtain a simple analytical description of nearly spherical, but triaxial, thin-tube models. In § 8 we use our analytical results to calculate mass fractions and angular momenta for a variety of perfect ellipsoids of different axis ratios, and we analyze the results in some detail. We discuss our results in § 9.

2. DYNAMICAL MODELS

In this section we first summarize the essential properties of orbits in a triaxial Stäckel model. Further details can be found in Papers I and III and in Statler (1987). We then derive the fundamental integral equation for the phase-space distribution function in terms of turning-point coordinates, and split it into four separate integral equations, one for each of the four different orbit families. We conclude with a brief outline of our strategy for solving these four integral equations.

2.1. Triaxial Stäckel Models

We define ellipsoidal coordinates (λ, μ, ν) in the usual way as the three roots for τ of

$$\frac{x^2}{\tau + \alpha} + \frac{y^2}{\tau + \beta} + \frac{z^2}{\tau + \gamma} = 1, \quad (2.1)$$

where (x, y, z) are Cartesian coordinates and α, β , and γ are constants. We choose $0 \leq -\gamma \leq \nu \leq -\beta \leq \mu \leq -\alpha \leq \lambda$. The coordinate surfaces are ellipsoids (λ) and hyperboloids of one (μ) and two (ν) sheets. There are three pairs of foci, two on the z -axis, at $z = \pm\sqrt{\gamma - \alpha}$ and at $z = \pm\sqrt{\gamma - \beta}$, and one on the y -axis, at $y = \pm\sqrt{\beta - \alpha}$. A choice of (λ, μ, ν) determines x^2, y^2 , and z^2 only, and generally corresponds to eight points $(\pm x, \pm y, \pm z)$. We consider mass models with a gravitational potential $V(\lambda, \mu, \nu)$ that is of Stäckel form, i.e.,

$$V = \frac{U(\lambda)}{(\lambda - \mu)(\lambda - \nu)} + \frac{U(\mu)}{(\mu - \nu)(\mu - \lambda)} + \frac{U(\nu)}{(\nu - \lambda)(\nu - \mu)}, \quad (2.2)$$

where

$$U(\tau) = -(\tau + \alpha)(\tau + \gamma)G(\tau), \quad (2.3)$$

and $G(\tau)$ is an arbitrary smooth function ($\tau = \lambda, \mu, \nu$). The density $\rho_m(\lambda, \mu, \nu)$ that corresponds to V can be found by application of Kuzmin's formula (Kuzmin 1973; de Zeeuw 1985b, eq. [14]) or directly from Poisson's equation. The potential (2.2) has the x -axis as long axis and the z -axis as short axis for all centrally concentrated mass models. This is usually true for the associated density also (de Zeeuw, Peletier, & Franx 1986).

2.2. Orbits

The momenta p_λ, p_μ , and p_ν conjugate to λ, μ , and ν are given by

$$p_\lambda = P^2 \dot{\lambda}, \quad p_\mu = Q^2 \dot{\mu}, \quad p_\nu = R^2 \dot{\nu}, \quad (2.4)$$

with metric coefficients

$$\begin{aligned} P^2 &= \frac{(\lambda - \mu)(\lambda - \nu)}{4(\lambda + \alpha)(\lambda + \beta)(\lambda + \gamma)}, \\ Q^2 &= \frac{(\mu - \lambda)(\mu - \nu)}{4(\mu + \alpha)(\mu + \beta)(\mu + \gamma)}, \\ R^2 &= \frac{(\nu - \mu)(\nu - \lambda)}{4(\nu + \alpha)(\nu + \beta)(\nu + \gamma)}. \end{aligned} \quad (2.5)$$

The equations of motion are

$$p_\tau^2 = \frac{B(\tau)}{2(\tau + \alpha)(\tau + \beta)(\tau + \gamma)} \quad (\tau = \lambda, \mu, \nu), \quad (2.6)$$

with

$$B(\tau) = (\tau + \alpha)(\tau + \gamma)H - (\tau + \gamma)I_2 - (\tau + \alpha)I_3 - U(\tau), \quad (2.7)$$

and H , I_2 , and I_3 are the three exact isolating integrals of motion admitted by V . H is the Hamiltonian, equal to the total orbital energy E , and I_2 and I_3 are nonclassical integrals which are quadratic expressions in the velocity components (e.g., de Zeeuw & Lynden-Bell 1985).

Motion separable in ellipsoidal coordinates is discussed in detail in Paper I. The motion in each of the coordinates is either an oscillation between turning points defined by $p_\tau^2 = 0$, or, if $p_\tau^2 > 0$ for all τ , a rotation ($\tau = \lambda, \mu, \nu$). Depending on the values of the three integrals H , I_2 , and I_3 , four different combinations of oscillation(s) and rotation(s) are possible, and these define the four bound orbit families: I- and O-tubes, S-tubes, and boxes.

Motion in a *box orbit* can be regarded as the sum of three oscillations, with turning points ν_0 , μ_0 , and λ_0 . The orbit generally fills the volume

$$-\gamma \leq \nu \leq \nu_0, \quad -\beta \leq \mu \leq \mu_0, \quad -\alpha \leq \lambda \leq \lambda_0. \quad (2.8)$$

Motion in an *inner long-axis tube* (I-tube) consists of a rotation in ν , and oscillations in μ and λ with turning points μ_1 , μ_2 , and λ_0 , so that such an orbit typically fills the volume

$$-\gamma \leq \nu \leq -\beta, \quad \mu_1 \leq \mu \leq \mu_2, \quad -\alpha \leq \lambda \leq \lambda_0. \quad (2.9)$$

Motion in an *outer long-axis tube* (O-tube) similarly consists of a rotation in ν , but the turning points for the μ - and λ -oscillations are μ_0 , λ_1 , and λ_2 , and the volume filled by the orbit is given by

$$-\gamma \leq \nu \leq -\beta, \quad \mu_0 \leq \mu \leq -\alpha, \quad \lambda_1 \leq \lambda \leq \lambda_2. \quad (2.10)$$

A *short-axis tube* (S-tube) has rotational motion in the μ -coordinate, and oscillations in ν and λ , with turning points ν_0 , λ_1 , and λ_2 , so that

$$-\gamma \leq \nu \leq \nu_0, \quad -\beta \leq \mu \leq -\alpha, \quad \lambda_1 \leq \lambda \leq \lambda_2. \quad (2.11)$$

A star in an I- or O-tube has a definite sense of rotation around the x -axis, while a star in an S-tube has a definite sense of rotation around the z -axis. In each case, both clockwise and counterclockwise motion may occur.

Statler (1987) illustrates the volumes in configuration space filled by the four different types of orbits (see also Binney & Tremaine 1987, p. 155); the reader may find it helpful to refer to this figure.

2.3. Turning Points and Divided Differences

Given values of the integrals of motion H , I_2 , and I_3 , the associated turning points τ_1 , τ_2 , and τ_3 are found by solving the equation $B(\tau) = 0$. We can express the integrals in terms of the turning points, two of which may coincide, as (see Paper III)

$$\begin{aligned} H &= U[\tau_1, \tau_2, \tau_3], \\ I_2 &= \frac{(\tau_1 + \alpha)(\tau_2 + \alpha)(\tau_3 + \alpha)}{\alpha - \gamma} U[-\alpha, \tau_1, \tau_2, \tau_3], \\ I_3 &= \frac{(\tau_1 + \gamma)(\tau_2 + \gamma)(\tau_3 + \gamma)}{\gamma - \alpha} U[-\gamma, \tau_1, \tau_2, \tau_3]. \end{aligned} \quad (2.12)$$

Square brackets here, and subsequently, denote *divided differences* of the function $U(\tau)$. The first-order divided difference $U[\tau_1, \tau_2]$ is given by

$$U[\tau_1, \tau_2] = \frac{U(\tau_1) - U(\tau_2)}{\tau_1 - \tau_2}, \quad (2.13)$$

and higher order differences are defined recursively by

$$U[\tau_1, \tau_2, \dots, \tau_n] = \frac{U[\tau_1, \tau_3, \dots, \tau_n] - U[\tau_2, \tau_3, \dots, \tau_n]}{\tau_1 - \tau_2}. \quad (2.14)$$

The divided differences are symmetric functions of their arguments. Their general properties are given in § IIb of Paper III. We remark that the potential V can be written as the divided difference

$$V(\lambda, \mu, \nu) = U[\nu, \mu, \lambda], \quad (2.15)$$

while the function $B(\tau)$ of equation (2.7) can be expressed as

$$B(\tau; \tau_1, \tau_2, \tau_3) = (\tau - \tau_1)(\tau - \tau_2)(\tau_3 - \tau)U[\tau, \tau_1, \tau_2, \tau_3]. \quad (2.16)$$

The velocity components $v_\lambda = P\dot{\lambda}$, $v_\mu = Q\dot{\mu}$, and $v_v = R\dot{v}$ are obtained by dividing the momenta by the metric coefficients to obtain

$$v_\lambda^2 = \frac{2B(\lambda; \tau_1, \tau_2, \tau_3)}{(\lambda - \mu)(\lambda - v)}, \quad v_\mu^2 = \frac{2B(\mu; \tau_1, \tau_2, \tau_3)}{(\mu - v)(\mu - \lambda)}, \quad v_v^2 = \frac{2B(v; \tau_1, \tau_2, \tau_3)}{(v - \lambda)(v - \mu)}. \quad (2.17)$$

We shall suppose throughout this paper that

$$U'''(\tau) > 0, \quad -\gamma \leq \tau, \quad (2.18)$$

as is easily seen to be the case for the perfect ellipsoid (Paper I) and the generalized Kuzmin-Kutuzov model (Paper III, § IVa; de Zeeuw & Pfenniger 1988). By integration of inequality (2.18), and application of the condition $U''(\infty) = 0$, which follows from equation (2.15) and the fact that the potential V vanishes at large distances, we find

$$U''(\tau) < 0, \quad -\gamma \leq \tau. \quad (2.19)$$

The value of H , a second-order divided difference of U , is therefore always negative, which is consistent with the fact that all our orbits with three finite turning points are bound. Furthermore, inequality (2.18) implies the positivity of all third-order divided differences:

$$U[\tau_1, \tau_2, \tau_3, \tau_4] > 0. \quad (2.20)$$

Consequently, the integral I_3 given in equation (2.12) is nonnegative, while the sign of I_2 depends on the location of the turning points relative to $-\alpha$ (see Paper I). Also, the equation $B(\tau) = 0$ can never have more than three roots (Kuzmin 1973).

2.4. Fundamental Integral Equation

A self-consistent phase-space distribution function $f \geq 0$ of a dynamical model with density ρ_m and potential V is a solution of the fundamental integral equation, which expresses ρ_m as the integral of f over the velocities. According to Jeans's (1915) theorem, f is a function of the isolating integrals H , I_2 , and I_3 only. The fundamental integral equation for triaxial Stäckel models is given in equation (117) of Paper I. Correcting for a factor of 8 that was erroneously omitted there, it is given by

$$\rho_m(\lambda, \mu, v) = \iiint f(H, I_2, I_3) \rho_{\text{orb}}(\lambda, \mu, v; H, I_2, I_3) dH dI_2 dI_3, \quad (2.21)$$

where $\rho_{\text{orb}}(\lambda, \mu, v; H, I_2, I_3)$ is the density of an individual orbit with integrals H , I_2 , and I_3 , and is given by

$$\rho_{\text{orb}}(\lambda, \mu, v; H, I_2, I_3) = \frac{2\sqrt{2}(\gamma - \alpha)}{\sqrt{B(\lambda)}\sqrt{-B(\mu)}\sqrt{B(v)}}. \quad (2.22)$$

The integration in equation (2.21) is over all values of H , I_2 , and I_3 for which the square roots in equation (2.22) are nonnegative.

The fundamental integral equation can be rewritten in terms of the turning points as

$$\rho_m(\lambda, \mu, v) = \iiint F(\tau_1, \tau_2, \tau_3) \rho_{\text{orb}}(\lambda, \mu, v; \tau_1, \tau_2, \tau_3) d\tau_1 d\tau_2 d\tau_3, \quad (2.23)$$

with

$$\rho_{\text{orb}}(\lambda, \mu, v; \tau_1, \tau_2, \tau_3) = \frac{2\sqrt{2}(\gamma - \alpha)}{\sqrt{B(\lambda; \tau_1, \tau_2, \tau_3)}\sqrt{-B(\mu; \tau_1, \tau_2, \tau_3)}\sqrt{B(v; \tau_1, \tau_2, \tau_3)}}, \quad (2.24)$$

where $B(\tau; \tau_1, \tau_2, \tau_3)$ is given in equation (2.16), and

$$F(\tau_1, \tau_2, \tau_3) = \left| \frac{\partial(H, I_2, I_3)}{\partial(\tau_1, \tau_2, \tau_3)} \right| f(H(\tau_1, \tau_2, \tau_3), I_2(\tau_1, \tau_2, \tau_3), I_3(\tau_1, \tau_2, \tau_3)). \quad (2.25)$$

The Jacobian determinant of the transformation $(H, I_2, I_3) \rightarrow (\tau_1, \tau_2, \tau_3)$ is derived in Paper III and is given by

$$\left| \frac{\partial(H, I_2, I_3)}{\partial(\tau_1, \tau_2, \tau_3)} \right| = \frac{(\tau_3 - \tau_1)(\tau_3 - \tau_2)(\tau_2 - \tau_1)}{(\gamma - \alpha)} h(\tau_1, \tau_2, \tau_3), \quad (2.26)$$

with the positive function h defined by

$$h(\tau_1, \tau_2, \tau_3) = U[\tau_1, \tau_1, \tau_2, \tau_3]U[\tau_1, \tau_2, \tau_2, \tau_3]U[\tau_1, \tau_2, \tau_3, \tau_3]. \quad (2.27)$$

Consequently, the Jacobian vanishes when and only when two turning points coincide.

Almost all points (λ, μ, v) in configuration space can be reached by orbits of all four families. However, the four different types of orbits give rise to different limits of integration in equation (2.23) on account of inequalities (2.8)–(2.11). We therefore split the given density $\rho_m(\lambda, \mu, v)$ into four component densities, one for each of the four orbit families:

$$\rho_m(\lambda, \mu, v) = \rho_{im}(\lambda, \mu, v) + \rho_{om}(\lambda, \mu, v) + \rho_{sm}(\lambda, \mu, v) + \rho_{bm}(\lambda, \mu, v), \quad (2.28)$$

where ρ_{im} , ρ_{om} , ρ_{sm} , and ρ_{bm} are due to the I-tubes, O-tubes, S-tubes, and box orbits, respectively, and correspond to distribution

functions f_{im} , f_{om} , f_{sm} , and f_{bm} . It follows that

$$f = f_{im} + f_{om} + f_{sm} + f_{bm}, \quad (2.29)$$

and the fundamental integral equation (2.21) or (2.23) splits into four separate integral equations, one for each of the four distribution functions. We find

$$\rho_{im}(\lambda, \mu, \nu) = \int_{-\beta}^{\mu} d\mu_1 \int_{\mu}^{-\alpha} d\mu_2 \int_{\lambda}^{\infty} d\lambda_0 F_{im}(\mu_1, \mu_2, \lambda_0) \rho_i(\lambda, \mu, \nu; \mu_1, \mu_2, \lambda_0), \quad (2.30)$$

where $\rho_i(\lambda, \mu, \nu; \mu_1, \mu_2, \lambda_0)$ is the density of the I-tube with turning points μ_1, μ_2 , and λ_0 , as given in equation (2.24). Similarly,

$$\rho_{om}(\lambda, \mu, \nu) = \int_{-\beta}^{\mu} d\mu_0 \int_{-\alpha}^{\lambda} d\lambda_1 \int_{\lambda}^{\infty} d\lambda_2 F_{om}(\mu_0, \lambda_1, \lambda_2) \rho_o(\lambda, \mu, \nu; \mu_0, \lambda_1, \lambda_2), \quad (2.31)$$

where $\rho_o(\lambda, \mu, \nu; \mu_0, \lambda_1, \lambda_2)$ is the density of an O-tube with turning points μ_0, λ_1 , and λ_2 , and

$$\rho_{sm}(\lambda, \mu, \nu) = \int_{\nu}^{-\beta} d\nu_0 \int_{-\alpha}^{\lambda} d\lambda_1 \int_{\lambda}^{\infty} d\lambda_2 F_{sm}(\nu_0, \lambda_1, \lambda_2) \rho_s(\lambda, \mu, \nu; \nu_0, \lambda_1, \lambda_2), \quad (2.32)$$

where $\rho_s(\lambda, \mu, \nu; \nu_0, \lambda_1, \lambda_2)$ is the density of an S-tube with turning points ν_0, λ_1 , and λ_2 . Finally,

$$\rho_{bm}(\lambda, \mu, \nu) = \int_{\nu}^{-\beta} d\nu_0 \int_{\mu}^{-\alpha} d\mu_0 \int_{\lambda}^{\infty} d\lambda_0 F_{bm}(\nu_0, \mu_0, \lambda_0) \rho_b(\lambda, \mu, \nu; \nu_0, \mu_0, \lambda_0), \quad (2.33)$$

where $\rho_b(\lambda, \mu, \nu; \nu_0, \mu_0, \lambda_0)$ is the density of a box orbit with turning points ν_0, μ_0 , and λ_0 . The functions F_{im} , F_{om} , F_{sm} , and F_{bm} are related to f_{im} , f_{om} , f_{sm} , and f_{bm} , respectively, by transformation (2.25).

2.5. Construction of Equilibrium Solutions with Thin Tube Orbits

Orbits of each of the four families are needed in the models that we shall construct. This is because the two degenerate coordinate surfaces $\mu = -\alpha$ and $\nu = -\beta$ are reached by certain orbit families only in extreme cases. Hence, finite distribution functions F of these orbits cannot provide the density needed there. Specifically, it follows from equations (2.30)–(2.33), and the expressions for the orbital densities, that for finite F 's, both ρ_{sm} and ρ_{bm} must vanish on $\nu = -\beta$, while both ρ_{im} and ρ_{bm} must vanish on $\mu = -\alpha$. A similar situation arises with the elliptic disk (Paper II), where the (S-) tube orbits are essential for providing the density on $\mu = -\alpha$.

The significant degenerate coordinate surfaces are illustrated in Figure 1, which is oriented consistently with the earlier figures of

FIG. 1.—Special surfaces $\mathcal{L}$ ($\nu = -\beta$) and $\mathcal{H}$ ($\mu = -\beta$) in the plane $y = 0$, and $\mathcal{S}$ ($\mu = -\alpha$) and $\mathcal{E}$ ($\lambda = -\alpha$) in the plane $x = 0$. The two top panels identify the four regions, and the bottom panel shows the three-dimensional configuration. The three pairs of foci are indicated by the open and filled circles and the filled squares. The focal hyperbola separates the regions $\mathcal{L}$ and $\mathcal{H}$. Its asymptotes (dashed lines) are given by $z = \pm x\sqrt{(1-T)/T}$, with T being the triaxiality parameter of eq. (2.35). The focal ellipse separates the regions $\mathcal{S}$ and $\mathcal{E}$. Long-axis tubes provide the entire density in region $\mathcal{L}$, while S- and O-tubes provide all the density in region $\mathcal{S}$. Correspondingly, S-tubes and boxes provide the entire density in region $\mathcal{H}$, while I-tubes and boxes do so in region $\mathcal{E}$.

Paper I and of Statler (1987). The surface $v = -\beta$, which we name surface $\mathcal{L}$, is the part of the plane $y = 0$ that lies outside the focal hyperbola

$$\frac{z^2}{(\gamma - \alpha)(1 - T)} - \frac{x^2}{T(\gamma - \alpha)} = 1. \quad (2.34)$$

The equation for the focal hyperbola has here been written in terms of the triaxiality parameter T defined as

$$T = \frac{\beta - \alpha}{\gamma - \alpha} = \frac{a^2 - b^2}{a^2 - c^2}, \quad (2.35)$$

which generally lies between the extreme of 0 for an oblate spheroid and 1 for a prolate spheroid. This is a natural parameter in the description of the projected properties of ellipsoids (Franx 1988a), and hence in the analysis of their intrinsic shapes (Franx, Illingworth, & de Zeeuw 1991). Surface $\mathcal{L}$ occupies the whole of the $y = 0$ plane in the prolate $T = 1$ limit, when the focal hyperbola is the x -axis, and none of it in the oblate $T = 0$ limit, when the focal hyperbola is two semi-infinite slits. We label as surface $\mathcal{H}$ the remainder of the $y = 0$ plane ($\mu = -\beta$) interior to the focal hyperbola. Surface $\mathcal{S}$ is our label for the surface $\mu = -\alpha$, which is the part of the plane $x = 0$ that lies outside the focal ellipse

$$\frac{y^2}{T(\gamma - \alpha)} + \frac{z^2}{(\gamma - \alpha)} = 1, \quad (2.36)$$

of eccentricity $\sqrt{1 - T}$. The interior of this ellipse, the degenerate ellipsoid $\lambda = -\alpha$, is labeled surface $\mathcal{E}$.

In this paper we shall restrict ourselves to using only those tube orbits that are infinitesimally thin, i.e., I-tubes with $\mu_1 = \mu_2$, O-tubes with $\lambda_1 = \lambda_2$, and S-tubes with $\lambda_1 = \lambda_2$. This simplifies the mathematics considerably, and will allow us to obtain explicit expressions for the tube-orbit distribution functions. Self-consistent solutions for the triaxial Stäckel models are so nonunique that, unlike the situation for the elliptic disk but like that for the prolate spheroid, many distribution functions can be obtained without using thick tube orbits. Figure 2 illustrates the shapes of the four kinds of orbits that we shall use in the rectified (λ, μ, v) -space of ellipsoidal coordinates. Note that a star on a thin tube orbit is restricted to (parts of) a two-dimensional surface instead of gradually filling a three-dimensional volume.

Our strategy to be implemented in §§ 3–5 is as follows. The first step is to find some combination of I- and O-tubes that provides the density $\rho_m(\lambda, \mu, -\beta)$ on surface $\mathcal{L}$ at which both ρ_{sm} and ρ_{bm} vanish. This problem is exactly the same as that of constructing a distribution function for a self-consistent prolate spheroid, which was discussed in detail in Paper IV, where surface $\mathcal{L}$ was an entire meridional plane. We found there that both types of long-axis tubes must be populated in the perfect prolate spheroid. The reason for this is that, if $\rho_m(\lambda, \mu, -\beta)$ is taken to be equal to either $\rho_{im}(\lambda, \mu, -\beta)$ or $\rho_{om}(\lambda, \mu, -\beta)$ everywhere, then not only would the resulting thin-tube distribution functions be singular, but they would also have ranges of negative values. The density at surface $\mathcal{L}$

FIG. 2.—Volumes in (λ, μ, v) -space filled by the three types of thin tube orbits, and by a box orbit. These are the building blocks for the self-consistent models constructed in this paper.

therefore must be split into components:

$$\rho_m(\lambda, \mu, -\beta) = \rho_{im}(\lambda, \mu, -\beta) + \rho_{om}(\lambda, \mu, -\beta), \quad (2.37)$$

before the distribution functions for the I- and O-tubes can be calculated. These splits are far from arbitrary, and the ones that we use depend upon a single parameter k . The distribution functions give expressions for the densities $\rho_{im}(\lambda, \mu, v)$ and $\rho_{om}(\lambda, \mu, v)$ throughout configuration space. We then know the density

$$\rho_{sm}(\lambda, -\alpha, v) = \rho_m(\lambda, -\alpha, v) - \rho_{om}(\lambda, -\alpha, v) \quad (2.38)$$

on surface $\mathcal{S}$ which must be provided exclusively by S-tubes. Finding a distribution of S-tubes which does so is exactly the same problem as that of constructing a distribution function for a self-consistent oblate spheroid given its density on a meridional plane. Its solution is available from Paper III and can be used to give the density $\rho_{sm}(\lambda, \mu, v)$ everywhere. The remaining density, $\rho_{bm}(\lambda, \mu, v)$, must be provided by box orbits, and is given by

$$\rho_{bm} = \rho_m - \rho_{im} - \rho_{om} - \rho_{sm}. \quad (2.39)$$

The distribution function F_{bm} can then be found by solving the Volterra integral equation (2.33), via a method that is a three-dimensional extension of that used in Paper II for the elliptic disk.

The next five sections give the technical details of the implementation of this strategy, and of some of its analytical consequences. Those uninterested in these details and interested only in the general features of the models that we have constructed should go to § 8.

3. DISTRIBUTION FUNCTIONS FOR THIN TUBE ORBITS

In this, the longest section, we calculate the two-dimensional distribution functions for the three families of thin tube orbits. The thin tube orbit distribution functions can be derived once densities have been specified on the two special surfaces $\mathcal{L}$ and $\mathcal{S}$. The derivations follow closely those presented in Papers IV and III for the I- and O-tubes and the S-tubes, respectively, but triaxiality now introduces certain additional features. Whereas in the axisymmetric cases the density on a meridional plane, which fully defines the distribution functions for the thin tube orbits, is the whole density, now the distributions of density of the tube-orbit families away from their special defining surfaces have to be calculated. We do this in closed form, and analyze certain of their properties. In particular, we show that it is not possible to have self-consistent ellipsoids of the type we are considering without some S-tubes.

3.1. I-Tubes

The density distribution of an infinitesimally thin I-tube with turning points $\mu_2 = \mu_1$ and λ_0 is given by

$$\rho_i(\lambda, \mu, v; \mu_1, \mu_1, \lambda_0) = \rho_{ii}(\lambda, \mu, v; \lambda_0) \delta(\mu - \mu_1), \quad (3.1)$$

with

$$\rho_{ii}(\lambda, \mu, v; \lambda_0) = \frac{2\pi\sqrt{2}(\gamma - \alpha)}{\sqrt{B(v; \mu, \mu, \lambda_0)}\sqrt{B(\lambda; \mu, \mu, \lambda_0)}\sqrt{(\lambda_0 - \mu)U[\mu, \mu, \mu, \lambda_0]}}. \quad (3.2)$$

A distribution of such thin I-tubes is described by a distribution function $F_{im}(\mu_1, \mu_2, \lambda_0)$ of the form

$$F_{im}(\mu_1, \mu_2, \lambda_0) = F_{iim}(\mu_1, \lambda_0) \delta(\mu_2 - \mu_1 - 0). \quad (3.3)$$

The -0 is included in the argument of the delta function to indicate the location of the spike of the delta function relative to the range of the two-dimensional integral of equation (2.30). Upon substitution of expressions (3.3) and (3.1), the fundamental integral equation (2.30) reduces to the equation

$$\rho_{im}(\lambda, \mu, v) = \int_{\lambda}^{\infty} d\lambda_0 F_{iim}(\mu, \lambda_0) \rho_{ii}(\lambda, \mu, v; \lambda_0), \quad (3.4)$$

for the two variable function F_{iim} . Unlike the prolate case of Paper IV, this equation will now overdetermine F_{iim} and have no solution for a general three-variable density $\rho_{im}(\lambda, \mu, v)$. However, F_{iim} can be determined from a given density $\rho_{im}(\lambda, \mu, -\beta)$ of I-tubes on the surface $\mathcal{L}$. The fundamental integral equation (3.4) can be rewritten at this surface, using equation (2.16), as

$$\rho_{im}(\lambda, \mu, -\beta) = \frac{1}{\lambda - \mu} \int_{\lambda}^{\infty} \frac{F_{iim}^*(\mu, \lambda_0) d\lambda_0}{\sqrt{(\lambda_0 - \lambda)U[\mu, \mu, \lambda, \lambda_0]}}, \quad (3.5)$$

after defining

$$F_{iim}^*(\mu, \lambda_0) = \frac{2\pi\sqrt{2}(\gamma - \alpha)F_{iim}(\mu, \lambda_0)}{(\mu + \beta)\sqrt{(\lambda_0 + \beta)U[-\beta, \mu, \mu, \lambda_0]}(\lambda_0 - \mu)U[\mu, \mu, \mu, \lambda_0]}. \quad (3.6)$$

The variable μ is a parameter in equation (3.5), so that this equation is merely a one-dimensional equation for $F_{iim}^*(\mu, \lambda_0)$ as a function of λ_0 . The physical reason for this simplification is that the only thin I-tubes that can provide density at a given μ are those with that value of μ (see Fig. 2).

Equation (3.5) is identical to the integral equation for thin I-tubes found for the prolate limit of Paper IV (eq. [2.32]) in which

$\gamma = \beta$ and $v = -\beta$ always. As shown there, it is of *Abel type* because

$$(\lambda_0 - \lambda)U[\mu, \mu, \lambda, \lambda_0] = U[\mu, \mu, \lambda_0] - U[\mu, \mu, \lambda], \quad (3.7)$$

and we can change from λ_0 to $U[\mu, \mu, \lambda_0]$ as the variable of integration. In view of much subsequent use of this second-order divided difference, we shall introduce the shorthand notation

$$U[\tau] = U[\mu, \mu, \tau], \quad (3.8)$$

in which the dependence on μ is suppressed. Note that the change of variables from λ to $U[\lambda]$ is always satisfactory because the derivative

$$U'[\lambda] = \frac{\partial U[\mu, \mu, \lambda]}{\partial \lambda} = U[\mu, \mu, \lambda, \lambda] \quad (3.9)$$

is always positive (see eq. [2.20] of § 2.3).

The solution of equation (3.5), in which the number of orbits with turning point λ_0 is found for successively smaller values of λ_0 , is given by

$$F_{im}^*(\mu, \lambda_0) = -\frac{1}{\pi} \frac{\partial}{\partial \lambda_0} \int_{\sigma=\lambda_0}^{\infty} \frac{(\sigma - \mu)\rho_{im}(\sigma, \mu, -\beta)dU[\sigma]}{\sqrt{U[\sigma] - U[\lambda_0]}}. \quad (3.10)$$

The differentiation in equation (3.10) can be carried out after an integration by parts, with the result

$$F_{im}^*(\mu, \lambda_0) = -\frac{U'[\lambda_0]}{\pi} \int_{\lambda_0}^{\infty} \frac{[\partial(\sigma - \mu)\rho_{im}(\sigma, \mu, -\beta)/\partial \sigma]d\sigma}{\sqrt{U[\sigma] - U[\lambda_0]}}, \quad (3.11)$$

provided that the mass model is sufficiently centrally concentrated, i.e., $\lambda\rho_{im}(\lambda, \mu, -\beta) \rightarrow 0$ as $\lambda \rightarrow \infty$. Further details can be found in § IIc of Paper IV. As there, a *necessary* and *sufficient* condition for a nonnegative f_{im} for the given $\rho_{im}(\lambda, \mu, -\beta)$ is that the integral of equation (3.11) be nonpositive for all $\lambda_0 \geq -\alpha$ and all μ in $[-\beta, -\alpha]$.

Now that we have obtained the unique distribution of thin I-tubes that exactly reproduces a given density $\rho_{im}(\lambda, \mu, -\beta)$ on the surface $\mathcal{L}$, we can use equation (3.4) to calculate the density $\rho_{im}(\lambda, \mu, v)$ of the I-tubes at all other points of space. Definitions (3.2) and (3.6) show that

$$F_{im}(\mu, \lambda_0)\rho_{it}(\lambda, \mu, v; \lambda_0) = \frac{1}{\lambda - \mu} \left(\frac{\mu + \beta}{\mu - v} \right) \frac{\sqrt{U[\lambda_0] - U[-\beta]} F_{im}^*(\mu, \lambda_0)}{\sqrt{U[\lambda_0] - U[v]} \sqrt{U[\lambda_0] - U[\lambda]}}. \quad (3.12)$$

Substitution of this expression into equation (3.4), and use of the solution (3.11) for F_{im}^* , then gives $\rho_{im}(\lambda, \mu, v)$ as a double integral. Interchange of the orders of integration, and changing from λ_0 to θ as the variable of integration, with $u_3 \cos^2 \theta = U[\lambda_0] - U[\lambda]$, yields

$$\rho_{im}(\lambda, \mu, v) = -\frac{2(\mu + \beta)}{\pi(\lambda - \mu)(\mu - v)} \int_{\lambda}^{\infty} d\sigma A_0(u_1, u_2, u_3) \frac{\partial}{\partial \sigma} [(\sigma - \mu)\rho_{im}(\sigma, \mu, -\beta)], \quad (3.13)$$

where we have defined

$$u_1 = U[\lambda] - U[-\beta], \quad u_2 = U[\lambda] - U[v], \quad u_3 = U[\sigma] - U[\lambda], \quad (3.14)$$

and

$$A_0(u_1, u_2, u_3) = \int_0^{\pi/2} d\theta \sqrt{\frac{u_1 + u_3 \cos^2 \theta}{u_2 + u_3 \cos^2 \theta}}. \quad (3.15)$$

The function $A_0(u_1, u_2, u_3)$ is a combination of complete elliptic integrals. It is the first of a sequence of similar integrals that we shall introduce subsequently. Expressions for all of them are given in Appendix A in terms of Carlson's (1979) symmetrized forms R_F , R_D , and R_J of the elliptic integrals of the first, second, and third kinds. These are both analytically and computationally convenient. The fact that $U[\tau]$ is an increasing function of τ , and that $\sigma \geq \lambda \geq -\beta \geq v$, implies that $u_3 \geq 0$ and $u_2 \geq u_1 > 0$, and hence $0 < \frac{1}{2}\pi(u_1/u_2)^{1/2} < A_0(u_1, u_2, u_3) \leq \pi/2$.

An alternative expression for the three-dimensional density $\rho_{im}(\lambda, \mu, v)$ is obtained by integrating equation (3.13) by parts to get

$$\rho_{im}(\lambda, \mu, v) = \left(\frac{\mu + \beta}{\mu - v} \right) \left\{ \rho_{im}(\lambda, \mu, -\beta) \sqrt{\frac{U[\lambda] - U[-\beta]}{U[\lambda] - U[v]}} + \frac{1}{\pi(\lambda - \mu)} \int_{\lambda}^{\infty} d\sigma A_1(u_1, u_2, u_3) (\sigma - \mu) \rho_{im}(\sigma, \mu, -\beta) U[\mu, \mu, \sigma, \sigma] \right\}, \quad (3.16)$$

with

$$A_1(u_1, u_2, u_3) = \int_0^{\pi/2} \frac{(u_2 - u_1) \cos^2 \theta d\theta}{(u_1 + u_3 \cos^2 \theta)^{1/2} (u_2 + u_3 \cos^2 \theta)^{3/2}}. \quad (3.17)$$

This second form for the density $\rho_{im}(\lambda, \mu, v)$ is convenient for numerical calculations, since it does not require the more lengthy evaluation of an R_j function (see Appendix A). We remark that equation (3.13), or its alternative form equation (3.16), allows us to calculate the density $\rho_{im}(\lambda, \mu, v)$ directly by means of a quadrature of $\rho_{im}(\lambda, \mu, -\beta)$, without the need of first calculating the explicit form of the distribution function f_{im} .

3.2. O-Tubes

Similar results can be derived for the thin O-tubes. The density of such a tube with turning points μ_0 and $\lambda_1 = \lambda_2$ is given by

$$\rho_o(\lambda, \mu, v; \mu_0, \lambda_1, \lambda_1) = \rho_{io}(\lambda, \mu, v; \mu_0) \delta(\lambda - \lambda_1), \quad (3.18)$$

with

$$\rho_{io}(\lambda, \mu, v; \mu_0) = \frac{2\pi\sqrt{2}(\gamma - \alpha)}{\sqrt{B(v; \mu_0, \lambda, \lambda)} \sqrt{-B(\mu; \mu_0, \lambda, \lambda)} \sqrt{(\lambda - \mu_0)U[\mu_0, \lambda, \lambda]}}. \quad (3.19)$$

A distribution of these orbits has a distribution function $F_{om}(\mu_0, \lambda_1, \lambda_2)$ of the form

$$F_{om}(\mu_0, \lambda_1, \lambda_2) = F_{tom}(\mu_0, \lambda_1) \delta(\lambda_2 - \lambda_1 - 0). \quad (3.20)$$

The fundamental integral equation (2.31) now reduces to

$$\rho_{om}(\lambda, \mu, v) = \int_{-\beta}^{\mu} d\mu_0 F_{tom}(\mu_0, \lambda) \rho_{io}(\lambda, \mu, v; \mu_0). \quad (3.21)$$

We fix F_{tom} also by the requirement that it reproduce a given density ρ_{om} in surface $\mathcal{L}$. Then the fundamental integral equation for F_{tom} is

$$\rho_{om}(\lambda, \mu, -\beta) = \frac{1}{\lambda - \mu} \int_{-\beta}^{\mu} \frac{F_{tom}^*(\mu_0, \lambda) d\mu_0}{\sqrt{(\mu - \mu_0)U[\mu_0, \mu, \lambda]}}, \quad (3.22)$$

with the definition

$$F_{tom}^*(\mu_0, \lambda) = \frac{2\pi\sqrt{2}(\gamma - \alpha)F_{tom}(\mu_0, \lambda)}{(\lambda + \beta)\sqrt{(\mu_0 + \beta)U[-\beta, \mu_0, \lambda]}(\lambda - \mu_0)U[\mu_0, \lambda, \lambda]}. \quad (3.23)$$

This is identical to the integral equation for the thin O-tubes in the prolate limit ($\gamma = \beta, v \equiv -\beta$), given in equation (2.48) of Paper IV. It is again of Abel type, since

$$(\mu - \mu_0)U[\mu_0, \mu, \lambda] = U[\mu, \lambda, \lambda] - U[\mu_0, \lambda, \lambda], \quad (3.24)$$

and it is convenient to use another increasing variable,

$$W[\tau] = U[\tau, \lambda, \lambda], \quad (3.25)$$

in terms of which the solution of equation (3.22) can be obtained as

$$F_{tom}^*(\mu_0, \lambda) = \frac{1}{\pi} \frac{\partial}{\partial \mu_0} \int_{\sigma=-\beta}^{\mu_0} \frac{(\lambda - \sigma)\rho_{om}(\lambda, \sigma, -\beta) dW[\sigma]}{\sqrt{W[\mu_0] - W[\sigma]}}. \quad (3.26)$$

If this expression is first integrated by parts and then differentiated, we obtain the alternative form

$$F_{tom}^*(\mu_0, \lambda) = \frac{W'[\mu_0]}{\pi} \left\{ \frac{(\lambda + \beta)\rho_{om}(\lambda, -\beta, -\beta)}{\sqrt{W[\mu_0] - W[-\beta]}} + \int_{-\beta}^{\mu_0} \frac{[\partial(\lambda - \sigma)\rho_{om}(\lambda, \sigma, -\beta)/\partial\sigma] d\sigma}{\sqrt{W[\mu_0] - W[\sigma]}} \right\}. \quad (3.27)$$

Note that $\rho_{om}(\lambda, -\beta, -\beta)$ need not vanish, even though the range of the integral in equation (3.22) vanishes when $\mu = -\beta$. The integrand, however, has an inverse square root singularity, and so does the term in F_{tom}^* in equation (3.27) that is proportional to $\rho_{om}(\lambda, -\beta, -\beta)$. A necessary and sufficient condition for f_{om} to be nonnegative is that the integral in equation (3.26) be a monotonically increasing function of μ_0 for every λ_1 .

For the calculation of $\rho_{om}(\lambda, \mu, v)$ away from surface $\mathcal{L}$, we use equation (3.21) with the distribution function that we just derived. Definitions (3.19) and (3.23) show that

$$F_{tom}(\mu_0, \lambda) \rho_{io}(\lambda, \mu, v; \mu_0) = \frac{1}{\lambda - \mu} \left(\frac{\lambda + \beta}{\lambda - v} \right) \frac{\sqrt{W[\mu_0] - W[-\beta]} F_{tom}^*(\mu_0, \lambda)}{\sqrt{W[\mu] - W[\mu_0]} \sqrt{W[\mu_0] - W[v]}}. \quad (3.28)$$

After substitution of this result into equation (3.21), use of expression (3.27), evaluation of the single integral, and interchange of the order of integration in the double integral followed by the trigonometric substitution $W[\mu_0] = W[\sigma] + w_3 \cos^2 \theta$, we find

$$\begin{aligned} \rho_{om}(\lambda, \mu, v) = \frac{2(\lambda + \beta)}{\pi(\lambda - \mu)(\lambda - v)} \left\{ (\lambda + \beta)\rho_{om}(\lambda, -\beta, -\beta) \arcsin \sqrt{\frac{W[\mu] - W[-\beta]}{W[\mu] - W[v]}} \right. \\ \left. + \int_{-\beta}^{\mu} d\sigma A_0(w_1, w_2, w_3) \frac{\partial}{\partial \sigma} [(\lambda - \sigma)\rho_{om}(\lambda, \sigma, -\beta)] \right\}, \quad (3.29) \end{aligned}$$

where $A_0(w_1, w_2, w_3)$ is the function defined in equation (3.15), and we have introduced the auxiliary variables

$$w_1 = W[\sigma] - W[-\beta], \quad w_2 = W[\sigma] - W[v], \quad w_3 = W[\mu] - W[\sigma]. \quad (3.30)$$

Because $\mu \geq \sigma \geq -\beta \geq v$ here, we have $w_2 \geq w_1 \geq 0$ and $w_3 \geq 0$, and hence $0 \leq \frac{1}{2}\pi(w_1/w_2)^{1/2} \leq A_0(w_1, w_2, w_3) \leq \pi/2$.

An alternative to expression (3.29) for the three-dimensional density $\rho_{om}(\lambda, \mu, v)$ of the unique distribution of infinitesimally thin O-tubes that has the given density $\rho_{om}(\lambda, \mu, -\beta)$ can be obtained via an integration by parts. The result is

$$\rho_{om}(\lambda, \mu, v) = \left(\frac{\lambda + \beta}{\lambda - v} \right) \left\{ \rho_{om}(\lambda, \mu, -\beta) \sqrt{\frac{W[\mu] - W[-\beta]}{W[\mu] - W[v]}} - \frac{1}{\pi(\lambda - \mu)} \int_{-\beta}^{\mu} d\sigma A_2(w_1, w_2, w_3)(\lambda - \sigma) \rho_{om}(\lambda, \sigma, -\beta) U[\sigma, \sigma, \lambda, \lambda] \right\}, \quad (3.31)$$

where we have defined

$$A_2(w_1, w_2, w_3) = \int_0^{\pi/2} \frac{(w_2 - w_1) \sin^2 \theta d\theta}{(w_1 + w_3 \cos^2 \theta)^{1/2} (w_2 + w_3 \cos^2 \theta)^{3/2}}. \quad (3.32)$$

Note that although the above alternative form for $\rho_{om}(\lambda, \mu, v)$ avoids the evaluation of an R_J function (see Appendix A), the integral A_2 does have a mild logarithmic singularity at the lower limit $\sigma = -\beta$ where $w_1 = 0$. The behavior of $\rho_{om}(\lambda, \mu, v)$ for $\mu \sim -\beta$, i.e., near surface $\mathcal{H}$, follows more immediately from equation (3.29), since the second term here vanishes as $\mu \downarrow -\beta$.

Important consequences for the density of S-tubes as well as that of O-tubes can be deduced from the alternative formula (3.31). The nonnegativity of the integrand and the fact that $W[\tau]$ is an increasing function of τ show that

$$\rho_{om}(\lambda, \mu, v) < \rho_{om}(\lambda, \mu, -\beta) \quad \text{for } v < -\beta. \quad (3.33)$$

This inequality can then be applied in surface $\mathcal{S}$ to get the further inequality

$$\rho_{om}(\lambda, -\alpha, v) < \rho_{om}(\lambda, -\alpha, -\beta) = \rho_m(\lambda, -\alpha, -\beta) \quad \text{for } v < -\beta, \quad (3.34)$$

with the equality coming from the fact that $\rho_{im} = 0$ at $\mu = -\alpha$ in surface $\mathcal{L}$. We can then show that centrally condensed models for which

$$\frac{\partial \rho_m(\lambda, \mu, v)}{\partial v} < 0, \quad (3.35)$$

which is the sufficient condition of Paper III for the existence of a thin-tube oblate model, must contain some S-tubes. This follows because inequality (3.35) gives

$$\rho_m(\lambda, -\alpha, -\beta) < \rho_m(\lambda, -\alpha, v) \quad \text{for } v < -\beta. \quad (3.36)$$

Combining inequalities (3.34) and (3.36) with equation (2.38) then shows that

$$\rho_{sm}(\lambda, -\alpha, v) > 0 \quad \text{for } v < -\beta, \quad (3.37)$$

hence the necessity for some S-tubes. Although Schwarzschild (1979, 1982) was able to construct some triaxial models without S-tubes, Statler (1987) was unable to do so for any of the perfect ellipsoids that he studied.

Because $\rho_{om}(\lambda, -\alpha, -\beta) = \rho_m(\lambda, -\alpha, -\beta)$, the first term on the right-hand side of equation (3.31), when applied at surface $\mathcal{S}$ to give $\rho_{om}(\lambda, -\alpha, v)$, is independent of the split of density between I- and O-tubes in surface $\mathcal{L}$. The second term does depend on $\rho_{om}(\lambda, \mu, -\beta)$, and hence this split, and in a seemingly paradoxical way, because the positivity of A_2 for $v < -\beta$ shows that an increase of $\rho_{om}(\lambda, \mu, -\beta)$ causes a decrease of $\rho_{om}(\lambda, -\alpha, v)$. For understanding, one must go back to the basic construction of a distribution function $F_{iom}^*(\mu_0, \lambda)$ for thin O-tubes. For any given λ , F_{iom}^* is determined sequentially from $\mu_0 = -\beta$ to $\mu_0 = -\alpha$ by the requirement that it reproduce the density $\rho_{om}(\lambda, \mu, -\beta)$. An increase of ρ_{om} locally at $\mu = \bar{\mu}$, say, will cause $F_{iom}^*(\bar{\mu}, \lambda)$ to increase, but also $F_{iom}^*(\mu_0, \lambda)$ to decrease for $\mu_0 > \bar{\mu}$, because a further part of the density in this region will be provided by the orbits with turning points at $\mu = \bar{\mu}$ that have been added (see Fig. 2). It is this corresponding decrease of orbits with larger $\bar{\mu}$, plus the limitation imposed by the fact that ρ_{om} is ρ_m at the intersection of surface $\mathcal{L}$ with surface $\mathcal{S}$, and so cannot be changed, which causes the negative correlation between $\rho_{om}(\lambda, -\alpha, v)$ and $\rho_{om}(\lambda, \mu, -\beta)$, and, hence, from equation (2.38), a positive correlation between $\rho_{sm}(\lambda, -\alpha, v)$ and $\rho_{om}(\lambda, \mu, -\beta)$.

3.3. S-Tubes

The density of a thin S-tube with turning points v_0 and $\lambda_1 = \lambda_2$ is given by

$$\rho_s(\lambda, \mu, v; v_0, \lambda_1, \lambda_1) = \rho_{is}(\lambda, \mu, v; v_0) \delta(\lambda - \lambda_1), \quad (3.38)$$

where we have defined

$$\rho_{is}(\lambda, \mu, v; v_0) = \frac{2\pi\sqrt{2}(\gamma - \alpha)}{\sqrt{B(v; v_0, \lambda, \lambda)}\sqrt{-B(\mu; v_0, \lambda, \lambda)}\sqrt{(\lambda - v_0)U[v_0, \lambda, \lambda, \lambda]}}. \quad (3.39)$$

A distribution of such tubes is described by a distribution function $F_{sm}(v_0, \lambda_1, \lambda_2)$ given by

$$F_{sm}(v_0, \lambda_1, \lambda_2) = F_{ism}(v_0, \lambda_1) \delta(\lambda_2 - \lambda_1 - 0). \quad (3.40)$$

Upon substitution of expressions (3.40) and (3.38), equation (2.32) reduces to

$$\rho_{sm}(\lambda, \mu, \nu) = \int_{\nu}^{-\beta} d\nu_0 F_{ism}(\nu_0, \lambda) \rho_{ts}(\lambda, \mu, \nu; \nu_0). \quad (3.41)$$

This gives the density $\rho_{sm}(\lambda, \mu, \nu)$ of the distribution of thin S-tubes described by a distribution function $F_{sm}(\nu_0, \lambda_1, \lambda_2)$ given in equation (3.40). Equation (3.41) is an integral equation for F_{ism} , and, like those for the I- and O-tubes, does not have a solution for general $\rho_{sm}(\lambda, \mu, \nu)$. We determine F_{ism} by specification of the density in surface $\mathcal{S}$. The fundamental integral equation then becomes

$$\rho_{sm}(\lambda, -\alpha, \nu) = \frac{1}{\lambda - \nu} \int_{\nu}^{-\beta} \frac{F_{ism}^*(\nu_0, \lambda) d\nu_0}{\sqrt{W[\nu_0] - W[\nu]}}, \quad (3.42)$$

where we have defined

$$F_{ism}^*(\nu_0, \lambda) = \frac{2\pi\sqrt{2}(\gamma - \alpha)F_{ism}(\nu_0, \lambda)}{(\lambda + \alpha)\sqrt{(-\alpha - \nu_0)U[\nu_0, -\alpha, \lambda, \lambda](\lambda - \nu_0)U[\nu_0, \lambda, \lambda, \lambda]}}. \quad (3.43)$$

These equations are identical to equations (2.31) and (2.32) of Paper III, for the distribution function of thin S-tubes in the oblate limit for which $\beta = \alpha$ and $\mu = -\alpha$ always. The solution of the one-dimensional integral equation of Abel type for F_{ism} can be written as

$$F_{ism}^*(\nu_0, \lambda) = -\frac{1}{\pi} \frac{\partial}{\partial \nu_0} \int_{\nu_0}^{-\beta} \frac{(\lambda - \sigma)\rho_{sm}(\lambda, -\alpha, \sigma)dW[\sigma]}{\sqrt{W[\sigma] - W[\nu_0]}}, \quad (3.44)$$

or

$$F_{ism}^*(\nu_0, \lambda) = -\frac{W'[\nu_0]}{\pi} \int_{\nu_0}^{-\beta} \frac{[\partial(\lambda - \sigma)\rho_{sm}(\lambda, -\alpha, \sigma)/\partial\sigma]d\sigma}{\sqrt{W[\sigma] - W[\nu_0]}}, \quad (3.45)$$

where $W[\tau]$ is again our shorthand notation for $U[\tau, \lambda, \lambda]$, as defined in equation (3.25). The fact that ρ_{sm} must vanish at surface $\mathcal{S}$ was used in deriving equation (3.45). A necessary and sufficient condition for f_{sm} to be nonnegative is that the integral in equation (3.44) be a monotonically decreasing function of ν_0 for all $\lambda \geq -\alpha$.

To calculate $\rho_{sm}(\lambda, \mu, \nu)$ away from surface $\mathcal{S}$, we use equations (3.39) and (3.43) to obtain

$$F_{ism}(\nu_0, \lambda)\rho_{ts}(\lambda, \mu, \nu; \nu_0) = \frac{1}{\lambda - \nu} \left(\frac{\lambda + \alpha}{\lambda - \mu} \right) \frac{\sqrt{W[-\alpha] - W[\nu_0]} F_{ism}^*(\nu_0, \lambda)}{\sqrt{W[\nu_0] - W[\nu]} \sqrt{W[\mu] - W[\nu_0]}}. \quad (3.46)$$

After substitution of this expression into equation (3.41), a further integration by parts, exchange of the order of integration, and the trigonometric substitution $W[\nu_0] = W[\nu] + w_2 \sin^2 \theta$, we obtain

$$\rho_{sm}(\lambda, \mu, \nu) = -\frac{2(\lambda + \alpha)}{\pi(\lambda - \mu)(\lambda - \nu)} \int_{\nu}^{-\beta} d\sigma A_0(w_4, w_3, w_2) \frac{\partial}{\partial \sigma} [(\lambda - \sigma)\rho_{sm}(\lambda, -\alpha, \sigma)], \quad (3.47)$$

where A_0 is again the function (3.15), w_2 and w_3 are given in equation (3.30), and we have defined

$$w_4 = W[-\alpha] - W[\sigma]. \quad (3.48)$$

We now have $A_0(w_4, w_3, w_2) \geq \pi/2$ because $-\alpha \geq \mu$ and hence $w_4 \geq w_3 \geq 0$. The integrand in equation (3.47) remains finite as $\nu \rightarrow -\beta$, so that ρ_{sm} vanishes in this limit, as it should. Equation (3.47) can also be integrated by parts to give

$$\rho_{sm}(\lambda, \mu, \nu) = \left(\frac{\lambda + \alpha}{\lambda - \mu} \right) \left\{ \rho_{sm}(\lambda, -\alpha, \nu) \sqrt{\frac{W[-\alpha] - W[\nu]}{W[\mu] - W[\nu]}} - \frac{1}{\pi(\lambda - \nu)} \int_{\nu}^{-\beta} d\sigma A_2(w_4, w_3, w_2)(\lambda - \sigma)\rho_{sm}(\lambda, -\alpha, \sigma)U[\sigma, \sigma, \lambda, \lambda] \right\}, \quad (3.49)$$

where the function A_2 was defined in equation (3.32) but is now nonpositive, since $w_3 \leq w_4$. Thus, the larger $\rho_{sm}(\lambda, -\alpha, \nu)$ is, the larger ρ_{sm} is elsewhere. Equation (3.49) cannot be used at surface $\mathcal{H}$ ($\mu = -\beta$) because the integral diverges there; w_3 vanishes at the upper limit of integration where A_2 has a nonintegrable singularity.

3.4. Actions and Distribution Functions

The fundamental arguments of the distribution function f are the action integrals J_λ , J_μ , and J_ν defined as (Paper I, § 8)

$$J_\tau = \frac{1}{2\pi} \oint p_\tau d\tau = \frac{\sqrt{2}}{\pi} \int_{\tau_-}^{\tau_+} \sqrt{\frac{B(\tau; \tau_1, \tau_2, \tau_3)}{(\tau + \alpha)(\tau + \beta)(\tau + \gamma)}} d\tau \quad (\tau = \lambda, \mu, \nu). \quad (3.50)$$

The limits of integration depend on the type of orbit, and can be read off from inequalities (2.8)–(2.11), respectively, for each of the four cases.

One action tends to zero for each of the thin-tube families. For a nearly thin I-tube, with turning points $\mu_1 \sim \mu_2 \sim \mu$ and λ_0 , it is J_μ , which can be expanded in powers of $\mu_2 - \mu_1$. To lowest order we find

$$J_\mu \sim \frac{\sqrt{(\lambda_0 - \mu_1)U[\mu_1, \mu_1, \mu_1, \lambda_0]}}{4\sqrt{2}\sqrt{(-\alpha - \mu_1)(\mu_1 + \beta)(\mu_1 + \gamma)}} (\mu_2 - \mu_1)^2 + O[(\mu_2 - \mu_1)^3], \quad (3.51)$$

so that $\mu_2 - \mu_1 \propto \sqrt{J_\mu}$. Similarly, for a nearly thin O-tube with turning points μ_0 and $\lambda_1 \sim \lambda_2 \sim \lambda$, we have

$$J_\lambda \sim \frac{\sqrt{(\lambda_1 - \mu_0)U[\mu_0, \lambda_1, \lambda_1, \lambda_1]}}{4\sqrt{2}\sqrt{(\lambda_1 + \alpha)(\lambda_1 + \beta)(\lambda_1 + \gamma)}} (\lambda_2 - \lambda_1)^2 + O[(\lambda_2 - \lambda_1)^3], \quad (3.52)$$

so that $\lambda_2 - \lambda_1 \propto \sqrt{J_\lambda}$. For a nearly thin S-tube with turning points v_0 and $\lambda_1 \sim \lambda_2 \sim \lambda$, we find

$$J_\lambda \sim \frac{\sqrt{(\lambda_1 - v_0)U[v_0, \lambda_1, \lambda_1, \lambda_1]}}{4\sqrt{2}\sqrt{(\lambda_1 + \alpha)(\lambda_1 + \beta)(\lambda_1 + \gamma)}} (\lambda_2 - \lambda_1)^2 + O[(\lambda_2 - \lambda_1)^3], \quad (3.53)$$

so that, again, $\lambda_2 - \lambda_1 \propto \sqrt{J_\lambda}$.

With the help of the above relations, we can rewrite the delta functions that occur in the expressions for each of the thin-tube distribution functions. Combining equations (2.25), (3.3), (3.6), and (3.51), we obtain

$$f_{im}(J_\lambda, J_\mu, J_v) = f_{im}(J_\lambda, J_v)\delta(J_\mu - 0), \quad (3.54)$$

with

$$f_{im}(J_\lambda, J_v) = \frac{\sqrt{\mu_1 + \beta}}{8\pi\sqrt{(-\alpha - \mu_1)(\mu_1 + \gamma)}} \frac{\sqrt{(\lambda_0 + \beta)U[-\beta, \mu_1, \mu_1, \lambda_0]} F_{im}^*(\mu_1, \lambda_0)}{(\lambda_0 - \mu_1)U[\mu_1, \mu_1, \lambda_0, \lambda_0]U[\mu_1, \mu_1, \mu_1, \lambda_0]}. \quad (3.55)$$

Similarly

$$f_{om}(J_\lambda, J_\mu, J_v) = f_{om}(J_\mu, J_v)\delta(J_\lambda - 0), \quad (3.56)$$

with

$$f_{om}(J_\mu, J_v) = \frac{\sqrt{\lambda_1 + \beta}}{8\pi\sqrt{(\lambda_1 + \alpha)(\lambda_1 + \gamma)}} \frac{\sqrt{(\mu_0 + \beta)U[-\beta, \mu_0, \lambda_1, \lambda_1]} F_{om}^*(\mu_0, \lambda_1)}{(\lambda_1 - \mu_0)U[\mu_0, \mu_0, \lambda_1, \lambda_1]U[\mu_0, \lambda_1, \lambda_1, \lambda_1]}, \quad (3.57)$$

and

$$f_{sm}(J_\lambda, J_\mu, J_v) = f_{sm}(J_\mu, J_v)\delta(J_\lambda - 0), \quad (3.58)$$

with

$$f_{sm}(J_\mu, J_v) = \frac{\sqrt{\lambda_1 + \alpha}}{8\pi\sqrt{(\lambda_1 + \beta)(\lambda_1 + \gamma)}} \frac{\sqrt{(-\alpha - v_0)U[v_0, -\alpha, \lambda_1, \lambda_1]} F_{sm}^*(v_0, \lambda_1)}{(\lambda_1 - v_0)U[v_0, v_0, \lambda_1, \lambda_1]U[v_0, \lambda_1, \lambda_1, \lambda_1]}. \quad (3.59)$$

Expressions for the turning points in terms of the action integrals generally require numerical inversion of the relations $J_\tau = J_\tau(\tau_1, \tau_2, \tau_3)$, as given by equations (3.50).

4. DIVISION OF DENSITY

In § 4.1 we consider how to implement the crucial first step of the strategy outlined in § 2.5 for constructing triaxial models. This is the split between I- and O-tubes of the density in surface $\mathcal{L}$. Then, in §§ 4.2 and 4.3, we analyze the behavior of the densities of each of the tube-orbit families near the degenerate surfaces $\mathcal{S}$ and $\mathcal{L}$, respectively, and, in particular, the behavior near the focal ellipse, the edge common to surfaces $\mathcal{S}$ and $\mathcal{E}$, and the focal hyperbola, the edge common to surfaces $\mathcal{L}$ and $\mathcal{H}$. These behaviors are significant for the determination of the residual box-orbit density and its distribution function, the latter being discussed in § 5. In the course of the discussion of § 4.3, we obtain explicit expressions for the S-tube distribution function F_{ism}^* and density ρ_{sm} in terms of the split of the I- and O-tubes defined in § 4.1.

4.1. Split between Long-Axis Tubes

The split (2.37) is constrained by the requirement that the distribution functions f_{im} and f_{om} be nonsingular at both $\mu_1 = -\alpha$ and $\lambda_1 = -\alpha$. Then, because surface $\mathcal{S}$ is reached by an I-tube at an extremity, if at all, we must have (see also eq. [2.30] and Fig. 2)

$$\rho_{im}(\lambda, -\alpha, v) = 0 \quad (\lambda \neq -\alpha). \quad (4.1)$$

Similarly, surface $\mathcal{E}$ can be reached only by an O-tube at an extremity, and

$$\rho_{om}(-\alpha, \mu, v) = 0 \quad (\mu \neq -\alpha). \quad (4.2)$$

It follows that both component densities $\rho_{im}(\lambda, \mu, -\beta)$ and $\rho_{om}(\lambda, \mu, -\beta)$ must be discontinuous and exhibit what we shall refer to as *radial behavior* at the focal ellipse $\lambda = \mu = -\alpha$; that is, their limiting values there depend on the direction in which it is approached.

For instance,

$$\lim_{\lambda \downarrow -\alpha} \rho_{im}(\lambda, -\alpha, -\beta) = 0, \quad \lim_{\mu \uparrow -\alpha} \rho_{im}(-\alpha, \mu, -\beta) = \rho_m(-\alpha, -\alpha, -\beta). \quad (4.3)$$

We showed in Paper IV that the constraints (4.1) and (4.2) can be satisfied by splits of the form

$$\begin{aligned} \rho_{im}(\lambda, \mu, -\beta) &= \left(\frac{-\alpha - \mu}{\lambda - \mu} \right) \left[1 + \left(\frac{\lambda + \alpha}{\beta - \alpha} \right) k(s) \right] \rho_m(\lambda, \mu, -\beta), \\ \rho_{om}(\lambda, \mu, -\beta) &= \left(\frac{\lambda + \alpha}{\lambda - \mu} \right) \left[1 - \left(\frac{-\alpha - \mu}{\beta - \alpha} \right) k(s) \right] \rho_m(\lambda, \mu, -\beta), \end{aligned} \quad (4.4)$$

where $k(s)$ is a function of $s = (-\alpha - \mu)/(\lambda - \mu)$. The factor $(-\alpha - \mu)$ in $\rho_{im}(\lambda, \mu, -\beta)$ becomes a factor of $F_{im}^*(\mu, \lambda_0)$ by equation (3.11), while $(\lambda + \alpha)$ becomes a factor of $F_{om}^*(\mu_0, \lambda)$ by equation (3.27). The resulting distribution functions $f_{im}(J_\lambda, J_\nu)$ and $f_{om}(J_\mu, J_\nu)$ vanish on the surfaces $\mu_1 = -\alpha$ and $\lambda_1 = -\alpha$, respectively. Because the actions are continuous functions of the turning points, the distribution functions f_{im} and f_{om} then are continuous functions everywhere except possibly at the focal ellipse ($\lambda_0 = \mu_1 = -\alpha$ for f_{im} ; $\lambda_1 = \mu_0 = -\alpha$ for f_{om}), where they generally have integrable inverse square root singularities. Such solutions are labeled *permissible*. We showed in § IIIb of Paper IV that there is sufficient freedom of choice to determine functions $k(s)$ that give *regular* solutions for which f_{im} and f_{om} are both bounded at the focal ellipse, although they do exhibit radial behavior there.

A complication that occurs with the split (4.4) is that it may predict impossible negative values of either ρ_{im} or ρ_{om} . For instance, negative values of ρ_{om} occur if $\mu < -\alpha - (\beta - \alpha)/k$ when k is a constant larger than unity. We avoid such negative densities by switching from the split (4.4) to a split in which $\rho_{om}(\lambda, \mu, -\beta) \equiv 0$ and $\rho_{im}(\lambda, \mu, -\beta) = \rho_m(\lambda, \mu, -\beta)$ at the curve in (λ, μ) -space at which equation (4.4) predicts that $\rho_{om}(\lambda, \mu, -\beta)$ vanishes. We call such a curve, or a similar one at which ρ_{im} vanishes, where we similarly switch to a split in which $\rho_{im}(\lambda, \mu, -\beta) \equiv 0$ and $\rho_{om}(\lambda, \mu, -\beta) = \rho_m(\lambda, \mu, -\beta)$, a *switch curve*. The use of such *switches* gives nonnegative density components $\rho_{im}(\lambda, \mu, -\beta)$ and $\rho_{om}(\lambda, \mu, -\beta)$ which are continuous at the switch curve, but whose gradients are discontinuous in the direction normal to the switch curve. These normal-derivative discontinuities are significant for the present triaxial case because, as equations (3.16) and (3.31) show, they carry through to the three-dimensional densities $\rho_{im}(\lambda, \mu, \nu)$ and $\rho_{om}(\lambda, \mu, \nu)$. The latter have normal-derivative discontinuities across the *switch surface* that is obtained by continuing the switch curve in (λ, μ) -space to all ν , and, most significantly, the discontinuities do not cancel in the sum $\rho_{im} + \rho_{om}$ for general values of ν on the switch surface as they do in surface $\mathcal{L}$. Hence the residual box-orbit density has a normal-derivative discontinuity across the switch surface, which in turn implies some lack of smoothness of the box-orbit distribution function F_{bm} . The rectangular shape of box orbits in ellipsoidal coordinate space makes it difficult for a box-orbit distribution function to produce a density field with a normal-derivative discontinuity across a curved surface in that space. In fact, such a discontinuity can be produced only by a box-orbit distribution function which has an infinite logarithmic singularity at the curved switch surface. Because we wish to avoid such singular box-orbit distribution functions, we shall not here use the regular solutions of Paper IV, which have curved switch surfaces when they are significantly nonspherical, but only the permissible solutions with constant values of k . Such solutions have switch surfaces if either $k > 1$ or $k < 0$, but ones that lie on hyperboloids of constant $\mu = -\alpha - (\beta - \alpha)/k$ for $k > 1$, or ellipsoids of constant $\lambda = -\alpha + (\beta - \alpha)/(-k)$ for $k < 0$. In either of these cases, the switch surfaces are flat and aligned with box-orbit boundaries in ellipsoidal coordinate space (see Fig. 2) and do not give rise to singular box-orbit distribution functions (see § 5 below). It should be noted that the problem just described with respect to the triaxial properties of many of the regular solutions of the kind considered in Paper IV (those for $\beta/\alpha < 0.7161$) is not inherently due to their regularity, but is due to our lack of care in making them sufficiently smooth. Regular solutions for the prolate case are nonunique, and smoother regular solutions could be found by small modifications of ours which smooth their sharp edges.

Once the split (2.37) has been accomplished for any given model with known density $\rho_m(\lambda, \mu, \nu)$, the three-dimensional densities of the two kinds of long-axis tubes are fixed by equations (3.16) and (3.31). The density ρ_{sm} in surface $\mathcal{S}$ is now fixed by the requirement (2.38) from § 2.5. The distribution function $f_{ism}(J_\mu, J_\nu)$ and the three-dimensional density $\rho_{sm}(\lambda, \mu, \nu)$ then follow from equations (3.59), (3.44), and (3.49), respectively, and the residual box-orbit density from equation (2.39).

4.2. Behavior of Density near Surface $\mathcal{S}$ and the Focal Ellipse

It is evident from equations (3.13) and (3.29) that the three-dimensional densities $\rho_{im}(\lambda, \mu, \nu)$ and $\rho_{om}(\lambda, \mu, \nu)$ exhibit radial behavior near the whole of the focal ellipse, because of their $(\lambda - \mu)^{-1}$ factors and the $(-\alpha - \mu)$ and $(\lambda + \alpha)$ factors of $\rho_{im}(\sigma, \mu, -\beta)$ and $\rho_{om}(\lambda, \sigma, -\beta)$, respectively, contained in the split (4.4). The integrals in these two equations are continuous in view of the boundedness of their A_0 terms. Near the focal ellipse, therefore,

$$\rho_{im}(\lambda, \mu, \nu) \sim \left(\frac{-\alpha - \mu}{\lambda - \mu} \right) \lim_{\mu \uparrow -\alpha} \rho_{im}(-\alpha, \mu, \nu), \quad (4.5)$$

and

$$\rho_{om}(\lambda, \mu, \nu) \sim \left(\frac{\lambda + \alpha}{\lambda - \mu} \right) \lim_{\lambda \downarrow -\alpha} \rho_{om}(\lambda, -\alpha, \nu). \quad (4.6)$$

The density $\rho_{sm}(\lambda, \mu, \nu)$ also exhibits radial behavior near the focal ellipse, as is seen from the $(\lambda + \alpha)/(\lambda - \mu)$ factor in equation (3.49). This equation contains the density ρ_{sm} in surface $\mathcal{S}$, which is continuous because the density $\rho_{om}(\lambda, -\alpha, \nu)$ is continuous. Thus, near

the focal ellipse,

$$\rho_{sm}(\lambda, \mu, \nu) \sim \left(\frac{\lambda + \alpha}{\lambda - \mu} \right) \lim_{\lambda \downarrow -\alpha} \rho_{sm}(\lambda, -\alpha, \nu). \quad (4.7)$$

The sum of the limits (4.6) and (4.7) is $\rho_m(-\alpha, -\alpha, \nu)$ by equation (2.38). The behavior of the box-orbit density near the focal ellipse is therefore described by (see eq. [2.39])

$$\rho_{bm}(\lambda, \mu, \nu) \sim \left(\frac{-\alpha - \mu}{\lambda - \mu} \right) \left[\rho_m(-\alpha, -\alpha, \nu) - \lim_{\mu \uparrow -\alpha} \rho_{im}(-\alpha, \mu, \nu) \right]. \quad (4.8)$$

For other points that are near surface $\mathcal{S}$ but not near the focal ellipse, the equations for ρ_{im} , ρ_{om} , and ρ_{sm} show that

$$\begin{aligned} \rho_{im}(\lambda, \mu, \nu) &= O(-\alpha - \mu) \\ \rho_{om}(\lambda, \mu, \nu) &= \rho_{om}(\lambda, -\alpha, \nu) + O(-\alpha - \mu) \quad (\lambda \neq -\alpha), \\ \rho_{sm}(\lambda, \mu, \nu) &= \rho_{sm}(\lambda, -\alpha, \nu) + O(-\alpha - \mu) \end{aligned} \quad (4.9)$$

and hence

$$\rho_{bm}(\lambda, \mu, \nu) = O(-\alpha - \mu) \quad (\lambda \neq -\alpha). \quad (4.10)$$

4.3. Behavior of Density near Surface $\mathcal{L}$ and the Focal Hyperbola

The behavior of ρ_{im} can be deduced from either equation (3.13) or equation (3.16). As $\nu \uparrow -\beta$, $u_2 - u_1 = O(-\beta - \nu)$, and so

$$\rho_{im}(\lambda, \mu, \nu) = \rho_{im}(\lambda, \mu, -\beta) + O(-\beta - \nu) \quad (\mu \neq -\beta), \quad (4.11)$$

while

$$\rho_{im}(\lambda, \mu, \nu) \sim \left(\frac{\mu + \beta}{\mu - \nu} \right) \lim_{\mu \downarrow -\beta} \rho_{im}(\lambda, \mu, -\beta), \quad (4.12)$$

near the focal hyperbola.

The behavior of ρ_{om} can most easily be deduced from equation (3.29). As a result of its arcsin term, we obtain, for $\mu \neq -\beta$,

$$\rho_{om}(\lambda, \mu, \nu) = \rho_{om}(\lambda, \mu, -\beta) - \frac{2\rho_{om}(\lambda, -\beta, -\beta)}{\pi} \left(\frac{\lambda + \beta}{\lambda - \mu} \right) \left[\frac{U[-\beta, -\beta, \lambda, \lambda]}{U[-\beta, \mu, \lambda, \lambda]} \left(\frac{-\beta - \nu}{\mu + \beta} \right) \right]^{1/2} + O(-\beta - \nu), \quad (4.13)$$

while near the focal hyperbola,

$$\rho_{om}(\lambda, \mu, \nu) \sim \frac{2}{\pi} \rho_{om}(\lambda, -\beta, -\beta) \arcsin \sqrt{\frac{\mu + \beta}{\mu - \nu}}. \quad (4.14)$$

In view of the radial behavior of ρ_{om} near the focal hyperbola, note that the quantity $\rho_{om}(\lambda, -\beta, -\beta)$, which first appeared unambiguously in equation (3.27) from an evaluation of $\rho_{om}(\lambda, \mu, -\beta)$, needs now to be defined more precisely by

$$\rho_{om}(\lambda, -\beta, -\beta) = \lim_{\mu \downarrow -\beta} \rho_{om}(\lambda, \mu, -\beta). \quad (4.15)$$

The sharp decline of the ρ_{om} density away from surface $\mathcal{L}$ described by the negative second component in equation (4.13) is due to O-tubes with μ_0 turning points close to $-\beta$ whose ν -motion is a rotation, but which linger near surface $\mathcal{L}$ because their v_ν -velocity is small there. Such orbits may not be populated, and are not in any of our models with $k > 1$, i.e., all of our models of § 8 for $b/a \lesssim 0.44$ and some for $b/a \lesssim 0.84$. However, when such tubes are present, their negative $O[(-\beta - \nu)^{1/2}]$ density component affects their density on surface $\mathcal{S}$, and is exactly counterbalanced by a positive density of S-tubes whose ν -motion is a libration, but which linger near ν_0 turning points which are close to surface $\mathcal{L}$. To see this mathematically, we need to relate the three-dimensional density $\rho_{sm}(\lambda, \mu, \nu)$ to $\rho_{om}(\lambda, \mu, -\beta)$. This is done by first using equations (2.38) and (3.45), with $\rho_{om}(\lambda, -\alpha, \nu)$ given by equation (3.29), to evaluate F_{ism}^* . After an interchange of orders of integration and an integration, we obtain

$$\begin{aligned} F_{ism}^*(\nu_0, \lambda) &= \frac{W'[\nu_0]}{\pi} \left\{ - \int_{\nu_0}^{-\beta} \frac{[\partial(\lambda - \sigma)\rho_m(\lambda, -\alpha, \sigma)/\partial\sigma]d\sigma}{\sqrt{W[\sigma] - W[\nu_0]}} \right. \\ &\quad + \left(\frac{\lambda + \beta}{\lambda + \alpha} \right) \sqrt{\frac{W[-\beta] - W[\nu_0]}{W[-\alpha] - W[\nu_0]}} \left[\frac{(\lambda + \beta)\rho_{om}(\lambda, -\beta, -\beta)}{\sqrt{W[-\beta] - W[\nu_0]}} \right. \\ &\quad \left. \left. + \int_{-\beta}^{-\alpha} \frac{[\partial(\lambda - \sigma)\rho_{om}(\lambda, \sigma, -\beta)/\partial\sigma]d\sigma}{\sqrt{W[\sigma] - W[\nu_0]}} \right] \right\}. \end{aligned} \quad (4.16)$$

The required density is then obtained when this evaluation is used with equations (3.41) and (3.46), and a further interchange of orders of integration and an integration, as

$$\begin{aligned} \rho_{sm}(\lambda, \mu, \nu) = \frac{2}{\pi(\lambda - \mu)(\lambda - \nu)} \left\{ -(\lambda + \alpha) \int_{\nu}^{-\beta} d\sigma A_0(w_4, w_3, w_2) \frac{\partial}{\partial \sigma} [(\lambda - \sigma)\rho_m(\lambda, -\alpha, \sigma)] \right. \\ \left. + (\lambda + \beta)^2 \rho_{om}(\lambda, -\beta, -\beta) \arcsin \sqrt{\frac{W[-\beta] - W[\nu]}{W[\mu] - W[\nu]}} \right. \\ \left. + (\lambda + \beta) \int_{-\beta}^{-\alpha} A_3(w_1, w_1 + w_3, w_2 - w_1) \frac{\partial}{\partial \sigma} [(\lambda - \sigma)\rho_{om}(\lambda, \sigma, -\beta)] d\sigma \right\}, \end{aligned} \quad (4.17)$$

where the auxiliary function

$$A_3(w_1, w_2, w_3) = \int_0^{\pi/2} \frac{w_3 \cos^2 \theta d\theta}{\sqrt{w_1 + w_3 \cos^2 \theta} \sqrt{w_2 + w_3 \cos^2 \theta}}, \quad (4.18)$$

is given in equation (A4), and requires the evaluation of an R_J function. The second term in equation (4.17) exactly counterbalances the $O[(-\beta - \nu)^{1/2}]$ term of equation (4.13), with the result that, for $\nu \sim -\beta$ and $\mu \neq -\beta$,

$$\begin{aligned} \rho_{im}(\lambda, \mu, \nu) + \rho_{om}(\lambda, \mu, \nu) + \rho_{sm}(\lambda, \mu, \nu) &= \rho_{im}(\lambda, \mu, -\beta) + \rho_{om}(\lambda, \mu, -\beta) + O(-\beta - \nu) \\ &= \rho_m(\lambda, \mu, -\beta) + O(-\beta - \nu). \end{aligned} \quad (4.19)$$

It then follows that the behavior of the box density near surface $\mathcal{L}$ is

$$\rho_{bm}(\lambda, \mu, \nu) = O(-\beta - \nu) \quad (\mu \neq -\beta). \quad (4.20)$$

The second term of equation (4.17) is also the dominant one near the focal hyperbola, both the other terms being small near it. It gives

$$\begin{aligned} \rho_{sm}(\lambda, \mu, \nu) &\sim \frac{2}{\pi} \rho_{om}(\lambda, -\beta, -\beta) \arcsin \sqrt{\frac{-\beta - \nu}{\mu - \nu}} \\ &= \frac{2}{\pi} \rho_{om}(\lambda, -\beta, -\beta) \left(\frac{\pi}{2} - \arcsin \sqrt{\frac{\mu + \beta}{\mu - \nu}} \right). \end{aligned} \quad (4.21)$$

Combining equations (4.12), (4.14), and (4.21), the box-orbit density in the neighborhood of the focal hyperbola is

$$\rho_{bm} \sim \left(\frac{-\beta - \nu}{\mu - \nu} \right) \lim_{\mu \rightarrow -\beta} \rho_{im}(\lambda, \mu, -\beta). \quad (4.22)$$

The rapid $O[(-\beta - \nu)^{1/2}]$ increase of ρ_{sm} from its zero value at surface $\mathcal{L}$, as described by equation (4.21), is one reason for S-tubes to be significant in ellipsoids that are close to, but not exactly, prolate. The latter feature was found by Levison & Richstone (1987) and by Statler (1987) in their numerical solutions, and it reappears in the solutions that we present in § 8, where we give a fuller analysis of it.

Two other important results can be obtained from equation (4.16) after integrating the term in square brackets by parts to get

$$\frac{(\lambda + \alpha)\rho_m(\lambda, -\alpha, -\beta)}{\sqrt{W[-\alpha] - W[\nu_0]}} + \frac{1}{2} \int_{\sigma=-\beta}^{-\alpha} \frac{(\lambda - \sigma)\rho_{om}(\lambda, \sigma, -\beta) dW[\sigma]}{\{W[\sigma] - W[\nu_0]\}^{3/2}}. \quad (4.23)$$

The first result is that the manifest positivity of both of these terms, together with the positivity of the first term of equation (4.16) that is shown in Paper III to be a consequence of inequality (3.35), shows that the latter inequality also guarantees the nonnegativity everywhere of our triaxial S-tube distribution function. The second result is that expression (4.23) contains all of the dependence of F_{ism}^* on the choice of $\rho_{om}(\lambda, \sigma, -\beta)$. Increasing $\rho_{om}(\lambda, \sigma, -\beta)$ therefore increases the distribution function f_{ism} everywhere except where it vanishes for turning points on surfaces $\mathcal{E}$ and $\mathcal{L}$.

Finally, we remark that either of the integrals of equation (4.17) may be integrated by parts. When both are integrated, we obtain the useful alternative formula

$$\begin{aligned} \rho_{sm}(\lambda, \mu, \nu) = \frac{2}{\pi(\lambda - \mu)(\lambda - \nu)} \left\{ \frac{\pi}{2} (\lambda + \alpha)(\lambda - \nu) \rho_m(\lambda, -\alpha, \nu) \sqrt{\frac{W[-\alpha] - W[\nu]}{W[\mu] - W[\nu]}} \right. \\ - (\lambda + \alpha)(\lambda + \beta) [W[-\alpha] - W[-\beta]] \rho_m(\lambda, -\alpha, -\beta) \\ \times A_4(W[-\alpha] - W[-\beta], W[\mu] - W[-\beta], W[-\beta] - W[\nu]) \\ - \frac{1}{2} (\lambda + \alpha) \int_{\nu}^{-\beta} d\sigma A_2(w_4, w_3, w_2) (\lambda - \sigma) \rho_m(\lambda, -\alpha, \sigma) U[\sigma, \sigma, \lambda, \lambda] \\ \left. - \frac{1}{2} (\lambda + \beta) \int_{-\beta}^{-\alpha} d\sigma \frac{w_2 - w_1}{w_3} A_1(w_1 + w_3, w_1, w_2 - w_1) (\lambda - \sigma) \rho_{om}(\lambda, \sigma, -\beta) U[\sigma, \sigma, \lambda, \lambda] \right\}, \end{aligned} \quad (4.24)$$

where

$$A_4(u_1, u_2, u_3) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{u_1 + u_3 \cos^2 \theta} \sqrt{u_2 + u_3 \cos^2 \theta}}. \quad (4.25)$$

This form also avoids the evaluation of an R_J function (see eq. [A4]). The w_3 in the denominator of the second integral of equation (4.24) is canceled by a w_3 factor of the A_1 which comes from the definition (3.17). Also, this A_1 is nonpositive, so that, as noted earlier in §§ 3.2 and 3.3, an increase of $\rho_{om}(\lambda, \mu, -\beta)$ causes an increase of ρ_{sm} everywhere. Equation (4.17), or the above alternative form (4.24), allows the calculation of $\rho_{sm}(\lambda, \mu, v)$ directly upon assigning the O-tube density ρ_{om} on surface $\mathcal{L}$, without the need of calculating any of the three thin-orbit distribution functions.

5. DISTRIBUTION FUNCTION FOR BOX ORBITS

We now investigate the solution of the integral equation (2.33) for the distribution function of the box orbits that must provide the remaining density $\rho_{bm} = \rho_m - \rho_{im} - \rho_{om} - \rho_{sm}$. The analysis is an extension of that presented in Paper II for the distribution function of the flat box orbits in the elliptic disk. We establish that, for a wide class of box-orbit densities ρ_{bm} possessing the properties derived in § 4, there is a unique distribution function $f_{bm}(v_0, \mu_0, \lambda_0)$ which is continuous everywhere except at the focal ellipse ($\lambda_0 = \mu_0 = -\alpha$) and at the focal hyperbola ($\mu_0 = v_0 = -\beta$). The proof requires the recasting, in § 5.1, of equation (2.33) into a Volterra equation of the second kind, and is first established for the whole of the boxlike domain of f_{bm} other than the neighborhoods of the sides $\lambda_0 = -\alpha$ and $\mu_0 = -\beta$ (see Fig. 2). The neighborhoods of the focal ellipse and the focal hyperbola, where the behavior of f_{bm} depends on the behavior of the density ρ_{bm} as derived in §§ 4.2 and 4.3, is studied next in §§ 5.2 and 5.3, respectively. Finally, in a brief § 5.4, the existence proof is extended to the remainder of the neighborhoods of the sides $\lambda_0 = -\alpha$ and $\mu_0 = -\beta$.

5.1. Existence of a Continuous Distribution Function

The integral equation (2.33) to be solved for the box orbits is of Volterra type, because the three lower and variable limits of integration match the arguments of the now known density ρ_{bm} . This equation is an initial-value problem, which starts from the limit $\lambda \rightarrow \infty$, $\mu \uparrow -\alpha$, $v \uparrow -\beta$, and then proceeds in the directions in which each of these coordinates decreases. In its solution, the populations of the most extensive box orbits, whose vertices lie on the z -axis as $z \rightarrow \infty$, are determined first, and the populations of successively less extensive box orbits are determined subsequently. This determination must be carried out numerically, although analytic approximations are possible in certain limiting cases. But standard theorems of the theory of integral equations can be applied to demonstrate the existence, uniqueness, and continuity of our solution.

The feature of the integral equation (2.33) that makes it amenable to analysis, though not to explicit solution, is that it is of generalized Abel type. That is, its kernel ρ_b can be written as

$$\rho_b(\lambda, \mu, v; v_0, \mu_0, \lambda_0) = \frac{2\sqrt{2}(\gamma - \alpha)G(\lambda, \mu, v; v_0, \mu_0, \lambda_0)}{\sqrt{(\lambda_0 - \lambda)(\mu_0 - \mu)(v_0 - v)}}, \quad (5.1)$$

where

$$G(\lambda, \mu, v; v_0, \mu_0, \lambda_0) = \{(\lambda - \mu_0)(\lambda - v_0)U[v_0, \mu_0, \lambda, \lambda_0]\}^{-1/2} \\ \times \{(\lambda_0 - \mu)(\mu - v_0)U[v_0, \mu, \mu_0, \lambda_0]\}^{-1/2} \{(\lambda_0 - v)(\mu_0 - v)U[v, v_0, \mu_0, \lambda_0]\}^{-1/2}. \quad (5.2)$$

The factor G is an analytic function of each argument and is continuous throughout all of the region $-\alpha \leq \lambda \leq \lambda_0$, $-\beta \leq \mu \leq \mu_0 \leq -\alpha$, $-\gamma \leq v \leq v_0 \leq -\beta$ in which a solution is required, except for the neighborhoods of surfaces $\mathcal{E}$ and $\mathcal{H}$. These neighborhoods require a special analysis, which is given in subsequent subsections. We here exclude them by restricting attention to the region

$$-\alpha + \delta_1 \leq \lambda, \quad -\beta + \delta_2 \leq \mu, \quad -\gamma \leq v \leq -\beta, \quad (5.3)$$

where δ_1 and δ_2 are small but fixed positive numbers.

The next step in our analysis is to replace the infinite range of the variable λ by the finite range of the new variable s , defined by

$$s = U[-\beta, -\alpha, \lambda]. \quad (5.4)$$

This s is the potential at the point where the ellipsoid $\lambda = \text{constant}$ meets the z -axis, and so is negative and increases monotonically to zero as $\lambda \rightarrow \infty$. In terms of the unique inverse $\lambda = \lambda(s)$ of equation (5.4), equation (2.33) becomes

$$\rho_{bm}(\lambda(s), \mu, v) = \int_v^{-\beta} dv_0 \int_{\mu}^{-\alpha} d\mu_0 \int_s^0 ds_0 \frac{G^*(\lambda(s), \mu, v; v_0, \mu_0, \lambda(s_0))F_{bm}(v_0, \mu_0, \lambda(s_0))}{\sqrt{(s_0 - s)(\mu_0 - \mu)(v_0 - v)}}, \quad (5.5)$$

where

$$G^*(\lambda, \mu, v; v_0, \mu_0, \lambda_0) = \frac{2\sqrt{2}(\gamma - \alpha)\sqrt{U[-\beta, -\alpha, \lambda, \lambda_0]}}{U[-\beta, -\alpha, \lambda_0, \lambda_0]} G(\lambda, \mu, v; v_0, \mu_0, \lambda_0). \quad (5.6)$$

The function G^* is also analytic and continuous in the region (5.3).

Equation (5.5) is a singular Volterra equation of the first kind, but can be converted to the theoretically simpler case of a

nonsingular Volterra equation of the second kind. This conversion is achieved as follows. We first multiply both sides of equation (5.5) by

$$\int_t^0 \frac{ds}{\sqrt{s-t}} \int_{\eta}^{-\alpha} \frac{d\mu}{\sqrt{\mu-\eta}} \int_{\zeta}^{-\beta} \frac{dv}{\sqrt{v-\zeta}}, \quad (5.7)$$

and then interchange the orders of all the right-hand side integrations. The result is a nonsingular equation of the first kind:

$$S_{bm}(t, \eta, \zeta) = \int_t^0 ds_0 \int_{\eta}^{-\alpha} d\mu_0 \int_{\zeta}^{-\beta} dv_0 F_{bm}(v_0, \mu_0, \lambda(s_0)) K(t, \eta, \zeta; v_0, \mu_0, s_0). \quad (5.8)$$

Its kernel K is given by

$$K(t, \eta, \zeta; v_0, \mu_0, s_0) = \int_t^{s_0} ds \int_{\eta}^{\mu_0} d\mu \int_{\zeta}^{v_0} dv \frac{G^*(\lambda(s), \mu, v; v_0, \mu_0, \lambda(s_0))}{\sqrt{(s_0-s)(s-t)(\mu_0-\mu)(\mu-\eta)(v_0-v)(v-\zeta)}}, \quad (5.9)$$

which is an analytic function of each argument provided that we restrict attention to the domain corresponding to relations (5.3) of

$$U[-\beta, -\alpha, -\alpha + \delta_1] \leq t \leq s_0 \leq 0, \quad -\beta + \delta_2 \leq \eta \leq \mu_0 \leq -\alpha, \quad -\gamma \leq \zeta \leq v_0 \leq -\beta. \quad (5.10)$$

The analyticity of K is easily seen by introducing trigonometric substitutions of the form $s-t = (s_0-t) \sin^2 \theta$ in each integration (cf. eqs. [3.11]–[3.13] of Paper II).

The left-hand side of equation (5.8) is the known function

$$S_{bm}(t, \eta, \zeta) = \int_t^0 \frac{ds}{\sqrt{s-t}} \int_{\eta}^{-\alpha} \frac{d\mu}{\sqrt{\mu-\eta}} \int_{\zeta}^{-\beta} \frac{dv}{\sqrt{v-\zeta}} \rho_{bm}(\lambda(s), \mu, v). \quad (5.11)$$

Because ρ_{bm} is $O(-\alpha - \mu)$ near surface $\mathcal{S}$ and ρ_{bm} is $O(-\beta - v)$ near surface $\mathcal{L}$ from the discussion of §§ 4.2 and 4.3, it follows that $S_{bm}(t, \eta, \zeta)$ is $O[(-\alpha - \eta)^{3/2}]$ and $O[(-\beta - \zeta)^{3/2}]$ near $\eta = -\alpha$ and $\zeta = -\beta$, respectively. The behavior of $S_{bm}(t, \eta, \zeta)$ as $t \rightarrow 0$ is related to the manner in which the density ρ_{bm} decays at large distances, where $\lambda \sim r^2$ and $s = O(1/r)$. Thus, if ρ_{bm} decays as λ^{-p} ($p = 2$ for the perfect ellipsoid), then $S_{bm} = O(t^{2p+1/2})$ as $t \rightarrow 0$. Hence, with any reasonable law of density decay and for a sufficiently smooth box density ρ_{bm} , we have a continuous left-hand side term when we differentiate equation (5.8) partially with respect to each variable to get the following nonsingular Volterra integral equation of the second kind:

$$\begin{aligned} \frac{\partial^3 S_{bm}}{\partial t \partial \eta \partial \zeta} = & -K(t, \eta, \zeta; \zeta, \eta, t) F_{bm}(\zeta, \eta, \lambda(t)) + \sum_{i=1}^3 SI_i - \sum_{i=1}^3 DI_i \\ & + \int_t^0 ds_0 \int_{\eta}^{-\alpha} d\mu_0 \int_{\zeta}^{-\beta} dv_0 F_{bm}(v_0, \mu_0, \lambda(s_0)) \frac{\partial^3}{\partial t \partial \eta \partial \zeta} K(t, \eta, \zeta; v_0, \mu_0, s_0). \end{aligned} \quad (5.12)$$

The two summations here are of single and double integrals, respectively, in the different possible combinations of variables. A typical single integral is

$$SI_1 = \int_t^0 ds_0 F_{bm}(\zeta, \eta, \lambda(s_0)) \frac{\partial}{\partial t} K(t, \eta, \zeta; \zeta, \eta, s_0), \quad (5.13)$$

while a typical double integral is

$$DI_1 = \int_{\eta}^{-\alpha} d\mu_0 \int_{\zeta}^{-\beta} dv_0 F_{bm}(v_0, \mu_0, \lambda(t)) \frac{\partial^2}{\partial \eta \partial \zeta} K(t, \eta, \zeta; v_0, \mu_0, t). \quad (5.14)$$

Equation (5.12) is of a form for which the existence of a unique and continuous solution for F_{bm} can be established, provided that its left-hand side term is continuous. The equation is a three-dimensional generalization of the two-dimensional equation (3.15) of Paper II. There we could establish the properties of F_{bm} by direct application of a theorem given in § 96 of Goursat (1964, Vol. 3). Although Goursat's proof is strictly for the two-dimensional case, even though it is contained in a section entitled "Extension to functions of several variables," it can be extended to the three-dimensional case in a straightforward manner and then applied to equation (5.12), provided only that the coefficient of F_{bm} in that equation is nonzero. This is easily seen to be the case using equation (5.9), which gives the evaluation

$$\begin{aligned} K(t, \eta, \zeta; \zeta, \eta, t) = & \pi^3 G^*(\lambda(t), \eta, \zeta; \zeta, \eta, \lambda(t)) \\ = & \frac{2\pi^3 \sqrt{2}(\gamma - \alpha)}{(\lambda - \eta)(\lambda - \zeta)(\eta - \zeta) \sqrt{U[-\beta, -\alpha, \lambda] U[\zeta, \eta, \lambda] U[\zeta, \eta, \eta] U[\zeta, \zeta, \eta, \lambda]}}. \end{aligned} \quad (5.15)$$

Hence the existence of a unique and continuous solution for $F_{bm}(v_0, \mu_0, \lambda_0)$ in the domain (5.3) is then proved. It is generally $O[(-\beta - v_0)^{1/2}]$ near $v_0 = -\beta$ and $O[(-\alpha - \mu_0)^{1/2}]$ near $\mu_0 = -\alpha$.

Any singularity of $\partial^3 S_{bm} / \partial t \partial \eta \partial \zeta$ is reflected in a corresponding singularity of the distribution function F_{bm} . It was for this reason that we restricted attention in § 4.1 to solutions for which the function k of equation (4.4) is constant, and excluded solutions for

which $\rho_{bm}(\lambda, \mu, \nu)$ has a normal-derivative discontinuity across a switch surface that is not aligned with a coordinate surface of λ or μ . A direct analysis shows that the double differentiation $\partial^2/\partial t \partial \eta$ of the S_{bm} that is obtained via the integration (5.7) from such a ρ_{bm} generates a logarithmic singularity. Solutions for which k is a constant greater than unity, and which are used below, have box densities for which $\partial \rho_{bm}/\partial \mu$ is discontinuous across the surface $\mu = \mu_s = -\alpha - (\beta - \alpha)/k$. This discontinuity is smoothed by the μ -part of the integration in expression (5.7), and then unsmoothed by the η -differentiation in equation (5.12). The net result is that there is a term proportional to $(\mu_s - \eta)^{1/2}$ for $\eta < \mu_s$ on the left-hand side of equation (5.12) and one proportional to $(\mu_s - \mu_0)^{1/2}$ for $\mu_0 < \mu_s$ in F_{bm} , but no discontinuity. Solutions for which k is a constant less than zero have similar behavior across the surface $\lambda = \lambda_s = -\alpha + (\beta - \alpha)/(-k)$.

5.2. Behavior near the Focal Ellipse

The analysis needed to understand the form of the solution of equation (2.33) near the focal ellipse is similar to that given in § IIIc of Paper II for the focal corner $\lambda = \mu = -\alpha$ of the elliptic disk. In view of this, we shall give an abbreviated discussion here, referring to our earlier analysis for fuller details. The essential steps in understanding the behavior of the integral equation in a slender box adjacent to the focal ellipse in which both $(\lambda + \alpha)$ and $(-\alpha - \mu)$ are small and $O(\epsilon)$ are as follows:

1. The density to be provided by the box orbits is that given by equation (4.8). It is $O(1)$ and has radial behavior in the slender box.
2. The width of the μ_0 -range of integration is $O(\epsilon)$.
3. The λ_0 -range of integration can be split into two parts, the first being the slender box region in which $(\lambda_0 + \alpha)$ is $O(\epsilon)$, and the second being the rest of the range.
4. The solution for $F_{bm}(\nu_0, \mu_0, \lambda_0)$ with λ_0 in this second region has already been determined before the focal ellipse is approached, in view of the Volterra form of the integral equation for F_{bm} . It is $O(\epsilon^{1/2})$, because the inhomogeneous term ρ_{bm} is $O(-\alpha - \mu_0) = O(\epsilon)$ when $(\lambda + \alpha)$ is not small, while the kernel ρ_b is $O(\epsilon^{-1/2})$ because of the $(\mu - \mu_0)^{-1/2}$ factor. The contribution of this second λ_0 -range to the right-hand side of equation (2.33) must also be $O(\epsilon)$ when λ is in the slender box. This contribution is too small, as well as qualitatively inappropriate, to produce the density field described in step 1.
5. But this density can be provided by some suitable and $O(1)$ distribution F_{bm} of orbits in the slender box region in which both $(\lambda_0 + \alpha)$ and $(-\alpha - \mu_0)$ are $O(\epsilon)$. Although the size of this region is only $O(\epsilon^2)$, the kernel ρ_b is $O(\epsilon^{-2})$ because its factor $[(\lambda_0 - \lambda)(\lambda - \mu_0)(\lambda_0 - \mu)(\mu_0 - \mu)]^{-1/2}$ is of this order.

For the slender box region, equation (2.33) can be simplified to

$$\left(\frac{-\alpha - \mu}{\lambda - \mu}\right) \left[\rho_m(-\alpha, -\alpha, \nu) - \lim_{\mu \uparrow -\alpha} \rho_{im}(\alpha, \mu, \nu) \right] = \int_{\mu}^{-\alpha} d\mu_0 \int_{\lambda}^{\infty} \frac{d\lambda_0}{\sqrt{(\lambda_0 - \lambda)(\mu_0 - \mu)(\lambda - \mu_0)(\lambda_0 - \mu)}} \times \frac{1}{(-\alpha - \nu)} \int_{\nu}^{-\beta} \frac{2\sqrt{2}(\gamma - \alpha)F_{bm}(\nu_0, \mu_0, \lambda_0)d\nu_0}{(-\alpha - \nu_0)U[\nu_0, -\alpha, -\alpha, -\alpha]\sqrt{U[\nu_0, -\alpha, -\alpha] - U[\nu, -\alpha, -\alpha]}}. \quad (5.16)$$

This simplification is achieved by using the estimate (4.8), and setting λ , λ_0 , μ , and μ_0 equal to $-\alpha$ in all factors of the kernel except those that are small. The solution of equation (5.16) is of the separated form

$$2\sqrt{2}(\gamma - \alpha)F_{bm}(\nu_0, \mu_0, \lambda_0) = F_{bm1}(\nu_0)F_{bm2}(\mu_0, \lambda_0), \quad (5.17)$$

with

$$(-\alpha - \nu) \left[\rho_m(-\alpha, -\alpha, \nu) - \lim_{\mu \uparrow -\alpha} \rho_{im}(-\alpha, \mu, \nu) \right] = \int_{\nu}^{-\beta} \frac{F_{bm1}(\nu_0)d\nu_0}{(-\alpha - \nu_0)U[\nu_0, -\alpha, -\alpha, -\alpha]\sqrt{U[\nu_0, -\alpha, -\alpha] - U[\nu, -\alpha, -\alpha]}}, \quad (5.18)$$

and

$$\left(\frac{-\alpha - \mu}{\lambda - \mu}\right) = \int_{\mu}^{-\alpha} d\mu_0 \int_{\lambda}^{\infty} \frac{F_{bm2}(\mu_0, \lambda_0)d\lambda_0}{\sqrt{(\lambda_0 - \lambda)(\mu_0 - \mu)(\lambda - \mu_0)(\lambda_0 - \mu)}}. \quad (5.19)$$

Equation (5.18) is yet another Abel equation, with solution

$$F_{bm1}(\nu_0) = -\frac{(-\alpha - \nu_0)U[\nu_0, -\alpha, -\alpha, -\alpha]U[\nu_0, \nu_0, -\alpha, -\alpha]}{\pi} \int_{\nu_0}^{-\beta} \frac{(\partial/\partial \nu)\{(-\alpha - \nu)[\rho_m(-\alpha, -\alpha, \nu) - \lim_{\mu \uparrow -\alpha} \rho_{im}(-\alpha, \mu, \nu)]\}d\nu}{\sqrt{U[\nu, -\alpha, -\alpha] - U[\nu_0, -\alpha, -\alpha]}}, \quad (5.20)$$

since the left-hand side of equation (5.18) vanishes at $\nu = -\beta$ by equation (4.3). Equation (5.19) is not an Abel equation, but it was solved in § IIIc of Paper II, where it was shown that, to sufficient accuracy,

$$F_{bm2}(\mu_0, \lambda_0) = \frac{2}{\pi^2} \frac{\sqrt{(\lambda_0 + \alpha)(-\alpha - \mu_0)}}{\lambda_0 - \mu_0}. \quad (5.21)$$

As could have been anticipated, the distribution function needed to produce a density ρ_{bm} with radial behavior near the focal ellipse itself has radial behavior there.

5.3. Behavior near the Focal Hyperbola

A similar treatment is needed to obtain the form of the distribution function in a slender box adjacent to the focal hyperbola in which both $(\mu + \beta)$ and $(-\beta - v)$ are $O(\epsilon)$. The density there is $O(1)$, has radial behavior, and is of the separated form of equation (4.22), with a function of λ multiplying $(-\beta - v)/(\mu - v)$. In the slender box the integral equation simplifies to

$$\left(\frac{-\beta - v}{\mu - v}\right) \lim_{\mu \downarrow -\beta} \rho_{im}(\lambda, \mu, -\beta) = \int_v^{-\beta} dv_0 \int_\mu^{-\alpha} \frac{d\mu_0}{\sqrt{(\mu_0 - \mu)(\mu - v_0)(\mu_0 - v)(v_0 - v)}} \\ \times \frac{1}{\lambda + \beta} \int_\lambda^\infty \frac{F_{bm}(v_0, \mu_0, \lambda_0) d\lambda_0}{(\lambda_0 + \beta)U[-\beta, -\beta, -\beta, \lambda_0] \sqrt{U[-\beta, -\beta, \lambda_0] - U[-\beta, -\beta, \lambda]}}. \quad (5.22)$$

It also has a solution of the separated form

$$2\sqrt{2}(\gamma - \alpha)F_{bm}(v_0, \mu_0, \lambda_0) = F_{bm3}(\lambda_0)F_{bm4}(v_0, \mu_0), \quad (5.23)$$

with

$$(\lambda + \beta) \lim_{\mu \downarrow -\beta} \rho_{im}(\lambda, \mu, -\beta) = \int_\lambda^\infty \frac{F_{bm3}(\lambda_0) d\lambda_0}{(\lambda_0 + \beta)U[-\beta, -\beta, -\beta, \lambda_0] \sqrt{U[-\beta, -\beta, \lambda_0] - U[-\beta, -\beta, \lambda]}} \quad (5.24)$$

and

$$\left(\frac{-\beta - v}{\mu - v}\right) = \int_v^{-\beta} dv_0 \int_\mu^{-\alpha} \frac{F_{bm4}(v_0, \mu_0) d\mu_0}{\sqrt{(\mu_0 - \mu)(\mu - v_0)(\mu_0 - v)(v_0 - v)}}. \quad (5.25)$$

Equation (5.24) is an Abel equation with solution

$$F_{bm3}(\lambda_0) = -\frac{(\lambda_0 + \beta)U[-\beta, -\beta, -\beta, \lambda_0]U[-\beta, -\beta, \lambda_0, \lambda_0]}{\pi} \int_{\lambda_0}^\infty \frac{\partial[(\lambda + \beta) \lim_{\mu \downarrow -\beta} \rho_{im}(\lambda, \mu, -\beta)]/\partial \lambda d\lambda}{\sqrt{U[-\beta, -\beta, \lambda] - U[-\beta, -\beta, \lambda_0]}}. \quad (5.26)$$

Equation (5.25) is, to the degree of approximation used here, just a relabeled form of equation (5.19) because the upper limit $-\alpha$ of its μ_0 integration is large, and effectively infinite. Hence its solution is, to a sufficient approximation,

$$F_{bm4}(v_0, \mu_0) = \frac{2}{\pi^2} \frac{\sqrt{(\mu_0 + \beta)(-\beta - v_0)}}{\mu_0 - v_0}, \quad (5.27)$$

and the box-orbit distribution of course has radial behavior near the focal hyperbola too.

The phase-space distribution function, which by equation (2.25) is given by

$$f_{bm}(v_0, \mu_0, \lambda_0) = \frac{(\gamma - \alpha)F_{bm}(v_0, \mu_0, \lambda_0)}{(\lambda_0 - \mu_0)(\lambda_0 - v_0)(\mu_0 - v_0)h(v_0, \mu_0, \lambda_0)}, \quad (5.28)$$

is singular at both the focal ellipse and the focal hyperbola. The distribution function f_{bm} for the elliptic disk of Paper II has a similar singularity at the focus. These singularities are integrable, and integrate to give a finite density and finite velocity moments.

5.4. Completion of Existence Proof

It is only after the solution for F_{bm} in the neighborhood of the focal curves has been obtained that we can obtain the solution for F_{bm} in the remainder of the region $-\alpha \leq \lambda, -\beta \leq \mu \leq -\alpha, -\gamma \leq v \leq -\beta$ that was excluded by inequalities (5.3). We again have a generalized Abel equation for F_{bm} , with a unique and continuous solution, as a straightforward extension of the discussion of § III d of Paper II shows.

6. MEAN STREAMING MOTIONS

The average over velocity space of any function Φ of the phase-space variables can be written in terms of the turning points, in the same manner as the fundamental integral equation (2.23), as

$$\rho\langle\Phi\rangle = \int \int \int \Phi(\lambda, \mu, v; \tau_1, \tau_2, \tau_3) F(\tau_1, \tau_2, \tau_3) \rho_{orb}(\lambda, \mu, v; \tau_1, \tau_2, \tau_3) d\tau_1 d\tau_2 d\tau_3. \quad (6.1)$$

This integral can be split into four parts, since F can be written as $F = F_i + F_o + F_s + F_b$, i.e., as the sum of the distribution functions for each of the four orbit families. The integration limits then are identical to those given in equations (2.30)–(2.33).

The velocity moments of order n of the distribution function f are defined as the averages $\langle v_\lambda^k v_\mu^\ell v_v^m \rangle$, with $k + \ell + m = n$. We restrict ourselves to the cases $n = 1$ and $n = 2$. The former are discussed in this section, while the latter are considered in Appendix B.

A number of the moments can be shown to vanish because of symmetries of the individual orbits. It follows from equation (2.17) that the magnitudes of v_λ , v_μ , and v_v , but not their signs, are fixed at each point (λ, μ, v) on a given orbit. If the motion in the coordinate τ is a libration, then one half of a phase-mixed population of stars will pass through (λ, μ, v) with $v_\tau > 0$ and the other half

with $v_\tau < 0$, so that the contribution to the associated first-order moment $\langle v_\tau \rangle$ is zero. If, on the other hand, the motion in τ is a rotation, then $\langle v_\tau \rangle \neq 0$ in general. The motion is a libration in all three coordinates for box orbits, so that these orbits do not contribute to the mean streaming motions in any coordinate. The motion is a rotation in v for I- and O-tubes, and a rotation in μ for S-tubes. As a result, both $\langle v_\mu \rangle$ and $\langle v_v \rangle$ will generally be nonzero, unless all clockwise and counterclockwise tubes are populated equally. In the next three subsections, we compute, for each tube-orbit family, the maximum mean streaming velocity that is attained when all orbits have the same positive sense of rotation. We find that the mean streaming velocities of both O- and S-tubes are less than the velocities of stars in closed elliptic orbits in surface $\mathcal{S}$ and the plane $z = 0$, respectively, while the maximum mean streaming velocities of I-tubes exceed those of stars in closed elliptic orbits in surface $\mathcal{E}$. In § 6.4 we give the integrals for the components of angular momentum corresponding to the maximum mean streaming motions.

6.1. Thin I-Tubes

The maximum value $\langle v_v \rangle_{\max}^i$ is obtained by direct evaluation of $\langle v_v \rangle$ via equation (6.1), using equations (3.11) and (3.12), as

$$\rho_{im} \langle v_v \rangle_{\max}^i(\lambda, \mu, v) = - \frac{2\sqrt{2}(\mu + \beta)}{\pi(\lambda - \mu)\sqrt{(\lambda - v)(\mu - v)}} \int_{\lambda}^{\infty} d\sigma B_0(u_1, u_3) \frac{\partial}{\partial \sigma} [(\sigma - \mu)\rho_{im}(\sigma, \mu, -\beta)], \quad (6.2)$$

where u_1 and u_3 are given in equation (3.14), and the function B_0 is the integral

$$B_0(u_1, u_3) = \int_0^{\pi/2} d\theta \sqrt{u_1 + u_3 \cos^2 \theta} \quad (6.3)$$

that was introduced in equation (3.12) of Paper III, and also appears in equation (4.5) of Paper IV, together with its explicit evaluation in terms of complete elliptic integrals. An alternative form of equation (6.2) that is obtained by partial integration is

$$\begin{aligned} \rho_{im} \langle v_v \rangle_{\max}^i(\lambda, \mu, v) = & \frac{\sqrt{2}(\mu + \beta)}{\sqrt{(\lambda - v)(\mu - v)}} \left\{ \rho_{im}(\lambda, \mu, -\beta) \sqrt{U[\lambda] - U[-\beta]} \right. \\ & \left. + \frac{1}{\pi(\lambda - \mu)} \int_{\lambda}^{\infty} d\sigma B_1(u_1, u_3) (\sigma - \mu) \rho_{im}(\sigma, \mu, -\beta) U[\mu, \mu, \sigma, \sigma] \right\}, \end{aligned} \quad (6.4)$$

where B_1 is the integral

$$B_1(u_1, u_3) = \int_0^{\pi/2} \frac{\cos^2 \theta d\theta}{\sqrt{u_1 + u_3 \cos^2 \theta}}, \quad (6.5)$$

introduced in equation (4.7) of Paper IV. Since $u_1 > 0$, it is everywhere finite with $0 \leq B_1(u_1, u_3) \leq \pi/(4u_1^{1/2})$. Equation (6.4) has a structure similar to that of equation (3.16) for ρ_{im} . Because the integrand of A_1 in the latter contains an additional factor of $(u_2 - u_1)/(u_2 + u_3 \cos^2 \theta)^{3/2}$ which is not present in B_1 , and this factor cannot exceed $u_2^{-1/2}$, it follows from direct comparison of the two equations that

$$\langle v_v \rangle_{\max}^i(\lambda, \mu, v) \geq \sqrt{2(\mu - v)U[v, \mu, \mu, \lambda]}. \quad (6.6)$$

Applied to surface $\mathcal{E}$ by setting $\lambda = -\alpha$, this inequality shows that the mean streaming of the thin I-tubes there exceeds the velocity of stars that are locally in closed elliptic orbits (as obtained by setting $\tau_3 = \lambda_0 = \lambda = -\alpha$ and $\tau_1 = \tau_2 = \mu$ in the third of eqs. [2.17]). A similar result was found in Paper IV for the prolate case, where the more rapid rotation of I-tubes which depart from surface $\mathcal{E}$ is explained as due to their angular momentum exceeding that of the stars that are confined to this surface.

Inspection of equations (6.2) and (6.4) shows that all of the v -dependence is in the factor outside the braces, and that

$$\rho_{im} \langle v_v \rangle_{\max}^i(\lambda, \mu, v) = \sqrt{\frac{(\lambda + \beta)(\mu + \beta)}{(\lambda - v)(\mu - v)}} \rho_{im} \langle v_v \rangle_{\max}^i(\lambda, \mu, -\beta). \quad (6.7)$$

This relation, which is a direct consequence of the equation of continuity, means that the integrals of equation (6.2) or equation (6.4) need to be evaluated only on surface $\mathcal{L}$.

6.2. Thin O-Tubes

The maximum possible streaming for O-tubes is obtained, using equations (6.1), (3.27), and (3.28), as

$$\begin{aligned} \rho_{om} \langle v_v \rangle_{\max}^o(\lambda, \mu, v) = & \frac{2\sqrt{2}(\lambda + \beta)}{\pi(\lambda - \mu)\sqrt{(\lambda - v)(\mu - v)}} \\ & \times \left\{ (\lambda + \beta) \rho_{om}(\lambda, -\beta, -\beta) \sqrt{W[\mu] - W[-\beta]} + \int_{-\beta}^{\mu} d\sigma B_0(w_1, w_3) \frac{\partial}{\partial \sigma} [(\lambda - \sigma) \rho_{om}(\lambda, \sigma, -\beta)] \right\}, \end{aligned} \quad (6.8)$$

where B_0 is the function defined in equation (6.3), and w_1 and w_3 are given in equation (3.30). Since $0 \leq w_1, w_3 \leq W[\mu] - W[-\beta]$,

the function $B_0(w_1, w_3)$ is bounded everywhere. Partial integration of equation (6.8) gives

$$\rho_{om} \langle v_v \rangle_{\max}^o(\lambda, \mu, v) = \frac{\sqrt{2}(\lambda + \beta)}{\sqrt{(\lambda - v)(\mu - v)}} \times \left\{ \rho_{om}(\lambda, \mu, -\beta) \sqrt{W[\mu] - W[-\beta]} - \frac{1}{\pi(\lambda - \mu)} \int_{-\beta}^{\mu} d\sigma B_2(w_1, w_3)(\lambda - \sigma) \rho_{om}(\lambda, \sigma, -\beta) U[\sigma, \sigma, \lambda, \lambda] \right\}, \quad (6.9)$$

where B_2 is the following integral also introduced in equation (3.14) of Paper III and equation (4.10) of Paper IV:

$$B_2(w_1, w_3) = \int_0^{\pi/2} \frac{\sin^2 \theta d\theta}{\sqrt{w_1 + w_3 \cos^2 \theta}}. \quad (6.10)$$

Note that B_2 diverges for $\mu \rightarrow -\beta$, since $w_1 \sim w_3 \rightarrow 0$. The mean streaming velocity at a general point (λ, μ, v) is again related to that at $(\lambda, \mu, -\beta)$ through the equation of continuity, the explicit relation being that of equation (6.7) with ρ_{im} replaced by ρ_{om} .

The mean streaming velocities of the O-tubes in surface $\mathcal{S}$ are found numerically to be less than the velocities of stars confined to this surface in closed elliptical orbits. This result, which can be established analytically for the prolate case by comparing equations (3.31) and (6.9), cannot now be established so simply, unlike the corresponding results for the two other tube-orbit families. The reason is that $\sqrt{W[-\alpha] - W[v]}$ times the A_2 in the integrand of equation (3.31), a term that is absent from the prolate case, may now exceed the B_2 of equation (6.9). However, this can happen only in a small part of the range of integration near $\sigma = -\beta$, which is canceled by the contributions from other parts of the range.

6.3. Thin S-Tubes

The maximum possible mean streaming, now in the μ -coordinate, is found by direct evaluation through equation (6.1) with the distribution function (3.45) as

$$\rho_{sm} \langle v_\mu \rangle_{\max}^s(\lambda, \mu, v) = - \frac{2\sqrt{2}(\lambda + \alpha)}{\pi(\lambda - v)\sqrt{(\lambda - \mu)(\mu - v)}} \int_v^{-\beta} d\sigma B_0(w_4, w_2) \frac{\partial}{\partial \sigma} [(\lambda - \sigma) \rho_{sm}(\lambda, -\alpha, \sigma)], \quad (6.11)$$

with B_0 given earlier in equation (6.3). The arguments w_2 and w_4 are defined in equations (3.30) and (3.48), respectively. Since $w_4 > 0$ and $0 \leq w_2 \leq W[-\beta] - W[v]$, the function $B_0(w_4, w_2)$ is bounded everywhere, and hence the integral (6.11) is well behaved. The alternative expression that is obtained via partial integration is

$$\rho_{sm} \langle v_\mu \rangle_{\max}^s(\lambda, \mu, v) = \frac{\sqrt{2}(\lambda + \alpha)}{\sqrt{(\lambda - \mu)(\mu - v)}} \times \left\{ \rho_{sm}(\lambda, -\alpha, v) \sqrt{W[-\alpha] - W[v]} - \frac{1}{\pi(\lambda - v)} \int_v^{-\beta} d\sigma B_2(w_4, w_2)(\lambda - \sigma) \rho_{sm}(\lambda, -\alpha, \sigma) U[\sigma, \sigma, \lambda, \lambda] \right\}. \quad (6.12)$$

The integral term here is nonpositive, whereas the integral term in the corresponding equation (3.49) for the density ρ_{sm} is nonnegative. The equations can therefore be combined to give the inequality

$$\langle v_\mu \rangle_{\max}^s(\lambda, \mu, v) \leq \sqrt{2(\lambda - \mu)U[v, \mu, \lambda, \lambda]}, \quad (6.13)$$

which, when applied in the plane $z = 0$ ($v = -\gamma$), shows that the mean streaming there is less than that of stars in closed elliptical orbits, in agreement with the similar result for the oblate limit (Paper III).

If we use the fact that $\rho_{sm}(\lambda, -\alpha, \sigma) = \rho_m(\lambda, -\alpha, \sigma) - \rho_{om}(\lambda, -\alpha, \sigma)$, then equation (6.11) can be rewritten as

$$\begin{aligned} \rho_{sm} \langle v_\mu \rangle_{\max}^s(\lambda, \mu, v) = & - \frac{2\sqrt{2}(\lambda + \alpha)}{\pi(\lambda - v)\sqrt{(\lambda - \mu)(\mu - v)}} \int_v^{-\beta} d\sigma B_0(w_4, w_2) \frac{\partial}{\partial \sigma} [(\lambda - \sigma) \rho_m(\lambda, -\alpha, \sigma)] \\ & + \frac{2\sqrt{2}(\lambda + \beta)^2}{\pi(\lambda - v)\sqrt{(\lambda - \mu)(\mu - v)}} \rho_{om}(\lambda, -\beta, -\beta) \sqrt{W[-\beta] - W[v]} \\ & + \frac{2\sqrt{2}(\lambda + \beta)(W[-\beta] - W[v])}{\pi(\lambda - v)\sqrt{(\lambda - \mu)(\mu - v)}} \int_{-\beta}^{-\alpha} d\sigma B_1(w_1, w_2 - w_1) \frac{\partial}{\partial \sigma} [(\lambda - \sigma) \rho_{om}(\lambda, \sigma, -\beta)], \end{aligned} \quad (6.14)$$

an alternative form of which can be obtained by integrating by parts. Just as with the density ρ_{sm} , the mean streaming in the S-tubes follows by a straightforward quadrature once the density ρ_{om} has been assigned on surface $\mathcal{L}$.

Finally, the μ -dependence of $\rho_{sm} \langle v_\mu \rangle_{\max}^s(\lambda, \mu, v)$ is entirely in the factor in front of the integral sign, and

$$\rho_{sm} \langle v_\mu \rangle_{\max}^s(\lambda, \mu, v) = \sqrt{\frac{(\lambda + \alpha)(-\alpha - v)}{(\lambda - \mu)(\mu - v)}} \rho_{sm} \langle v_\mu \rangle_{\max}^s(\lambda, -\alpha, v), \quad (6.15)$$

in agreement with the equation of continuity.

6.4. Total Angular Momenta

The components of the total angular momentum of the model are obtained by a volume integration of the mean streaming of each family of tube orbits. The mean streaming in v of the I-tubes contributes an x -component of angular momentum of amount

$$\begin{aligned}\langle L_x \rangle_T^i &= \iiint (y \langle v_z \rangle^i - z \langle v_y \rangle^i) \rho_{im} dx dy dz \\ &= 8 \iiint \frac{1}{R} \left(y \frac{\partial z}{\partial v} - z \frac{\partial y}{\partial v} \right) \rho_{im} \langle v_v \rangle_{\max}^i PQR d\lambda d\mu dv.\end{aligned}\quad (6.16)$$

The metric coefficients from equation (2.5), expressions for Cartesian coordinates in terms of ellipsoidal coordinates from equation (8) of Paper I, and equation (6.7) for the mean streaming, are needed to evaluate the integral. The simple v -dependence in the latter equation causes the v -integration to be elementary, and equation (6.16) then simplifies to

$$\langle L_x \rangle_T^i = \frac{\pi}{\sqrt{(\beta - \alpha)(\gamma - \alpha)}} \int_{-\beta}^{-\alpha} d\mu \int_{-\alpha}^{\infty} d\lambda \sqrt{\frac{(\lambda + \beta)(\mu + \beta)}{(\lambda + \alpha)(-\alpha - \mu)}} (\lambda - \mu) \rho_{im} \langle v_v \rangle_{\max}^i(\lambda, \mu, -\beta). \quad (6.17)$$

At first sight, this equation seems to differ from equation (4.15) of Paper IV for $\langle L_x \rangle_T^i$ in the prolate case only in the initial constant factor because equation (6.2) for $\rho_{im} \langle v_v \rangle_{\max}^i(\lambda, \mu, -\beta)$ matches equation (4.3) of Paper IV for $\rho_{im} \langle v_x \rangle_{\max}^i(\lambda, \mu)$. However, there is a hidden difference in that the potential function $U(\tau)$ for the triaxial case generally differs from that for the prolate case, so that angular momenta do have to be recomputed.

The mean streaming in v of the O-tubes contributes a component $\langle L_x \rangle_T^o$ to the x -component of angular momentum given by the same integration as that of equation (6.17) but with $\rho_{im} \langle v_v \rangle_{\max}^i$ replaced by $\rho_{om} \langle v_v \rangle_{\max}^o$.

The mean streaming in μ of the S-tubes gives rise to a z -component of angular momentum, which can be evaluated in a similar manner using equation (6.15), as

$$\begin{aligned}\langle L_z \rangle_T &= \iiint (x \langle v_y \rangle^s - y \langle v_x \rangle^s) \rho_{sm} dx dy dz \\ &= 8 \iiint \left(x \frac{\partial y}{\partial \mu} - y \frac{\partial x}{\partial \mu} \right) \rho_{sm} \langle v_\mu \rangle_{\max}^s PR d\lambda d\mu dv \\ &= \frac{\pi}{\sqrt{(\gamma - \alpha)(\gamma - \beta)}} \int_{-\gamma}^{-\beta} dv \int_{-\alpha}^{\infty} d\lambda \sqrt{\frac{(\lambda + \alpha)(-\alpha - v)}{(\lambda + \gamma)(v + \gamma)}} (\lambda - v) \rho_{sm} \langle v_\mu \rangle_{\max}^s(\lambda, -\alpha, v).\end{aligned}\quad (6.18)$$

Equation (6.18) is closely related to equation (4.32) of Paper III for $\langle L_z \rangle_T$ for the oblate spheroids, and reduces to equation (4.53) of Paper II for the maximum angular momentum of the elliptic disk.

7. NEARLY SPHERICAL FIGURES

Here we derive analytical approximations to the mass and momentum densities, and their integrated totals, for nearly spherical ellipsoids. The integrated totals are found to depend only on the triaxiality parameter T , defined in equation (2.35).

Only O-tubes rotating about one axis, and S-tubes rotating about a mutually perpendicular axis, remain as the spherical limit is approached. Both β and γ tend to α , so that the ranges of the coordinates μ and v , and all the w 's, become small. The focal ellipse disappears, and the focal hyperbola becomes the two straight lines $z = \pm x\sqrt{(1 - T)/T}$. I-tubes, and hence the choice of splits, become irrelevant. Analytically we can set each of μ , v , $-\beta$, and $-\gamma$ equal to $-\alpha$ except in small terms involving differences between these nearly equal quantities. Approximations for the two remaining mass densities are obtained most simply from equations (3.29) and (4.17), which have single $O(1)$ terms involving the ratios of two small quantities, and are

$$\begin{aligned}\rho_{om}(\lambda, \mu, v) &\sim \frac{2}{\pi} \arcsin \sqrt{\frac{W[\mu] - W[-\beta]}{W[\mu] - W[v]}} \rho_m(r) \\ &\sim \frac{2}{\pi} \arcsin \sqrt{\frac{\mu + \beta}{\mu - v}} \rho_m(r)\end{aligned}\quad (7.1)$$

and

$$\rho_{sm}(\lambda, \mu, v) \sim \frac{2}{\pi} \arcsin \sqrt{\frac{-\beta - v}{\mu - v}} \rho_m(r), \quad (7.2)$$

where $\rho_m(r)$ equals $\rho_m(\lambda, -\alpha, -\alpha)$, the radial density profile, with $r^2 = \lambda + \alpha$. Approximations for the remaining momentum densities are obtained similarly from equations (6.8) and (6.14) as

$$\rho_{om} \langle v_v \rangle_{\max}^o(\lambda, \mu, v) \sim \frac{2}{\pi} \sqrt{\frac{\mu + \beta}{\mu - v}} \rho_m(r) v_c(r) \quad (7.3)$$

and

$$\rho_{sm} \langle v_\mu \rangle_{\max}^s(\lambda, \mu, \nu) \sim \frac{2}{\pi} \sqrt{\frac{(-\beta - \nu)}{(\mu - \nu)}} \rho_m(r) v_c(r), \quad (7.4)$$

where $v_c(r) = \sqrt{2(\lambda + \alpha)U[-\alpha, -\alpha, \lambda, \lambda]}$ is the circular velocity of the spherical limit.

Each of our approximate densities is of the form of an initial geometric factor multiplying a spherically symmetric density. Consequently their integrated totals are products of geometric factors, which depend only on the triaxiality parameter T , with a mass or angular momentum of the spherical limit. The relative sizes of the masses M_{om} and M_{sm} of the O- and S-tubes as fractions of the total mass in the spherical limit can be obtained by integrating equation (7.1) over configuration space, though this integration is not elementary, and appears to require complex variable methods. The final result,

$$M_{om} = \frac{2}{\pi} \left(\arctan \sqrt{\frac{T}{1-T}} \right) M, \quad M_{sm} = \left(1 - \frac{2}{\pi} \arctan \sqrt{\frac{T}{1-T}} \right) M, \quad (7.5)$$

shows that the ratio M_{om}/M is the proportion of the plane $y = 0$ occupied by the now double triangular form of the surface $\mathcal{L}$ on which $\rho_{om}/\rho_m = 1$. The density ρ_{om} vanishes on the remainder $\mathcal{H}$ of this plane where $\rho_{sm}/\rho_m = 1$. Because of the isotropy of the spatial distribution of the purely circular orbits of the spherical limit, the relative areas of the unique central plane in which the two densities are fully segregated gives their relative total masses.

The two components of angular momentum are derived more simply from the use of approximations (7.3) and (7.4) in equations (6.17) and (6.18) and elementary integrations to get

$$\langle L_x \rangle_T = \sqrt{T} \langle L \rangle, \quad \langle L_z \rangle_T = \sqrt{1-T} \langle L \rangle, \quad (7.6)$$

where

$$\langle L \rangle = 2\pi \int_0^\infty dr r^3 v_c(r) \rho_m(r) \quad (7.7)$$

is the angular momentum of a spherical galaxy composed of stars in circular orbits with the same sense of rotation about a single axis. Equations (7.6) show that the total angular momentum of a slightly triaxial figure with thin tube orbits is also $\langle L \rangle$ to a first approximation, and that it is inclined to the short axis at an angle

$$\psi_{\text{int}} = \arctan \sqrt{\frac{T}{1-T}}. \quad (7.8)$$

Equations (7.3) and (7.4) show that the total momentum density at a point (λ, μ, ν) of the figure is

$$\left\{ \sqrt{\frac{\mu + \beta}{\mu - \nu}} \hat{e}_\nu + \sqrt{\frac{-\beta - \nu}{\mu - \nu}} \hat{e}_\mu \right\} \rho_m(r) \frac{2}{\pi} v_c(r). \quad (7.9)$$

The first component of expression (7.9), which combines the unit vectors $\hat{e}_\nu$ and $\hat{e}_\mu$, is itself a unit vector in the nearly spherical surface $\lambda = \text{constant}$. A straightforward evaluation of its scalar product with the unit vector $\sqrt{T} \hat{e}_x + \sqrt{1-T} \hat{e}_y$ in the direction of the total angular momentum vector shows that these vectors become perpendicular in the spherical limit. The factor $2/\pi$ times the circular velocity in expression (7.9) gives the mean velocity of an isotropic combination of circular orbits with the same sense of rotation about a single axis. Consequently, the slightly triaxial figures all approach a single dynamical limit, and it is only its inclination that varies.

The direction in the (x, z) -plane defined by equation (7.8) is that from which the triaxial Stäckel model appears round (Franx 1988a). The above results show that, when observed from this direction, the line-of-sight radial velocity will vanish along any position angle when the figure is itself spherical, although it will not generally do so when the figure is triaxial (Levison 1987).

8. CONSTRUCTION OF MODELS

We come at last to the construction of specific triaxial Stäckel models, using the formalism developed in the preceding sections. Like Statler (1987), we restrict ourselves to the perfect ellipsoid of Paper I, for which we have constructed models for a wide range of axis ratios, which include his. The results of our computations are given in §§ 8.2–8.4, after a brief listing in § 8.1 of properties of perfect ellipsoids.

8.1. The Perfect Ellipsoid

The density distribution of the perfect ellipsoid is given by

$$\rho_m = \frac{\rho_0}{(1 + m^2)^2}, \quad m^2 = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}. \quad (8.1)$$

This density is stratified on similar concentric ellipsoids with semiaxes ma , mb , and mc , and is

$$\rho_m(\lambda, \mu, \nu) = \frac{\rho_0 \alpha^2 \beta^2 \gamma^2}{\lambda^2 \mu^2 \nu^2}, \quad (8.2)$$

in terms of the ellipsoidal coordinates. The central density is ρ_0 , and ρ_m falls off at large radii as λ^{-2} . The total mass M of the ellipsoid is equal to $\pi^2 \rho_0 abc$. The units of mass, length, and time used in our computations are such that

$$GM = 1, \quad \rho_0 = 1, \quad a = 1. \quad (8.3)$$

The potential of the ellipsoid is of Stäckel form with

$$U(\tau) = \frac{GM}{\pi} \int_0^\infty \frac{(\tau + \alpha)(\tau + \gamma)}{\tau + \sigma} \sqrt{\frac{\sigma - \beta}{(\sigma - \alpha)(\sigma - \gamma)}} d\sigma, \quad (8.4)$$

and

$$\alpha = -a^2, \quad \beta = -b^2, \quad \gamma = -c^2. \quad (8.5)$$

The function $U(\tau)$ can be expressed in terms of incomplete elliptic integrals, and is readily evaluated numerically (Paper I, eq. [B1]; de Zeeuw & Pfenniger 1988, eq. [2.2]). Its third-order divided difference can be written explicitly as

$$U[\tau_1, \tau_2, \tau_3, \tau_4] = \frac{GM}{\pi} \int_0^\infty \frac{\sqrt{(\sigma - \alpha)(\sigma - \beta)(\sigma - \gamma)}}{(\sigma + \tau_1)(\sigma + \tau_2)(\sigma + \tau_3)(\sigma + \tau_4)} d\sigma, \quad (8.6)$$

but is most easily computed by taking repeated divided differences of the function (8.4).

8.2. Construction of Models

The crucial first step in the strategy described in § 2.5 is the split of the density,

$$\rho_m(\lambda, \mu, -\beta) = \frac{\rho_0 \alpha^2 \gamma^2}{\lambda^2 \mu^2}, \quad (8.7)$$

in surface $\mathcal{S}$ into components $\rho_{im}(\lambda, \mu, -\beta)$ and $\rho_{om}(\lambda, \mu, -\beta)$ for the I- and O-tubes, respectively. We use a split of the form (4.4) with a constant value of k , for the reasons discussed in § 4.1. As in Paper IV, we find that such splits give nonnegative distribution functions $F_{tim}(\mu_1, \lambda_0)$ and $F_{tom}(\mu_0, \lambda_1)$ for values of k in some range $[k_o, k_i]$, this range depending on geometrical shape. The present equations (3.11) and (3.27) for F_{tim} and F_{tom} are formally the same as equations (2.39) and (2.49) of Paper IV for these distribution functions when the prolate density is taken to be $\rho_m(\lambda, \mu, -\beta)$. The extreme values k_o and k_i of the k -range occur because regions in which distribution functions are negative arise for values of k outside the range. They can be computed using equations (3.11) and either (3.15) or (3.19) of Paper IV. These same equations apply here and give the range of possible values of k for perfect ellipsoids. However, the values of k_i and k_o must be recomputed because the function $U(\tau)$ for the triaxial case, as given by equation (8.4), depends on both γ and β , and so differs from the function $U(\tau)$ of the prolate case in which $\gamma = \beta$. As Table 1 shows, the k_i and k_o values depend primarily on the parameter $b/a = (\beta/\alpha)^{1/2}$ of the prolate case, and little on the extra parameter $c/a = (\gamma/\alpha)^{1/2}$ of the triaxial case.

The distribution function for S-tubes can be computed from equations (3.59) and (4.16), and its nonnegativity, which was established analytically in § 4.2, verified. Figure 3 shows what is in fact a typical contour plot of a nonnegative f_{ism} for a triaxial model in the (ℓ_1, n_0) -plane (with $\ell_1 = \sqrt{\lambda_1}$ and $n_0 = \sqrt{\nu_0}$). It is that for a nearly oblate model with the same c/a ratio and choice of coordinates as those for the f_{ism} contour plot of Figure 2 of Paper III. Despite the fact that the plot of Paper III is for a

TABLE 1
VALUES OF k_i AND k_o ^a

c/a	b/a						
	0.125	0.250	0.375	0.500	0.625	0.750	0.875
0.875							0.826722 -0.303700
0.750						1.533667 0.249711	0.824146 -0.305212
0.625					2.121924 0.657976	1.528479 0.246917	0.821312 -0.306984
0.500				2.593668 0.922203	2.114474 0.654438	1.522942 0.243672	0.818270 -0.309024
0.375			2.952409 1.062755	2.584869 0.918849	2.106925 0.650422	1.517290 0.240038	0.815144 -0.311285
0.250		3.202892 1.153198	2.943817 1.059807	2.576748 0.915256	2.099888 0.646186	1.511976 0.236262	0.812184 -0.313605
0.125	3.350324 1.205847	3.196506 1.150650	2.937285 1.057201	2.570487 0.912018	2.094401 0.642438	1.507795 0.232976	0.809837 -0.315599

^a In each pair of values, the top one is k_i and the bottom one is k_o .

FIG. 3.—Contours of f_{ism} in the $(\ell_1 = \sqrt{\lambda_1}, n_0 = \sqrt{v_0})$ -plane for a triaxial, but nearly oblate, model with $k = k_i = 0.069648$, $b/a = 0.99$, and $c/a = 0.5$. This c/a ratio is the same as that used in the f_{ism} contour plot of Fig. 2 of Paper III, and contours are again spaced logarithmically at intervals of $2^{1/3}$. The upper boundary is now $n_0 = b = 0.99$ rather than the $n_0 = a = 1$ of the earlier plot.

Kuzmin-Kutuzov model, whereas the present plot is for a perfect ellipsoidal density distribution, the plots are similar everywhere except near the left and upper boundaries at which $\ell_1 = a$ and $n_0 = b$. As equations (3.59) and (4.16) show, f_{ism} now vanishes on both of these boundaries, whereas in the oblate case it does not. This accounts for the accumulation of the logarithmically spaced contours adjacent to those boundaries.

The final step that must be carried out computationally to verify that an acceptable triaxial model exists is the calculation of the residual box density, and the verification that it is everywhere nonnegative. We find that ρ_{bm} , like ρ_m , increases as each of λ , μ , and v decrease toward the center of the ellipsoid where $\rho_{bm} = \rho_m$. In fact, the ratio ρ_{bm}/ρ_m increases as each coordinate decreases, with only a few minor exceptions near surface $\mathcal{H}$ for some fairly prolate objects. Then the existence of a nonnegative box distribution function must be verified by a numerical solution of the integral equation (2.33), in the manner outlined in the first paragraph of § 5.1. For the most part, these distribution functions likewise increase as λ , μ , and v decrease.

8.3. Statler's Set of Ellipsoids

Statler constructed models for all 21 possible triaxial combinations of the ratios b/a and c/a between $1/8$ and $7/8$ in steps of $1/8$. We have computed two enlarged sets of models using the same axis ratios and adding in their prolate, oblate, and disk neighbors. This was done for the two extreme values k_i and k_o of k in the split (4.4), and the results for the mass fractions $\Delta M/M$ and angular momenta of the different orbit families are displayed in Tables 2–5. The entries for the spherical model are the limit of the oblate sequence.

Certain properties of our models change monotonically with the parameter k . Increasing k increases $\rho_{im}(\lambda, \mu, -\beta)$ and, hence, by equations (3.16) and (6.4), $\rho_{im}(\lambda, \mu, v)$ and $\rho_{im}\langle v_\nu \rangle_{\max}^i(\lambda, \mu, v)$ everywhere except for certain special surfaces. The total mass and angular momentum of I-tubes therefore both increase with k , and are largest when $k = k_i$. Increasing k has a more complicated effect on ρ_{om} , since ρ_{om} then decreases in some regions, e.g., surface $\mathcal{L}$, but increases in others, e.g., surface $\mathcal{S}$, as discussed in § 3.3. Integration is necessary to find the net effect, and, in all the cases tabulated, the total mass of the O-tubes is less at the larger value of k . Figure 4 presents a detailed study of how overall properties vary with k for one specific axis ratio. The most remarkable feature shown here is the occurrence of a maximum of long-axis angular momentum at an intermediate value of k . This same feature was found for prolate models in Paper IV and arises because of the relatively large x -angular momentum of flat O-tubes that remain close to the $x = 0$ plane, and hence is dependent on how many of these there are. The maximum angular momentum about the long axis therefore is generally not that given in either of our tables. But the maximum angular momentum about the short axis and the maximum mass of the S-tubes are those given in Tables 4 and 5 for $k = k_o$. This is because increasing k decreases f_{ism} , as discussed in § 4.2, and hence decreases both $\rho_{sm}(\lambda, \mu, v)$ and $\rho_{sm}\langle v_\mu \rangle_{\max}^s(\lambda, \mu, v)$.

Because our solutions with $k = k_o$ have the largest $\langle L_z \rangle_T$, the properties listed in our Tables 4 and 5 should be compared with those given in Statler's Figure 4 and Table 2 for his $zmax$ solutions. Precise agreement is not to be expected, for at least two reasons. One is that Statler's models were truncated at 95% of their mass. The other is that our $k = k_o$ models maximize $\langle L_z \rangle_T$ only relative to models for which all the tubes are thin and the specific split (4.4) is made. Despite the coarseness of Statler's gridding and the specialness of our models, there is good general agreement with respect to trends and often quite close numerical agreement. Most of our solutions have more box orbits than his and fewer long-axis tubes, except near the prolate hypotenuse, where his solutions have fewer I- and O-tubes but more S-tubes. Consequently, his nearly prolate solutions have larger $\langle L_z \rangle_T$ than ours, though our nearly oblate models have a few more S-tubes and slightly larger $\langle L_z \rangle_T$ than his. All of our $\langle L_x \rangle_T$ values lie within Statler's fairly wide limits, but well below his maximum, and even our intermediate $\langle L_x \rangle_T$ maximum of 0.2643 from Figure 4 is significantly less than his maximum of 0.4412 for this shape. The mass fractions and angular momenta for the triaxial models appear to merge smoothly with those for the oblate ($b/a = 1$) and disk ($c/a = 1$) limits. Our values for the disk angular momenta agree with those of Teuben (1987) only after his values are corrected by a factor of $(a/b)^{1/2}$.

TABLE 2
MASS FRACTIONS FOR THE FOUR ORBITAL FAMILIES^a WHEN $k = k_i$

c/a	b/a								
	0.000	0.125	0.250	0.375	0.500	0.625	0.750	0.875	1.000
1.000									0.0000
									0.0000
									0.0000
									1.0000
0.875								0.0000	0.0000
								0.3722	0.0000
								0.6278	0.0000
								0.0000	1.0000
0.750							0.0000	0.2056	0.0000
							0.6546	0.0929	0.0000
							0.3454	0.2239	0.0000
							0.0000	0.4777	1.0000
0.625						0.0000	0.3460	0.2475	0.0000
						0.7958	0.2039	0.0376	0.0000
						0.2042	0.1560	0.1019	0.0000
						0.0000	0.2941	0.6129	1.0000
0.500					0.0000	0.4570	0.4366	0.2754	0.0000
					0.8817	0.2488	0.0794	0.0154	0.0000
					0.1183	0.0892	0.0684	0.0449	0.0000
					0.0000	0.2049	0.4155	0.6643	1.0000
0.375				0.0000	0.5763	0.5945	0.4853	0.2973	0.0000
				0.9376	0.2387	0.0841	0.0282	0.0056	0.0000
				0.0624	0.0447	0.0339	0.0260	0.0171	0.0000
				0.0000	0.1402	0.2876	0.4605	0.6800	1.0000
0.250			0.0000	0.7154	0.7424	0.6588	0.5164	0.3141	0.0000
			0.9733	0.1797	0.0587	0.0218	0.0075	0.0015	0.0000
			0.0267	0.0172	0.0123	0.0094	0.0072	0.0047	0.0000
			0.0000	0.0878	0.1865	0.3100	0.4689	0.6797	1.0000
0.125		0.0000	0.8767	0.8746	0.8022	0.6899	0.5356	0.3256	0.0000
		0.9935	0.0764	0.0192	0.0066	0.0025	0.0009	0.0002	0.0000
		0.0065	0.0032	0.0021	0.0015	0.0011	0.0009	0.0006	0.0000
		0.0000	0.0438	0.1042	0.1897	0.3064	0.4626	0.6737	1.0000
0.000	1.0000	0.9894	0.9569	0.9006	0.8170	0.7010	0.5443	0.3312	0.0000
	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	0.0000	0.0106	0.0431	0.0994	0.1830	0.2990	0.4557	0.6688	1.0000

^a In each group of four values the order (top to bottom) is B, I, O, S.

8.4. Four Families of Models

Figure 5 displays “Ellipsoidland,” i.e., the plane of axis ratios of the perfect ellipsoids. The 21 points corresponding to Statler’s models are indicated. Their peculiar spacing in the diagram is caused by our use of b^2/a^2 and c^2/a^2 as variables, rather than b/a and c/a , in which Statler’s grid is regular (see Tables 2–5). Our choice of variables is motivated by the fact that the squares of the axis ratios are related more directly to the dynamics of the models, since they appear in the expressions for the focal distances of the ellipsoidal coordinate system in which the motion separates. Our choice of variables for Figure 5 results in curves of constant triaxiality T which are straight lines. Detailed studies of the intrinsic shapes of elliptical galaxies show that $0.5 \leq c/a \leq b/a \leq 1$ (Binney 1985; Franx, Illingworth, & de Zeeuw 1991), so that these galaxies occupy the portion of the triangle above the lower of the two horizontal dashed lines.

For additional insight into the behavior of our families of solutions, we computed both $k = k_o$ maximum $\langle L_z \rangle_T$ models and $k = k_i$ minimum $\langle L_z \rangle_T$ models for the four cuts across Ellipsoidland shown in Figure 5. Figures 6 and 7 show the orbital mass fractions and the total angular momenta for the constant values 0.5 and 0.75 of c/a , all the way from the prolate ($T = 1$) to the oblate ($T = 0$) limits. These models display the abrupt changes in internal dynamics that occur near the prolate limit that were noted earlier by Levison & Richstone (1987) as well as by Statler. These abrupt changes are due to a combination of dynamical and geometrical effects which alter the orbital densities of tube orbits. We consider in turn how each of the different orbital densities are affected when v becomes a variable with a narrow range rather than the constant $-\beta$.

The orbital density (3.2) of thin I-tubes is affected primarily by the denominator term $\sqrt{B(v; \mu, \mu, \lambda_o)} = (\mu - v)\sqrt{U[\lambda_o] - U[\lambda]}$ due to the rotational v -motion. It causes the triaxial ρ_{im} density to contain a factor $(\mu + \beta)/(\mu - v)$ (cf eqs. [3.13] and [3.16]) which is no longer unity as in the prolate case. The effect is a geometrical one caused by the changed shape of the I-tubes, because the v_v velocity is not greatly changed. The density ρ_{im} must now vanish on the surface $\mathcal{H}$, which slices through the central core of the mass distribution. This surface $\mathcal{H}$ opens up rapidly when T drops below unity, the vertices of the focal hyperbola lying at

TABLE 3
ANGULAR MOMENTA $\langle L_x \rangle_T^i$, $\langle L_x \rangle_T^o$, AND $\langle L_z \rangle_T^a$ IN UNITS OF $\sqrt{GM^3 a}$ WHEN $k = k_i$

c/a	b/a								
	0.000	0.125	0.250	0.375	0.500	0.625	0.750	0.875	1.000
1.000									0.0000 0.0000 0.8567
0.875								0.0564 0.7097 0.0000 0.0000	0.0000 0.0000 0.9778
0.750							0.1404 0.5260 0.0000	0.0251 0.3233 0.6412	0.0000 0.0000 1.0896
0.625						0.2012 0.3384 0.0000	0.0665 0.2551 0.4332	0.0119 0.1568 0.8471	0.0000 0.0000 1.1935
0.500					0.2294 0.2050 0.0000	0.0894 0.1545 0.3220	0.0296 0.1166 0.6356	0.0053 0.0716 0.9648	0.0000 0.0000 1.2900
0.375				0.2248 0.1112 0.0000	0.0866 0.0800 0.2336	0.0339 0.0603 0.4752	0.0113 0.0456 0.7426	0.0020 0.0280 1.0460	0.0000 0.0000 1.3794
0.250			0.1865 0.0483 0.0000	0.0607 0.0312 0.1550	0.0236 0.0225 0.3319	0.0093 0.0170 0.5458	0.0031 0.0128 0.8031	0.0006 0.0079 1.1084	0.0000 0.0000 1.4617
0.125		0.1126 0.0119 0.0000	0.0214 0.0058 0.0829	0.0071 0.0038 0.2009	0.0028 0.0027 0.3641	0.0011 0.0021 0.5763	0.0004 0.0016 0.8406	0.0001 0.0010 1.1598	0.0000 0.0000 1.5360
0.000	0.0000 0.0000 0.0000	0.0000 0.0000 0.0223	0.0000 0.0000 0.0904	0.0000 0.0000 0.2063	0.0000 0.0000 0.3726	0.0000 0.0000 0.5924	0.0000 0.0000 0.8683	0.0000 0.0000 1.2032	0.0000 0.0000 1.6000

^a In each group of three values, the order (top to bottom) is I, O, S.

FIG. 4a

FIG. 4b

FIG. 4.—(a) Mass fractions of the four orbit families and (b) angular momenta of the thin tube-orbit families in self-consistent permissible solutions for one perfect ellipsoid (Statler model 4), as they vary through the range $[k_o, k_i]$ of the split parameter k .

TABLE 4
MASS FRACTIONS FOR THE FOUR ORBITAL FAMILIES^a WHEN $k = k_o$

c/a	b/a								
	0.000	0.125	0.250	0.375	0.500	0.625	0.750	0.875	1.000
1.000									0.0000
									0.0000
									0.0000
									1.0000
0.875								0.0000	0.0000
								0.0484	0.0000
								0.9516	0.0000
								0.0000	1.0000
0.750							0.0000	0.1368	0.0000
							0.2542	0.0111	0.0000
							0.7458	0.2555	0.0000
							0.0000	0.5966	1.0000
0.625						0.0000	0.2800	0.2084	0.0000
						0.4951	0.0751	0.0043	0.0000
						0.5049	0.2183	0.1137	0.0000
						0.0000	0.4266	0.6737	1.0000
0.500					0.0000	0.4229	0.3989	0.2564	0.0000
					0.6910	0.1463	0.0286	0.0017	0.0000
					0.3090	0.1501	0.0902	0.0496	0.0000
					0.0000	0.2807	0.4823	0.6923	1.0000
0.375				0.0000	0.5634	0.5770	0.4692	0.2897	0.0000
				0.8353	0.1777	0.0486	0.0100	0.0006	0.0000
				0.1647	0.0905	0.0532	0.0334	0.0188	0.0000
				0.0000	0.1684	0.3212	0.4874	0.6909	1.0000
0.250			0.0000	0.7126	0.7374	0.6533	0.5117	0.3119	0.0000
			0.9292	0.1539	0.0430	0.0124	0.0026	0.0002	0.0000
			0.0708	0.0421	0.0235	0.0142	0.0091	0.0052	0.0000
			0.0000	0.0914	0.1961	0.3200	0.4766	0.6827	1.0000
0.125		0.0000	0.8764	0.8741	0.8015	0.6892	0.5351	0.3253	0.0000
		0.9826	0.0713	0.0163	0.0048	0.0014	0.0003	0.0000	0.0000
		0.0174	0.0082	0.0049	0.0027	0.0017	0.0011	0.0006	0.0000
		0.0000	0.0441	0.1048	0.1909	0.3076	0.4635	0.6740	1.0000
0.000	1.0000	0.9894	0.9569	0.9006	0.8170	0.7010	0.5443	0.3312	0.0000
	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	0.0000	0.0106	0.0431	0.0994	0.1830	0.2990	0.4557	0.6688	1.0000

^a In each group of four values, the order (top to bottom) is B, I, O, S.

FIG. 5.—Ellipsoidland: the plane of all possible triaxial ellipsoids with axis ratios b/a and c/a . The limiting cases with more symmetry which were studied in Papers II, III, and IV are indicated. The dots correspond to Statler's 21 models, with the encircled dot being the model of Fig. 4. The two horizontal short-dashed lines indicate sequences of models with $c/a = 0.5$ and $c/a = 0.75$, for which orbital mass fractions and total angular momenta are given in Figs. 6 and 7. The long-dashed lines correspond to sequences with triaxiality parameter T equal to $\frac{1}{3}$ and $\frac{2}{3}$, and their mass fractions and angular momenta are displayed in Figs. 8 and 9.

TABLE 5
ANGULAR MOMENTA $\langle L_x \rangle_T^I$, $\langle L_x \rangle_T^O$, AND $\langle L_z \rangle_T^S$ IN UNITS OF $\sqrt{GM^3a}$ WHEN $k = k_o$

c/a	b/a								
	0.000	0.125	0.250	0.375	0.500	0.625	0.750	0.875	1.000
1.000									0.0000
									0.0000
									0.8567
0.875								0.0037	0.0000
								0.6594	0.0000
								0.0000	0.9778
0.750							0.0435	0.0016	0.0000
							0.4676	0.3005	0.0000
							0.0000	0.7667	1.0896
0.625						0.1029	0.0205	0.0008	0.0000
						0.3361	0.2264	0.1458	0.0000
						0.0000	0.6000	0.9066	1.1935
0.500					0.1495	0.0455	0.0090	0.0003	0.0000
					0.2488	0.1528	0.1032	0.0666	0.0000
					0.0000	0.4223	0.7161	0.9911	1.2900
0.375				0.1709	0.0562	0.0172	0.0034	0.0001	0.0000
				0.1768	0.0962	0.0594	0.0402	0.0260	0.0000
				0.0000	0.2726	0.5187	0.7743	1.0560	1.3794
0.250			0.1599	0.0459	0.0153	0.0047	0.0009	0.0000	0.0000
			0.0856	0.0493	0.0268	0.0166	0.0113	0.0073	0.0000
			0.0000	0.1606	0.3450	0.5585	0.8120	1.1112	1.4617
0.125		0.1056	0.0182	0.0053	0.0018	0.0006	0.0001	0.0000	0.0000
		0.0222	0.0103	0.0059	0.0032	0.0020	0.0014	0.0009	0.0000
		0.0000	0.0834	0.2019	0.3658	0.5779	0.8417	1.1601	1.5360
0.000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	0.0000	0.0223	0.0904	0.2063	0.3726	0.5924	0.8683	1.2032	1.6000

* In each group of three values, the order (top to bottom) is I, O, S.

$(0, 0, \pm\sqrt{(\gamma - \alpha)(1 - T)})$ and the angle between its asymptotes being $2 \arctan \sqrt{(1 - T)/T}$. The consequent rapid decline of ΔM_I is evident in the figures.

The orbital density (3.19) of thin O-tubes is also affected primarily by changes to the denominator term arising from the rotational v -motion, which here is $\sqrt{B(v; \mu_0, \lambda, \bar{\lambda})} = (\lambda - v)\sqrt{W[\mu_0] - W[v]}$. This term is small and causes significant changes for orbits with μ_0 turning points that are close to $-\beta$. This is the dynamical effect that we discussed in § 4.3, and is due to orbits which nearly fill an ellipsoidal shell and get close to and linger near the surface $\mathcal{L}$. Such orbits have no counterparts in the prolate case, in which the v -motion cannot vary with v . Because these orbits do not occur for our models with $k > 1$, it is only the solid curve for $k_o < 1$ that shows the rapid $\sqrt{1 - T}$ decline of ΔM_O near $T = 1$ in Figures 6 and 7. The milder declines of ΔM_O shown by the dashed $k = k_i > 1$ curves are due to the general lessening of ρ_{om} away from the surface $\mathcal{L}$ that we demonstrated in § 3.2.

Both dynamical and geometrical effects can cause significant changes to the orbital density of S-tubes. The dynamical effects, also noted in § 4.3, arise from the v -motions of orbits which almost fill an ellipsoidal shell, and which linger near turning points close to the surface $\mathcal{L}$. They cause the denominator component $\sqrt{B(v; v_0, \lambda, \bar{\lambda})} = (\lambda - v)\sqrt{W[v_0] - W[v]}$ of the orbital density (3.39) to be small. These orbits compensate for the density that outer tubes with $\mu_0 \sim -\beta$ fail to provide on surface $\mathcal{S}$, and account for the sharp rise of the solid $k_o < 1$ curves for ΔM_S in Figure 6 and 7. (This is an interchange between orbit families which both have zero J_λ action.) The steep, though less sharp, rises of the other ΔM_S curves are due to a geometrical property of the μ -motions, for which $\sqrt{-B(\mu; v_0, \lambda, \bar{\lambda})} = (\lambda - \mu)\sqrt{W[\mu] - W[v_0]}$. This term is small and the orbital density large for a nearly prolate figure when μ is near $-\beta$, i.e., near surface $\mathcal{H}$. This locally large orbital density is due to the squeezing of its shape that an S-tube must undergo to be able to cross the plane $y = 0$ through the narrow gap provided by surface $\mathcal{H}$, rather than to v_μ velocity variations. The fact that the density missing from surface $\mathcal{S}$ must be provided by S-tubes whose densities are substantially larger near surface $\mathcal{H}$ than elsewhere accounts for a substantial buildup of ΔM_S in all cases.

The changes in the mass fractions that occur more generally from left to right across Figures 6 and 7 can be understood in terms of changes to the special ellipsoidal coordinate surfaces (see Fig. 1). The length scale $\sqrt{\gamma - \alpha}$ is fixed, but the shapes of the focal ellipse and hyperbola change with T . Surface $\mathcal{L}$ gets progressively smaller as the asymptotes of the hyperbola open, accounting for the steadily diminishing fraction of long-axis tubes. The resulting decrease in O-tubes together with the growth of surface $\mathcal{S}$ as the focal ellipse thins both contribute to the increase of ΔM_S . Space becomes available for box orbits once surface $\mathcal{H}$ appears, but then disappears as the focal ellipse flattens. Hence ΔM_B first rises and then falls. Much of its rise near $T = 1$ goes to compensate for the decrease of ΔM_I that occurs as surface $\mathcal{H}$ appears; this is an interchange between I-tubes with zero J_μ and box orbits with small J_μ .

The angular momentum components in the right halves of Figures 6 and 7 generally correlate with the changing mass components. They also reflect the facts that I-tubes carry substantially less angular momentum than O-tubes, and that flat O-tubes

FIG. 6a

FIG. 6b

FIG. 6.—(a) Mass fractions and (b) angular momenta of the different orbit families for $k = k_o$ solutions (solid curves; maximum angular momentum about the short axis) and $k = k_i$ solutions (dashed curves; minimum angular momentum about the short axis) for perfect ellipsoids with $c/a = 0.5$ as functions of the triaxiality parameter T , defined in eq. (2.35).

FIG. 7a

FIG. 7b

FIG. 7.—Same as Fig. 6, but for perfect ellipsoids with $c/a = 0.75$

FIG. 8a

FIG. 8b

FIG. 8.—Same as Fig. 6, but for perfect ellipsoids with $T = \frac{1}{3}$, and as functions of c^2/a^2

FIG. 9a

FIG. 9b

FIG. 9.—Same as Fig. 8, but for perfect ellipsoids with $T = \frac{2}{3}$

which stay close to surface $\mathcal{S}$ carry substantially more angular momentum than the extensive ones with μ_0 close to $-\beta$ (cf. Paper IV, § IVb). The latter is the reason why the marked drop in ΔM_O near $T = 1$, caused by such extensive tubes, is not reflected in the O-tube angular momenta.

Figures 8 and 9 both show variations along paths of constant T across the triangular parameter space of Figure 5, from its elliptic disk base to its spherical apex. These variations may also be understood by reference to Figure 1. The shapes of the focal hyperbola and ellipse, which depend on T , are now unchanged, but their scale, which is proportional to $\sqrt{\gamma - \alpha} = a\sqrt{1 - c^2/a^2}$, diminishes from left to right with increasing c^2/a^2 . Changes in the total density field, from disklike to spherical, are also much more significant now. For instance, the density at the important reference points $(0, 0, \pm\sqrt{\gamma - \alpha})$ (the vertices of the focal ellipse and the foci of the focal hyperbola) is $(M/\pi^2 a^2 b)(c/a)^3$, and increases strongly with c^2/a^2 . Consequently, the growth of surface $\mathcal{L}$ and the density on it give rise to a steady growth of O-tubes. There is also some growth of I-tubes, but the geometrical effects discussed earlier cause these tubes to be relatively unimportant except when T is near 1. Moreover, as pointed out in Paper IV, the splits between the I- and O-tubes that we have been using already favor I-tubes fairly heavily. The mass fraction ΔM_B falls steadily as the density in the plane $y = 0$ migrates from $\mathcal{H}$ to $\mathcal{L}$, $\mathcal{L}$ grows in size, and $\mathcal{E}$ shrinks. Changes in ΔM_S reflect the conflicting trends of the growth of the surface $\mathcal{S}$ and the density on it, and the increase in O-tube mass. Figures 8 and 9 also confirm the analysis of § 7 of the ways in which the spherical limit, in which the dynamical distinction between S- and O-tubes disappears, is approached. This feature is ignored in Tables 2–5, where the uppermost entries refer to the spherical limit of oblate models. Similarly, the lowermost left entries for the needle limit in the tables refer to its approach through disks. As Tremaine & de Zeeuw (1987) have shown, there is no dynamical distinction between boxes and I-tubes in this limit, because the only orbits that remain inside the needle are the x-axis oscillations.

9. DISCUSSION

This paper is the culmination of a sequence of five. Paper I is concerned primarily with the properties of orbits in a Stäckel potential, but does formulate the problem of finding a self-consistent distribution function for such a potential in its § 7. The solution of this problem, using analytical methods as far as possible, has been the main preoccupation of subsequent papers. The main focus in Paper II on the elliptic disk was on showing, as rigorously as we could and with as little reliance on numerics as possible, that more than one self-consistent equilibrium solution exists. The elliptic disk allows both box orbits and S-tubes, and both are needed to construct a self-consistent model. Both contribute density throughout space, and the self-consistent problem separates into two parts, one for each orbit family. The parts can largely be tackled independently, which implies a certain level of flexibility to alter the density field of one orbit family, and then readjust the distribution of the other orbit family to compensate for the difference. The understanding of the existence of this flexibility, which we gained in Paper II and which is due to the presence of more than one orbit family, led us there to suspect strongly that self-consistent triaxial equilibria are not unique. Thereafter the issue of nonuniqueness became less prominent.

In Paper II we used some of the flexibility available to adjust the distribution of S-tubes and constructed two detailed models, one with infinitesimally thin tubes only, but another with some thicker tubes. Our subsequent papers have used only thin tube orbits. We have done this primarily because of the progress that it has allowed us to make, in this the final paper of the sequence, in the construction of the four separate distribution functions for the four orbit families of the triaxial case. As we showed in Paper III for the oblate case, explicit analytical expressions can be found for two-parameter thin-tube distribution functions. The self-consistent solutions of Paper III are unique because there is only one way in which the one family of thin S-tubes can populate the two-parameter density field. More general oblate Stäckel models are nonunique because the S-tubes can be chosen in many ways from their three-integral family to populate the two-parameter density field (Vandervoort 1984; Bishop 1986). The oblate thin-tube models are undoubtedly extreme and special in ways that we do not yet fully understand. But standard axisymmetric models which use no third integral other than energy and angular momentum are also special.

The prolate Stäckel models of Paper IV are nonunique, even with the restriction to thin tube orbits, because of the possibility of exchanging density between the two families of I- and O-tubes. We found that both families are needed for a self-consistent equilibrium, and investigated the variety of models that occur with one particular method of splitting the density between the two families of long-axis tubes. This same method has been used with the models of the present paper. Paper IV also taught us that the relative numbers of I- and O-tubes can have important kinematic effects. O-tubes have substantially more angular momentum than I-tubes, while I-tubes have rapid rotational velocities, larger than those for closed orbits, in the inner part of the equatorial plane (see Fig. 9 of Paper IV). This same feature persists in the triaxial case (§ 6).

In the present paper we have proved the existence of many self-consistent triaxial equilibrium models, and we have presented a manageable method for constructing them, which determines, in turn, distribution functions for the I- and O-tubes, then the S-tubes, and, last, the box orbits. The density distributions of all four orbit families can be calculated by straightforward quadratures once the density split between the I- and O-tubes has been chosen on the special surface $\mathcal{L}$ (Fig. 1). The velocity moments of the three tube orbit families can be found in a similar manner. Our approach, which seeks out two of the special surfaces in space at which the density is all provided by just two of the four orbit families, may not be the same as that used by Nature to construct elliptical galaxies. Nevertheless, it has allowed us to obtain a number of analytical results, and to show that orbits of all four families are needed in triaxial models. It has allowed us to construct a one-parameter range of self-consistent perfect ellipsoids with the same shape, along which many physical properties were shown to change monotonically (see Fig. 4). The more I-tubes we choose at the first step, the fewer S-tubes there are, and the less is the maximum possible angular momentum about the short axis. The effect of our initial choice on the population of the long-axis tubes, and on the possible angular momentum about the long axis, is more complex.

We have also investigated the set of 21 ellipsoids of different shape considered earlier by Statler (1987). Although our models show the same general trends, they are less varied than his. Even with our restriction to thin tube orbits, all of our models have less angular momentum than Statler's about the long axis, and many have less angular momentum about the short axis. We suspect that

the lesser long-axis angular momentum of our models may be partly due to their relatively high populations of I-tubes, but it remains to be seen whether all of these differences are due to our set of models being too restricted rather than Statler's being too broad.

Aided by our analytical formulae, we have been able to understand the variation of orbital mass fractions and total angular momenta along the four continuous sets of models of widely varying shapes presented in § 8.4. The sometimes rapid variations are due to a combination of dynamical effects and the geometry of the ellipsoids: they are caused by changes in orbital densities and in the geometry of special coordinate surfaces and do not rely in any fundamental way on the fact that the only tube orbits that we use are thin. Hence the same variations can be expected for more general self-consistent triaxial models.

Our models have been used to calculate maximum possible streaming motions around both the long and the short axis. This is useful for the interpretation of the observed velocity fields of elliptical galaxies, especially in view of the recent discovery of some ellipticals that appear to rotate around the long axis at large radii but around the short axis in the central regions (Wagner, Bender, & Möllenhof 1988; Bender 1988; Jedrzejewski & Schechter 1989; Franx, Illingworth, & Heckman 1989). Furthermore, a proper statistical analysis of the intrinsic shapes of elliptical galaxies, using kinematic data along different position angles, also requires knowledge of the internal stellar streaming field (Binney 1985; Franx 1988b; Franx, Illingworth, & de Zeeuw 1991). A major omission of this paper, which we plan to remedy in the future, is the lack of study of observable properties of any of our models. As Statler (1987, 1991) has shown, the projected velocity fields of triaxial galaxies contain much information on their internal dynamical structure. Another omission is the lack of use of the results for second velocity moments which we developed in Appendix B. We expect these results to become useful in connection with work in progress on the direct solution of the Jeans equations for Stäckel models, extending the results already obtained by Evans & Lynden-Bell (1989) for spheroidal models. Such solutions are not restricted to the use of thin tube orbits.

Whether our triaxial models are stable or not is still an open question. Merritt & Stiavelli (1990) have found oblate thin-tube models of all axis ratios to be unstable to a large-scale $m = 1$ distortion. An increase in the radial pressure, i.e., a thickening of the thin S-tubes, stabilizes these models. These authors also found that strongly flattened oblate models are liable to ring instabilities, a result that has recently been confirmed and sharpened by de Zeeuw & Schwarzschild (1991). But stability may be a more complicated matter when orbits of more than one family are present. The first indication of this came from de Zeeuw & Schwarzschild's (1989) finding of the different contributions of the four orbit families to the stability of triaxial galaxies to barlike disturbances. They found box orbits to be always destabilizing and O-tubes to be always stabilizing, while members of the other two orbit families can act in either manner. Merritt & Hernquist (1991) have found the regular prolate spheroids of Paper IV to be liable to bending instabilities if they are more elongated than E6. The great majority of the orbits of such prolate models, whose overall properties are close to those with $k = k_i$, are I-tubes. As Figures 6 and 7 and the discussion of § 8.4 show, such a predominance of I-tubes can occur only with very nearly prolate ellipsoids. It seems likely that the triaxial thin tube orbit models will be afflicted by similar instabilities, but we expect that the nearly round models relevant for elliptical galaxies can be stabilized by a modest increase of the radial thickness of the thin tubes. In this context it is worth noting that N -body simulations of the merging of two spiral disks can result in stable triaxial equilibria with a large fraction of the stars occupying fairly thin tube orbits (Barnes 1990). It is clearly useful to develop a wider range of triaxial models that are not restricted to thin tube orbits. Work along these lines is in progress (e.g., Dejonghe & Laurent 1991).

It is a pleasure to thank Martin Schwarzschild for stimulating discussions, Tom Statler for a referee's report which led to substantial improvement, and Melissa Smith for competent help with the typing of this long manuscript. This research was supported in part by an RCA Fellowship to T. dZ., and by NSF grants DMS 8701128 and DMS 9001404 to C. H. The final rewriting of the manuscript was aided by a Bezoekersbeurs by NWO, the Netherlands Organization for Scientific Research, to C. H.

APPENDIX A

ELLIPTIC INTEGRALS

The various weight functions that occur in the expressions for the density, maximum mean streaming velocity, and second moments are all combinations of complete elliptic integrals. They are most conveniently expressed in terms of the symmetrized forms of the elliptic integrals introduced by Carlson (1979), which are available in the standard numerical libraries. He defined the incomplete elliptic integrals of the three kinds as

$$\begin{aligned} R_F(x, y, z) &= \frac{1}{2} \int_0^\infty \frac{dt}{\sqrt{(x+t)(y+t)(z+t)}}, \\ R_D(x, y, z) &= \frac{3}{2} \int_0^\infty \frac{dt}{\sqrt{(x+t)(y+t)(z+t)^3}}, \\ R_J(x, y, z, p) &= \frac{3}{2} \int_0^\infty \frac{dt}{\sqrt{(x+t)(y+t)(z+t)(p+t)}}. \end{aligned} \quad (\text{A1})$$

It is evident that R_F and R_J are symmetric in x , y , and z , and that R_D is symmetric in its first two arguments. The following scaling relationships hold:

$$\begin{aligned} R_F(x, y, z) &= \frac{1}{s^{1/2}} R_F\left(\frac{x}{s}, \frac{y}{s}, \frac{z}{s}\right), \\ R_D(x, y, z) &= \frac{1}{s^{3/2}} R_D\left(\frac{x}{s}, \frac{y}{s}, \frac{z}{s}\right), \\ R_J(x, y, z, p) &= \frac{1}{s^{3/2}} R_J\left(\frac{x}{s}, \frac{y}{s}, \frac{z}{s}, \frac{p}{s}\right). \end{aligned} \quad (\text{A2})$$

Also

$$xR_D(y, z, x) + yR_D(z, x, y) + zR_D(x, y, z) = 3R_F(x, y, z). \quad (\text{A3})$$

The complete elliptic integrals are Carlson functions with one zero argument. Relations with the classical Legendre forms can be found in many places (see, e.g., Appendix B of de Zeeuw & Pfenniger 1988).

We shall work with the following basic trigonometric integrals, to which all the three-argument integrals newly introduced in this paper can be related. They become $x = 0$ cases of Carlson's integrals with the substitution $t = \tan^2 \theta$, as follows:

$$\begin{aligned} A_1(u_1, u_2, u_3) &= \int_0^{\pi/2} \frac{(u_2 - u_1) \cos^2 \theta d\theta}{(u_1 + u_3 \cos^2 \theta)^{1/2} (u_2 + u_3 \cos^2 \theta)^{3/2}} = \frac{(u_2 - u_1)}{3u_1^{1/2}u_2^{3/2}} R_D\left(0, 1 + \frac{u_3}{u_1}, 1 + \frac{u_3}{u_2}\right), \\ A_2(u_1, u_2, u_3) &= \int_0^{\pi/2} \frac{(u_2 - u_1) \sin^2 \theta d\theta}{(u_1 + u_3 \cos^2 \theta)^{1/2} (u_2 + u_3 \cos^2 \theta)^{3/2}} = \frac{(u_2 - u_1)(u_1 + u_3)}{3(u_1 u_2)^{3/2}} R_D\left(0, 1 + \frac{u_3}{u_2}, 1 + \frac{u_3}{u_1}\right), \\ A_3(u_1, u_2, u_3) &= \int_0^{\pi/2} \frac{u_3 \cos^2 \theta d\theta}{\sqrt{u_1 + u_3 \cos^2 \theta} \sqrt{u_2 + u_3 \cos^2 \theta}} = \frac{u_3}{3\sqrt{u_1 u_2}} R_J\left(0, 1 + \frac{u_3}{u_1}, 1 + \frac{u_3}{u_2}, 1\right), \\ A_4(u_1, u_2, u_3) &= \int_0^{\pi/2} \frac{d\theta}{\sqrt{u_1 + u_3 \cos^2 \theta} \sqrt{u_2 + u_3 \cos^2 \theta}} = \frac{1}{\sqrt{u_1 u_2}} R_F\left(0, 1 + \frac{u_3}{u_1}, 1 + \frac{u_3}{u_2}\right), \end{aligned} \quad (\text{A4})$$

Note that the order of the second and third arguments in the R_D that appears in the expression for A_2 is the opposite of that in the expression for A_1 . Only three of the integrals (A4) are independent, because the interrelation

$$u_2 A_2(u_1, u_2, u_3) + (u_2 + u_3) A_1(u_1, u_2, u_3) = (u_2 - u_1) A_4(u_1, u_2, u_3) \quad (\text{A5})$$

follows immediately from their definitions. More compact expressions are obtained when we use A_1 , A_3 , and A_4 for the integrals associated with the I-tubes but A_2 , A_3 , and A_4 for the integrals associated with the O- and S-tubes. The integral A_0 can immediately be expressed in terms of the basic ones:

$$A_0(u_1, u_2, u_3) = \int_0^{\pi/2} d\theta \sqrt{\frac{u_1 + u_3 \cos^2 \theta}{u_2 + u_3 \cos^2 \theta}} = A_3(u_1, u_2, u_3) + u_1 A_4(u_1, u_2, u_3). \quad (\text{A6})$$

Consider next the integrals that arise in connection with the second velocity moments of the thin I-tube distributions, discussed in Appendix B below. An integration by parts, in which a single $\cos \theta$ factor is integrated, is needed to reduce

$$C_0(u_1, u_2, u_3) = 2u_3 \int_0^{\pi/2} d\theta \cos^2 \theta \sqrt{\frac{u_1 + u_3 \cos^2 \theta}{u_2 + u_3 \cos^2 \theta}} \quad (\text{A7})$$

to our basic forms, and to yield the result

$$C_0(u_1, u_2, u_3) = (u_2 + u_3)u_3 A_1(u_1, u_2, u_3) + (u_1 - u_2 + u_3)A_3(u_1, u_2, u_3) + u_1 u_3 A_4(u_1, u_2, u_3). \quad (\text{A8})$$

The integrals C_1 , E_0 , and E_1 can then all be reduced using definitions and formulae already determined as follows:

$$\begin{aligned} C_1(u_1, u_2, u_3) &= \int_0^{\pi/2} \frac{\cos^2 \theta [2(u_1 + u_3 \cos^2 \theta)(u_2 + u_3 \cos^2 \theta) + u_3(u_2 - u_1) \cos^2 \theta] d\theta}{(u_1 + u_3 \cos^2 \theta)^{1/2} (u_2 + u_3 \cos^2 \theta)^{3/2}} \\ &= \frac{1}{u_3} [C_0(u_1, u_2, u_3) + (u_2 - u_1)A_3(u_1, u_2, u_3)] - u_2 A_1(u_1, u_2, u_3) \\ &= u_3 A_1(u_1, u_2, u_3) + A_3(u_1, u_2, u_3) + u_1 A_4(u_1, u_2, u_3) \\ &= A_0(u_1, u_2, u_3) + u_3 A_1(u_1, u_2, u_3), \end{aligned} \quad (\text{A9})$$

$$\begin{aligned} E_0(u_1, u_2, u_3) &= 2 \int_0^{\pi/2} d\theta \sqrt{u_1 + u_3 \cos^2 \theta} \sqrt{u_2 + u_3 \cos^2 \theta} \\ &= 2u_2 A_0(u_1, u_2, u_3) + C_0(u_1, u_2, u_3) \\ &= (u_2 + u_3)u_3 A_1(u_1, u_2, u_3) + (u_1 + u_2 + u_3)A_3(u_1, u_2, u_3) + u_1(2u_2 + u_3)A_4(u_1, u_2, u_3), \end{aligned} \quad (\text{A10})$$

and

$$\begin{aligned}
 E_1(u_1, u_2, u_3) &= \int_0^{\pi/2} \frac{(u_1 + u_2 + 2u_3 \cos^2 \theta) \cos^2 \theta d\theta}{\sqrt{u_1 + u_3 \cos^2 \theta} \sqrt{u_2 + u_3 \cos^2 \theta}} \\
 &= \frac{1}{u_3} [C_0(u_1, u_2, u_3) + (u_2 - u_1)A_3(u_1, u_2, u_3)] \\
 &= (u_2 + u_3)A_1(u_1, u_2, u_3) + A_3(u_1, u_2, u_3) + u_1 A_4(u_1, u_2, u_3) \\
 &= A_0(u_1, u_2, u_3) + (u_2 + u_3)A_1(u_1, u_2, u_3). \tag{A11}
 \end{aligned}$$

Since $u_2 \geq u_1 > 0$ and $u_3 \geq 0$ in our applications, the functions C_1 and E_1 are everywhere bounded. Partial differentiation with respect to u_3 shows C_1 is an increasing and E_1 a decreasing function of u_3 . Hence we have

$$\frac{\pi}{2} \sqrt{\frac{u_1}{u_2}} \leq C_1(u_1, u_2, u_3) \leq \frac{\pi}{2}, \quad \frac{\pi}{4} \frac{(u_1 + u_2)}{\sqrt{u_1 u_2}} \geq E_1(u_1, u_2, u_3) \geq \frac{\pi}{2}. \tag{A12}$$

The formulae for O- and S-tubes given in Appendix B involve integrals with different sets of arguments. With A_2 rather than A_1 as a basic integral, equation (A10) becomes

$$E_0(w_1, w_2, w_3) = -w_2 w_3 A_2(w_1, w_2, w_3) + (w_1 + w_2 + w_3)A_3(w_1, w_2, w_3) + w_2(2w_1 + w_3)A_4(w_1, w_2, w_3). \tag{A13}$$

Expressions for the remaining functions can all be obtained using reductions given earlier, and equation (A5). This results in

$$\begin{aligned}
 D_0(w_1, w_2, w_3) &= 2w_3 \int_0^{\pi/2} d\theta \sin^2 \theta \sqrt{\frac{w_1 + w_3 \cos^2 \theta}{w_2 + w_3 \cos^2 \theta}} \\
 &= 2(w_2 + w_3)A_0(w_1, w_2, w_3) - E_0(w_1, w_2, w_3) \\
 &= w_2 w_3 A_2(w_1, w_2, w_3) - (w_1 - w_2 - w_3)A_3(w_1, w_2, w_3) + w_3(2w_1 - w_2)A_4(w_1, w_2, w_3), \tag{A14}
 \end{aligned}$$

$$\begin{aligned}
 D_2(w_1, w_2, w_3) &= \int_0^{\pi/2} \frac{\sin^2 \theta [2(w_1 + w_3 \cos^2 \theta)(w_2 + w_3 \cos^2 \theta) - w_3(w_2 - w_1) \sin^2 \theta] d\theta}{(w_1 + w_3 \cos^2 \theta)^{1/2} (w_2 + w_3 \cos^2 \theta)^{3/2}} \\
 &= \frac{1}{w_3} [D_0(w_1, w_2, w_3) - (w_2 - w_1)A_3(w_1, w_2, w_3)] \\
 &\quad + (w_2 + w_3)A_1(w_1, w_2, w_3) - w_3 A_2(w_1, w_2, w_3) \\
 &= -w_3 A_2(w_1, w_2, w_3) + A_3(w_1, w_2, w_3) + w_1 A_4(w_1, w_2, w_3) \\
 &= A_0(w_1, w_2, w_3) - w_3 A_2(w_1, w_2, w_3), \tag{A15}
 \end{aligned}$$

$$\begin{aligned}
 E_2(w_1, w_2, w_3) &= \int_0^{\pi/2} \frac{(w_1 + w_2 + 2w_3 \cos^2 \theta) \sin^2 \theta d\theta}{\sqrt{w_1 + w_3 \cos^2 \theta} \sqrt{w_2 + w_3 \cos^2 \theta}} \\
 &= A_0(w_1, w_2, w_3) + A_0(w_2, w_1, w_3) - E_1(w_1, w_2, w_3) \\
 &= w_2 A_2(w_1, w_2, w_3) + A_3(w_1, w_2, w_3) + w_1 A_4(w_1, w_2, w_3) \\
 &= A_0(w_1, w_2, w_3) + w_2 A_2(w_1, w_2, w_3), \tag{A16}
 \end{aligned}$$

and

$$\begin{aligned}
 F_0(w_1, w_2, w_3) &= \int_0^{\pi/2} \frac{2w_3^2 \sin^2 \theta \cos^2 \theta d\theta}{\sqrt{w_1 + w_3 \cos^2 \theta} \sqrt{w_2 + w_3 \cos^2 \theta}} \\
 &= \frac{1}{w_2 - w_1} [(w_1 + w_2)D_0(w_1, w_2, w_3) - 2w_1 w_3 E_2(w_1, w_2, w_3)] \\
 &= w_2 w_3 [A_2(w_1, w_2, w_3) - A_4(w_1, w_2, w_3)] + (w_1 + w_2 + w_3)A_3(w_1, w_2, w_3). \tag{A17}
 \end{aligned}$$

Yet other interesting relations exist between the various elliptic integrals used in this paper. The consistency of expressions (3.29) and (4.17) for the densities ρ_{om} and ρ_{sm} when evaluated on surface $\mathcal{S}$ with the interrelation (2.38) follows easily if the relation

$$A_0(w_1, w_2, w_4) + A_3(w_1, w_1 + w_4, w_2 - w_1) = \frac{\pi}{2} \tag{A18}$$

is used. This relation, which is valid for $w_2 > w_1 > 0$ and $w_4 > 0$, can be established by using a special case of a quadratic transformation of elliptic integrals due to Zill & Carlson (1970).

APPENDIX B

SECOND MOMENTS

The Jeans equations for the second velocity moments are given in § B1. Their solutions for the three thin tube orbit families are given in the next three subsections as one-dimensional weighted integrals of the densities on the two special surfaces $\mathcal{L}$ and $\mathcal{S}$ (Fig. 1).

B1. THE JEANS EQUATIONS

The symmetries that cause certain first-order moments in § 6 to vanish also cause all the mixed second-order velocity moments to vanish:

$$\langle v_\lambda v_\mu \rangle = \langle v_\mu v_\nu \rangle = \langle v_\nu v_\lambda \rangle = 0. \quad (\text{B1})$$

The remaining second-order moments $\langle v_\lambda^2 \rangle$, $\langle v_\mu^2 \rangle$, and $\langle v_\nu^2 \rangle$ are related by the Jeans equations, also referred to as the equations of stellar hydrodynamics. These are (Lynden-Bell 1960)

$$\frac{\partial T_{\lambda\lambda}}{\partial \lambda} + \frac{T_{\lambda\lambda} - T_{\mu\mu}}{2(\lambda - \mu)} + \frac{T_{\lambda\lambda} - T_{\nu\nu}}{2(\lambda - \nu)} = -\rho \frac{\partial V}{\partial \lambda} = -\rho(\lambda, \mu, \nu)U[v, \mu, \lambda], \quad (\text{B2a})$$

$$\frac{\partial T_{\mu\mu}}{\partial \mu} + \frac{T_{\lambda\lambda} - T_{\mu\mu}}{2(\lambda - \mu)} + \frac{T_{\mu\mu} - T_{\nu\nu}}{2(\mu - \nu)} = -\rho \frac{\partial V}{\partial \mu} = -\rho(\lambda, \mu, \nu)U[v, \mu, \lambda], \quad (\text{B2b})$$

$$\frac{\partial T_{\nu\nu}}{\partial \nu} + \frac{T_{\lambda\lambda} - T_{\nu\nu}}{2(\lambda - \nu)} + \frac{T_{\mu\mu} - T_{\nu\nu}}{2(\mu - \nu)} = -\rho \frac{\partial V}{\partial \nu} = -\rho(\lambda, \mu, \nu)U[v, \nu, \mu], \quad (\text{B2c})$$

where we have written $T_{\tau\tau} = \rho \langle v_\tau^2 \rangle$. Note that ρ here can be any density distribution that corresponds to a physical distribution function in the potential V , that for any single orbit family, for example, and does not have to be the self-consistent ρ_m .

Equations (B2) can be verified by substitution of expressions (2.17) in definition (6.1), but it is sometimes easier to determine second-order moments by solving the Jeans equations rather than by direct evaluation of the integral (6.1). In order to avoid unphysical singularities in the velocity dispersions at the focal curves of the ellipsoidal coordinates, where normals to the coordinate surfaces change directions discontinuously, we require that

$$T_{\lambda\lambda}(-\alpha, -\alpha, \nu) = T_{\mu\mu}(-\alpha, -\alpha, \nu), \quad T_{\mu\mu}(\lambda, -\beta, -\beta) = T_{\nu\nu}(\lambda, -\beta, -\beta). \quad (\text{B3})$$

The above results show that *the principal axes of the velocity dispersion tensor in a Stäckel model are always aligned with the ellipsoidal coordinates in which the potential is of Stäckel form*. This result was first obtained by Eddington (1915). In fact, Eddington assumed that the principal axes of the velocity ellipsoid are aligned with the coordinates, and derived the coordinate systems in which this happens for potentials that admit integrals of motion that are quadratic in the velocities. This yielded—of course—precisely the ellipsoidal coordinates and the Stäckel potentials. See also Chandrasekhar (1942).

B2. THIN I-TUBES

We denote the second-order velocity moments associated with the thin I-tubes as $T_{\tau\tau}^i$ ($\tau = \lambda, \mu, \nu$). We have $T_{\mu\mu}^i \equiv 0$, since there is now no motion in the μ -coordinate. As a result, equation (B2b) reduces to an algebraic relation between $T_{\lambda\lambda}^i$ and $T_{\nu\nu}^i$, given by

$$T_{\nu\nu}^i = 2(\mu - \nu)\rho_{im}(\lambda, \mu, \nu)U[v, \mu, \mu, \lambda] + \left(\frac{\mu - \nu}{\lambda - \mu}\right)T_{\lambda\lambda}^i. \quad (\text{B4})$$

This relation can be used to eliminate $T_{\nu\nu}^i$ and $T_{\lambda\lambda}^i$ from the remaining Jeans equations (B2a) and (B2c), respectively, which then become

$$\frac{\partial}{\partial \lambda} [(\lambda - \nu)T_{\lambda\lambda}^i] = -(\lambda - \mu)\rho_{im}(\lambda, \mu, \nu)U[\mu, \mu, \lambda, \lambda], \quad (\text{B5a})$$

$$\frac{\partial}{\partial \nu} [(\lambda - \nu)T_{\nu\nu}^i] = -(\mu - \nu)\rho_{im}(\lambda, \mu, \nu)U[v, \nu, \mu, \mu]. \quad (\text{B5b})$$

The first of these equations can be integrated to give

$$T_{\lambda\lambda}^i(\lambda, \mu, \nu) = \frac{1}{\lambda - \nu} \int_\lambda^\infty (\sigma - \mu)\rho_{im}(\sigma, \mu, \nu)U[\mu, \mu, \sigma, \sigma]d\sigma. \quad (\text{B6})$$

Here we have used the fact that $\lambda T_{\lambda\lambda}^i \rightarrow 0$ as $\lambda \rightarrow \infty$, since both the density ρ_{im} and the velocity v_λ tend to zero at large radii. $T_{\nu\nu}^i$ can now be calculated by substitution of equation (B6) in the relation (B4). Alternatively, integration of the other Jeans equation, equation (B5b), gives

$$T_{\nu\nu}^i(\lambda, \mu, \nu) = \frac{1}{\lambda - \nu} \int_\nu^{-\beta} (\mu - \sigma)\rho_{im}(\lambda, \mu, \sigma)U[\sigma, \sigma, \mu, \mu]d\sigma + \left(\frac{\lambda + \beta}{\lambda - \nu}\right)T_{\nu\nu}^i(\lambda, \mu, -\beta). \quad (\text{B7})$$

The boundary value $T_{vv}^i(\lambda, \mu, -\beta)$ follows from relation (B4):

$$T_{vv}^i(\lambda, \mu, -\beta) = 2(\mu + \beta)\rho_{im}(\lambda, \mu, -\beta)U[-\beta, \mu, \mu, \lambda] + \left(\frac{\mu + \beta}{\lambda - \mu}\right)T_{\lambda\lambda}^i(\lambda, \mu, -\beta), \quad (\text{B8})$$

where $T_{\lambda\lambda}^i(\lambda, \mu, -\beta)$ can be obtained from equation (B6).

Expressions (B6) and (B7) give $T_{\lambda\lambda}^i$ and T_{vv}^i as weighted integrals over the density ρ_{im} at a general point (λ, μ, v) . We have seen in § 3.1 that $\rho_{im}(\lambda, \mu, v)$ can be written as an integral over $\rho_{im}(\lambda, \mu, -\beta)$, so that equations (B6) and (B7) then give $T_{\lambda\lambda}^i$ and T_{vv}^i as double integrals over $\rho_{im}(\lambda, \mu, -\beta)$. However, the inner integrals can be done in closed form after an exchange of orders of integration, and we are again left with single integrals. The same results are obtained by direct evaluation of $T_{\lambda\lambda}^i$ and T_{vv}^i via definition (6.1). Using the form of the distribution function given by equations (3.11) and (3.12), we find

$$T_{\lambda\lambda}^i(\lambda, \mu, v) = -\frac{2(\mu + \beta)}{\pi(\lambda - v)(\mu - v)} \int_{\lambda}^{\infty} d\sigma C_0(u_1, u_2, u_3) \frac{\partial}{\partial \sigma} [(\sigma - \mu)\rho_{im}(\sigma, \mu, -\beta)] \quad (\text{B9})$$

and

$$T_{vv}^i(\lambda, \mu, v) = -\frac{2(\mu + \beta)}{\pi(\lambda - \mu)(\lambda - v)} \int_{\lambda}^{\infty} d\sigma E_0(u_1, u_2, u_3) \frac{\partial}{\partial \sigma} [(\sigma - \mu)\rho_{im}(\sigma, \mu, -\beta)], \quad (\text{B10})$$

where C_0 and E_0 are the auxiliary functions given in equations (A7) and (A10), and u_1, u_2 , and u_3 were defined in equation (3.14). Alternative forms are obtained by integrating equation (B10) by parts to give

$$T_{\lambda\lambda}^i(\lambda, \mu, v) = \frac{2(\mu + \beta)}{\pi(\lambda - v)(\mu - v)} \int_{\lambda}^{\infty} d\sigma C_1(u_1, u_2, u_3)(\sigma - \mu)\rho_{im}(\sigma, \mu, -\beta)U[\mu, \mu, \sigma, \sigma] \quad (\text{B11})$$

and

$$T_{vv}^i(\lambda, \mu, v) = \frac{2(\mu + \beta)}{\lambda - v} \left\{ \rho_{im}(\lambda, \mu, -\beta)\sqrt{u_1 u_2} + \frac{1}{\pi(\lambda - \mu)} \int_{\lambda}^{\infty} d\sigma E_1(u_1, u_2, u_3)(\sigma - \mu)\rho_{im}(\sigma, \mu, -\beta)U[\mu, \mu, \sigma, \sigma] \right\}, \quad (\text{B12})$$

with C_1 and E_1 given in equations (A9) and (A11).

B3. THIN O-TUBES

The moment $T_{\lambda\lambda}^o \equiv 0$ for O-tubes, and equation (B2a) then reduces to

$$T_{vv}^o = 2(\lambda - v)\rho_{om}(\lambda, \mu, v)U[v, \mu, \lambda, \lambda] - \left(\frac{\lambda - v}{\lambda - \mu}\right)T_{\mu\mu}^o. \quad (\text{B13})$$

Substitution in equations (B2b) and (B2c) gives

$$\frac{\partial}{\partial \mu} [(\mu - v)T_{\mu\mu}^o] = (\lambda - \mu)\rho_{om}(\lambda, \mu, v)U[\mu, \mu, \lambda, \lambda], \quad (\text{B14a})$$

$$\frac{\partial}{\partial v} [(\mu - v)T_{vv}^o] = (v - \lambda)\rho_{om}(\lambda, \mu, v)U[v, v, \lambda, \lambda]. \quad (\text{B14b})$$

Integration of the first of these equations gives

$$T_{\mu\mu}^o(\lambda, \mu, v) = \frac{1}{\mu - v} \int_{-\beta}^{\mu} d\sigma (\lambda - \sigma)\rho_{om}(\lambda, \sigma, v)U[\sigma, \sigma, \lambda, \lambda], \quad (\text{B15})$$

after using the fact that now $v_{\mu} = 0$ at $\mu = -\beta$, and hence $T_{\mu\mu}^o(\lambda, -\beta, v) = 0$. The second moment T_{vv}^o can be evaluated by means of equation (B13) or by direct integration of equation (B14b). The latter gives

$$T_{vv}^o(\lambda, \mu, v) = \frac{1}{\mu - v} \int_v^{-\beta} d\sigma (\lambda - \sigma)\rho_{om}(\lambda, \mu, \sigma)U[\sigma, \sigma, \lambda, \lambda] + \left(\frac{\mu + \beta}{\mu - v}\right)T_{vv}^o(\lambda, \mu, -\beta), \quad (\text{B16})$$

where (see eq. [B13])

$$T_{vv}^o(\lambda, \mu, -\beta) = 2(\lambda + \beta)\rho_{om}(\lambda, \mu, -\beta)U[-\beta, \mu, \lambda, \lambda] - \left(\frac{\lambda + \beta}{\lambda - \mu}\right)T_{\mu\mu}^o(\lambda, \mu, -\beta). \quad (\text{B17})$$

Here the value of $T_{\mu\mu}^o$ follows from equation (B15).

Just as for the I-tubes, a change of the order of integration after the density $\rho_{om}(\lambda, \mu, v)$ has been written as a single integral over $\rho_{om}(\lambda, \mu, -\beta)$ (see § 3.2) leads to single integrals. The resulting expressions are most easily obtained by direct evaluation of the

second-order moments via definition (6.1) and the distribution function given by equations (3.27) and (3.28). They are

$$T_{\mu\mu}^o = \frac{2(\lambda + \beta)}{\pi(\lambda - \nu)(\mu - \nu)} \left((\lambda + \beta)\rho_{om}(\lambda, -\beta, -\beta) \left[\{DW[\mu] - W[\nu]\} \arcsin \sqrt{\frac{W[\mu] - W[-\beta]}{W[\mu] - W[\nu]}} - \sqrt{W[\mu] - W[-\beta]}\sqrt{W[-\beta] - W[\nu]} \right] + \int_{-\beta}^{\mu} d\sigma D_0(w_1, w_2, w_3) \frac{\partial}{\partial \sigma} [(\lambda - \sigma)\rho_{om}(\lambda, \sigma, -\beta)] \right) \quad (B18)$$

and

$$T_{\nu\nu}^o = \frac{2(\lambda + \beta)}{\pi(\lambda - \mu)(\mu - \nu)} \left((\lambda + \beta)\rho_{om}(\lambda, -\beta, -\beta) \left[\{W[\mu] - W[\nu]\} \arcsin \sqrt{\frac{W[\mu] - W[-\beta]}{W[\mu] - W[\nu]}} + \sqrt{W[\mu] - W[-\beta]}\sqrt{W[-\beta] - W[\nu]} \right] + \int_{-\beta}^{\mu} d\sigma E_0(w_1, w_2, w_3) \frac{\partial}{\partial \sigma} [(\lambda - \sigma)\rho_{om}(\lambda, \sigma, -\beta)] \right), \quad (B19)$$

where w_1 , w_2 , and w_3 were defined in equation (3.30), and the weight functions D_0 and E_0 are given in equations (A14) and (A10), respectively.

Alternative forms obtained by integration by parts are given by

$$T_{\mu\mu}^o(\lambda, \mu, \nu) = \frac{2(\lambda + \beta)}{\pi(\lambda - \nu)(\mu - \nu)} \int_{-\beta}^{\mu} d\sigma D_2(w_1, w_2, w_3)(\lambda - \sigma)\rho_{om}(\lambda, \sigma, -\beta)U[\sigma, \sigma, \lambda, \lambda] \quad (B20)$$

and

$$T_{\nu\nu}^o(\lambda, \mu, \nu) = \frac{2(\lambda + \beta)}{\mu - \nu} \left\{ \rho_{om}(\lambda, \mu, -\beta)\sqrt{W[\mu] - W[-\beta]}\sqrt{W[\mu] - W[\nu]} - \frac{1}{\pi(\lambda - \mu)} \int_{-\beta}^{\mu} d\sigma E_2(w_1, w_2, w_3)(\lambda - \sigma)\rho_{om}(\lambda, \sigma, -\beta)U[\sigma, \sigma, \lambda, \lambda] \right\}, \quad (B21)$$

with D_2 and E_2 given in equations (A15) and (A16).

B4. THIN S-TUBES

Thin S-tubes have $T_{\lambda\lambda}^s \equiv 0$, and the resulting Jeans equations are similar to those for O-tubes. We find

$$T_{\nu\nu}^s = 2(\lambda - \nu)\rho_{sm}(\lambda, \mu, \nu)U[\nu, \mu, \lambda, \lambda] - \left(\frac{\lambda - \nu}{\lambda - \mu}\right)T_{\mu\mu}^s. \quad (B22)$$

Integration of the remaining equations yields

$$T_{\nu\nu}^s(\lambda, \mu, \nu) = \frac{1}{\mu - \nu} \int_{\nu}^{-\beta} d\sigma (\lambda - \sigma)\rho_{sm}(\lambda, \mu, \sigma)U[\sigma, \sigma, \lambda, \lambda] \quad (B23)$$

and

$$T_{\mu\mu}^s(\lambda, \mu, \nu) = -\frac{1}{\mu - \nu} \int_{\mu}^{-\alpha} d\sigma (\lambda - \sigma)\rho_{sm}(\lambda, \sigma, \nu)U[\sigma, \sigma, \lambda, \lambda] + \left(\frac{-\alpha - \nu}{\mu - \nu}\right)T_{\mu\mu}^s(\lambda, -\alpha, \nu), \quad (B24)$$

with

$$T_{\mu\mu}^s(\lambda, -\alpha, \nu) = 2(\lambda + \alpha)\rho_{sm}(\lambda, -\alpha, \nu)U[\nu, -\alpha, \lambda, \lambda] - \left(\frac{\lambda + \alpha}{\lambda - \nu}\right)T_{\nu\nu}^s(\lambda, -\alpha, \nu), \quad (B25)$$

and $T_{\nu\nu}^s(\lambda, -\alpha, \nu)$ follows immediately from equation (B23).

Direct evaluation of the second moments in terms of $\rho_{sm}(\lambda, -\alpha, \nu)$ via equation (6.1) gives

$$T_{\mu\mu}^s(\lambda, \mu, \nu) = -\frac{2(\lambda + \alpha)}{\pi(\lambda - \nu)(\mu - \nu)} \int_{\nu}^{-\beta} d\sigma E_0(w_4, w_3, w_2) \frac{\partial}{\partial \sigma} [(\lambda - \sigma)\rho_{sm}(\lambda, -\alpha, \sigma)] \quad (B26)$$

and

$$T_{\nu\nu}^s(\lambda, \mu, \nu) = -\frac{2(\lambda + \alpha)}{\pi(\lambda - \mu)(\mu - \nu)} \int_{\nu}^{-\beta} d\sigma D_0(w_4, w_3, w_2) \frac{\partial}{\partial \sigma} [(\lambda - \sigma)\rho_{sm}(\lambda, -\alpha, \sigma)], \quad (B27)$$

where D_0 and E_0 are given in equations (A14) and (A10), respectively, and w_2 , w_3 , and w_4 were given in equations (3.30) and (3.48). Alternative forms are

$$T_{\nu\nu}^s(\lambda, \mu, \nu) = \frac{2(\lambda + \alpha)}{\pi(\lambda - \mu)(\mu - \nu)} \int_{\nu}^{-\beta} d\sigma D_2(w_4, w_3, w_2)(\lambda - \sigma)\rho_{sm}(\lambda, -\alpha, \sigma)U[\sigma, \sigma, \lambda, \lambda] \quad (B28)$$

and

$$T_{\mu\mu}^s(\lambda, \mu, \nu) = \frac{2(\lambda + \alpha)}{\mu - \nu} \left\{ \rho_{sm}(\lambda, -\alpha, \nu) \sqrt{W[-\alpha] - W[\nu]} \sqrt{W[\mu] - W[\nu]} \right. \\ \left. - \frac{1}{\pi(\lambda - \nu)} \int_{\nu}^{-\beta} d\sigma E_2(w_4, w_3, w_2) \rho_{sm}(\lambda, -\alpha, \sigma) (\lambda - \sigma) U[\sigma, \sigma, \lambda, \lambda] \right\}, \quad (\text{B29})$$

where D_2 and E_2 are defined in equations (A15) and (A16).

Just as was done for $\rho_{sm}(\lambda, \mu, \nu)$ in § 4.3, we can make expressions (B26) and (B27) more explicit in terms of our density split by using equation (2.38) with $\rho_{om}(\lambda, -\alpha, \sigma)$ given in equation (3.29). We obtain

$$T_{\mu\mu}^s(\lambda, \mu, \nu) = -\frac{2(\lambda + \alpha)}{\pi(\lambda - \nu)(\mu - \nu)} \int_{\nu}^{-\beta} d\sigma E_0(w_4, w_3, w_2) \frac{\partial}{\partial \sigma} [(\lambda - \sigma) \rho_m(\lambda, -\alpha, \sigma)] \\ + \frac{2(\lambda + \beta)^2 \rho_{om}(\lambda, -\beta, -\beta)}{\pi(\lambda - \nu)(\mu - \nu)} \left[\sqrt{W[\mu] - W[-\beta]} \sqrt{W[-\beta] - W[\nu]} + \{W[\mu] - W[\nu]\} \arcsin \sqrt{\frac{W[-\beta] - W[\nu]}{W[\mu] - W[\nu]}} \right] \\ + \frac{2(\lambda + \beta)}{\pi(\lambda - \nu)(\mu - \nu)} \int_{-\beta}^{-\alpha} d\sigma C_0(w_1 + w_3, w_1, w_2 - w_1) \frac{\partial}{\partial \sigma} [(\lambda - \sigma) \rho_{om}(\lambda, \sigma, -\beta)] \quad (\text{B30})$$

and

$$T_{\nu\nu}^s(\lambda, \mu, \nu) = -\frac{2(\lambda + \alpha)}{\pi(\lambda - \mu)(\mu - \nu)} \int_{\nu}^{-\beta} d\sigma D_0(w_4, w_3, w_2) \frac{\partial}{\partial \sigma} [(\lambda - \sigma) \rho_m(\lambda, -\alpha, \sigma)] \\ + \frac{2(\lambda + \beta)^2 \rho_{om}(\lambda, -\beta, -\beta)}{\pi(\lambda - \mu)(\mu - \nu)} \left[-\sqrt{W[\mu] - W[-\beta]} \sqrt{W[-\beta] - W[\nu]} + \{W[\mu] - W[\nu]\} \arcsin \sqrt{\frac{W[-\beta] - W[\nu]}{W[\mu] - W[\nu]}} \right] \\ + \frac{2(\lambda + \beta)}{\pi(\lambda - \mu)(\mu - \nu)} \int_{-\beta}^{-\alpha} d\sigma F_0(w_1, w_1 + w_3, w_2 - w_1) \frac{\partial}{\partial \sigma} [(\lambda - \sigma) \rho_{om}(\lambda, \sigma, -\beta)], \quad (\text{B31})$$

where F_0 is given in equation (A17). Use of these expressions avoids the need for first calculating ρ_{sm} at $\mu = -\alpha$.

REFERENCES

- Barnes, J. E. 1990, talk presented at Aspen Center for Physics Workshop on Stellar Dynamics and Galaxy Formation
 Bender, R. 1988, A&A, 202, L5
 Binney, J. J. 1985, MNRAS, 212, 767
 Binney, J. J., & Tremaine, S. D. 1987, Galactic Dynamics (Princeton: Princeton Univ. Press)
 Bishop, J. L. 1986, ApJ, 305, 14
 ———. 1987, ApJ, 322, 618
 Carlson, B. C. 1979, Numer. Math., 33, 1
 Chandrasekhar, S. 1942, Principles of Stellar Dynamics (Chicago: Univ. Chicago Press)
 Davies, R. L., & Birkinshaw, M. 1988, ApJS, 68, 409
 Dejonghe, H., & Laurent, D. 1991, MNRAS, 252, 606
 de Zeeuw, P. T. 1985a, MNRAS, 216, 273 (Paper I)
 ———. 1985b, MNRAS, 216, 599
 de Zeeuw, P. T., & Hunter, C. 1990, ApJ, 356, 365 (Paper III)
 de Zeeuw, P. T., Hunter, C., & Schwarzschild, M. 1987, ApJ, 317, 607 (Paper II)
 de Zeeuw, P. T., & Lynden-Bell, D. 1985, MNRAS, 215, 713
 de Zeeuw, P. T., Peletier, R. F., & Franx, M. 1986, MNRAS, 221, 1001
 de Zeeuw, P. T., & Pfenniger, D. 1988, MNRAS, 235, 949
 de Zeeuw, P. T., & Schwarzschild, M. 1989, ApJ, 345, 84
 ———. 1991, ApJ, 369, 57
 Eddington, A. S. 1915, MNRAS, 76, 37
 Evans, N. W., & Lynden-Bell, D. 1989, MNRAS, 236, 801
 Franx, M. 1988a, MNRAS, 231, 285
 ———. 1988b, Ph.D. thesis, Univ. Leiden
 Franx, M., Illingworth, G. D., & de Zeeuw, P. T. 1991, ApJ, 383, 112
 Franx, M., Illingworth, G. D., & Heckman, T. 1989, ApJ, 344, 613
 Goursat, E. 1964, A Course in Mathematical Analysis, Vol. 3 (New York: Dover)
 Hunter, C., de Zeeuw, P. T., Park, Ch., & Schwarzschild, M. 1990, ApJ, 363, 367 (Paper IV)
 Jeans, J. H. 1915, MNRAS, 76, 71
 Jedrzejewski, R. I., & Schechter, P. L. 1989, AJ, 98, 147
 Kuzmin, G. G. 1973, in Proc. All-Union Conf., Dynamics of Galaxies and Clusters, ed. T. B. Omarov (Alma Ata: Akad. Nauk Kazakhskoj SSR), 71 (English transl. in IAU Symp. 127, Structure and Dynamics of Elliptical Galaxies, ed. P. T. de Zeeuw [Dordrecht: Reidel], 553)
 Levison, H. F. 1987, ApJ, 320, L93
 Levison, H. F., & Richstone, D. O. 1987, ApJ, 314, 476
 Lynden-Bell, D. 1960, Ph.D. thesis, Cambridge Univ.
 Merritt, D. R., & Hernquist, L. 1991, ApJ, 376, 439
 Merritt, D. R., & Stiavelli, M. 1990, ApJ, 358, 399
 Schwarzschild, M. 1979, ApJ, 232, 263
 ———. 1982, ApJ, 236, 599
 Statler, T. S. 1987, ApJ, 321, 113
 ———. 1991, AJ, 101, 882
 Teuben, P. J. 1987, MNRAS, 227, 815
 Tremaine, S. D., & de Zeeuw, P. T. 1987, in IAU Symp. 127, Structure and Dynamics of Elliptical Galaxies, ed. P. T. de Zeeuw (Dordrecht: Reidel), 493
 Vandervoort, P. O. 1984, ApJ, 287, 475
 Wagner, S. J. 1990, in Dynamics and Interactions of Galaxies, ed. R. Wielen (Berlin: Springer-Verlag), 244
 Wagner, S. J., Bender, R., & Möllenhof, C. 1988, A&A, 195, L5
 Wilkinson, A., Sharples, R. M., Fosbury, R. A. E., & Wallace, P. T. 1986, MNRAS, 218, 297
 Zill, D. G., & Carlson, B. C. 1970, Math. Comput., 24, 199