Glassy dynamics of pinned charge-density waves

A. Erzan* and E. Veermans
Solid State Physics Laboratory, University of Groningen, Melkweg 1, 9718 EP Groningen, The Netherlands

R. Heijungs
Laboratory of Physiology and Physiological Physics, University of Leiden, Post Office Box 9604, 2300 RC Leiden, The Netherlands

L. Pietronero
Dipartimento di Fisica, Universita di Roma La Sapeinza, Piazzale Aldo Moro 2, 00185 Roma, Italy

(Received 11 January 1990)

Stretched-exponential relaxation behavior observed in a one-dimensional model of pinned charge-density waves is shown to arise from anomalous deterministic diffusion confined to a closed portion of phase space. Simulation results and scaling arguments are used to relate the index of the stretched exponential to the mean-field value of the anomalous-random-walk exponent on directed percolation clusters.

I. INTRODUCTION

Nonexponential relaxation in random systems, polymers, and glasses[1] has been an active field of study for many years. More recently, pinned–charge-density-wave (CDW) systems have been added[2] to this list.

A wide variety of mechanisms can be shown to give rise to either stretched-exponential or algebraic relaxation patterns.[3–6] Although it is now well established[6] that a scale-invariant distribution of relaxation times underlies these diverse phenomena, one is rarely able to determine the form of the relevant distribution from a microscopic theory. The demonstration by Mezard *et al.*[7] of the ultrametric topology of the phase space of the infinite-range Ising spin glass, with the free energies of the quasidegenerate states behaving as *independent* random variables, has led to a flurry of activity to model the relaxation of glassy systems in terms of diffusion on ultrametric spaces, with a hierarchy of relaxation times.[8–12] More often than not, however, such constructions involve *ad hoc* assumptions as to the form of the distributions, and are unable to account for the apparent universality observed in a broad class of physical systems.[13]

Ogielski[14] and Campbell and co-workers[13,15–17] have recently proposed that the stretched-exponential relaxation observed in short-range Ising spin glasses can be understood in terms of the random motion performed by the phase point in Ising phase space, which consists of the vertices of an N-dimensional hypercube. Without making any detailed assumption on the distribution of waiting times or relaxation rates, they are thus able to avail themselves directly of the scaling results for random walks on percolation clusters.[18–20] These authors argue that in an Ising spin glass, for temperatures below the Griffiths temperature T_c, certain edges will be removed from this hypercubic network due to frustration, and that as one lowers the temperature even further one ends up, at the freezing temperature T_g, with a percolating cluster of connected vertices. The stretched exponential arises naturally as the way in which the asymptotic distribution of phase points on this closed, fractal hyperspace depends upon time.

We address ourselves in this paper to the relaxation behavior displayed by a classical, one-dimensional many-body model of randomly pinned CDW's elaborated by Pietronero and Strässler.[21] We have performed extensive simulations on this model to determine the form of the relaxation patterns in the pinned phase and found a stretched-exponential dependence of the polarization and the current upon time, with a fractional power $\sim\frac{1}{3}$ for the exponent. We show that the phase space of this model shows a close resemblance to that of short-range Ising spin glasses. However, due to the presence of an external field, the relaxation problem is akin to random walks on *directed* percolation clusters.[22] We relate the observed fractional power to the appropriate anomalous-random-walk dimension. Moreover, we point out that the diffusive motion arises as a property of the microscopic equations of motion[21] describing the dynamics of the system, which may be cast in the form of a coupled-map lattice,[23] and no additional assumptions on the randomness of the motion in phase space are necessary.

In Sec. II, we will present the model and motivate the discussion. In Sec. III, we will present the results of our numerical simulations together with a discussion of our findings in terms of waiting time distributions. Section IV contains scaling arguments and a comparison with our numerical results, while in Sec. V we discuss our model within the context of coupled-map lattices and deterministic diffusion.

II. THE MODEL

The classical one-dimensional model of charge-density waves (CDW's) at zero temperature, with pinning due to impurities, leads[21] to an equation of motion for the phases ψ_i at each impurity site i, $i=1,\ldots,N$,

$$\frac{d\psi_i}{dt}=B\left[\frac{\psi_{i+1}-\psi_i}{r_{i+1}-r_i}-\frac{\psi_i-\psi_{i-1}}{r_i-r_{i-1}}\right] -\sin[2\pi(qr_i+\psi_i)]+EQ_i \ . \quad (1)$$

The r_i are the random positions of the impurities, q is the wave number of the CDW, $Q_i=\frac{1}{2}(r_{i+1}-r_{i-1})$, and B and E are dimensionless constants corresponding to the diffusive coupling and the external field, respectively. The strength of the sinusoidal pinning potential has been uniformly normalized to unity.

For simplicity, let us introduce the quenched random variables $\phi_i=qr_i$, $0<\phi_i<1$, and otherwise take the intervals between the impurity sites to be uniform. The equation of motion now becomes

$$\frac{d\psi_i}{dt}=B(\psi_{i+1}+\psi_{i-1}-2\psi_i)-\sin[2\pi(\phi_i+\psi_i)]+E \ . \quad (2)$$

Numerical simulations are to be performed on this set of equations by taking discrete time steps, $t=n\delta t$. What we have in effect, then, is a set of recurrence relations for the values $\psi_i(n)$ of the phases at the lattices sites i, at time steps n. These recurrence relations may be expressed as a set of diffusively coupled one-dimensional maps, or a coupled-map lattice[23,24] for the ψ_i. Define

$$f_i(x)=(1-2B\delta t)x-\delta t\sin[2\pi(\phi_i+x)]+E\delta t \ . \quad (3)$$

Then,

$$\psi_i(n+1)=f_i(\psi_i(n))+B\delta t(\psi_{i+1}(n)+\psi_{i-1}(n)) \ . \quad (4)$$

The model given by Eqs. (2) or (4) exhibits[23,25] a "sliding" and a "pinned" phase for E above or below a threshold field $E_{\text{th}}(B)$, where a dynamic phase transition[26] takes place. Below the threshold, an applied field gives rise to a polarization,

$$P(t)=\frac{1}{N}\sum_i \psi_i(t) \ , \quad (5)$$

which, on the average, saturates as $t\to\infty$ to some P_∞ such that

$$P_\infty-P(t)\sim\exp(-t^\beta) \ . \quad (6)$$

The polarization current,

$$J(t)=\frac{dP(t)}{dt} \ , \quad (7)$$

also exhibits stretched-exponential decay, viz.,

$$J(t)\sim\exp(-t^\beta) \quad (8)$$

with time. For $E>E_{\text{th}}$, the sustained current shows both broad- and narrow-band noise.[21]

Besides nonexponential relaxation in the pinned phase, the present system has a number of other properties that are analogous to those of spin glasses, like hysteresis and metastability[23,25,26] as well as strong sample dependence in quantities such as P_∞ and the threshold field E_{th} up to sample sizes of the order of 10^4.

To understand the dynamics of the system represented by Eq. (4), it is instructive to plot $f(x)$ for different values of the parameters appearing in Eq. (3). In Fig. 1, we display two curves corresponding to the symmetric case of $\phi_i=0$, $E=0$ and to $E>0$; $B=1$ and $\delta t=0.1$. [Note that *at any given moment,* one may replace E in Eq. (3) with an effective field $E_{\text{eff}}=E+B(\psi_{i+1}+\psi_{i-1})$, where one has absorbed the coupling.] The first curve exhibits two attractive fixed points at x_1 and x_2, while the second, in the presence of a sufficiently large field E, has only one (upper) attractive fixed point. The stable fixed point at x_1 and the unstable fixed point at the origin have disappeared via an inverse tangent bifurcation. In the first case, the dynamical variable x will relax exponentially to one of these two stable fixed points depending upon the initial value. If the phase point were to be found in the basin of attraction of x_1, an increase in E that makes the fixed point x_1 disappear would cause a transition to take place to the upper, albeit slightly displaced, stable fixed point. This transition will proceed extremely slowly, almost imperceptibly, at first, while x iterates through the narrow neck[27] (indicated by the open arrow in Fig. 1), and then much more rapidly as the maximum is neared. If the phase point were in the basin of attraction of x_2, this increase in E would result in an exponential relaxation to the slightly displaced stable fixed point.

An inspection of the motion of the ψ_i under the action of the set of coupled maps in Eq. (4), for a generic set of $\{\phi_i\}$ and random initial conditions shows this scenario to be typical for fields up to about 10% of E_{th}. In Fig. 2 different $\psi_i(t)$ in a chain of 500 points are plotted against t (here $B=1$, $\delta t=0.01$, $E=0.5$). One sees single abrupt transitions[25,26] between what appear to be two relatively stable values separated by a step typically of the order of unity. This prompts us, in the light of the preceding paragraph, to identify the two basins of attraction of $f_i(\psi_i)$ with two stable "states" of the variable ψ_i. The precise location of these basins of attraction is immateri-

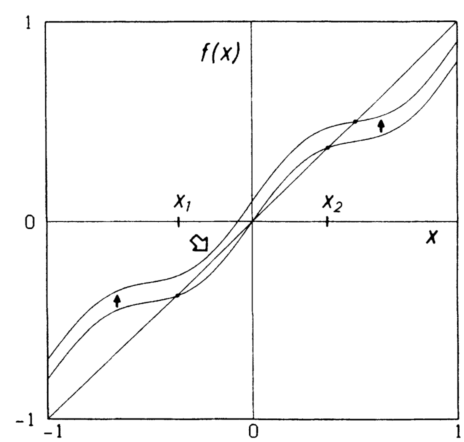

FIG. 1. Plot of the modified *sine* map appearing in Eq. (3), for $\phi=0$. The lower curve has $E=0$ and the upper curve $E>0$. See the text.

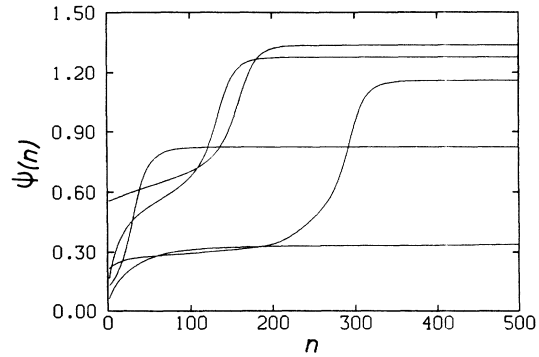

FIG. 2. Trajectories of a number of individual phases ψ_i out of a chain of $N=500$, against n, the number of time steps δt. $B=1$, $\delta t=0.01$, and $E=0.5$.

al. What interests us here is that each dynamical variable is in one or the other of its allowed states and that the relaxation proceeds via transitions that take place over time scales much shorter than the scale of observation times of such quantities as the polarization or the current.

The identification of the distinct basins of attraction of the continuous variable ψ_i with discrete states leads to a "coarse-grained" picture of phase space, where the infinite-dimensional phase space of a chain of N points is replaced by the 2^N vertices of an N-dimensional hypercube. In the presence of a positive applied field, the transitions are overwhelmingly from the "lower" to the "upper" states, so that the edges of the hypercube may be replaced with diodes pointing in the positive direction. It should also be clear that because of the random pinning, some vertices will be permanently removed (some ψ_i unable to make a transition to the next state), while for higher values of the applied field the concentration of permanently removed vertices will be correspondingly lower.

In the next section we will provide numerical evidence that, for time scales large compared to the transition times, we may view the relaxation of a pinned CDW in terms of a random walk of the phase point over a directed percolation network on the surface of this hypercube, although in microscopic detail the dynamics is fully deterministic.

III. NUMERICAL RESULTS

In this section we would like to present quantitative results on the relaxation behavior of the pinned CDW [Eq. (4)]. For a fixed value of the coupling strength B [chosen to have the same magnitude as the coefficient of the pinning term in Eq. (4)], we have explored the dependence of the scaling behavior on the applied field E. For relatively small fields we find satisfactory agreement with expectations based on the picture presented earlier, while anomalies show up in the neighborhood of the threshold, where the system starts to depart significantly from the simplified model for the phase space outlined in the preceding section.

We have performed extensive simulations of Eq. (4) with random initial conditions for ψ_i, to compute the polarization current,

$$J(t)=\left\langle \frac{1}{N}\sum_i^N \dot{\psi}_i(t)\right\rangle_{\text{config}} \tag{9}$$

with N typically 10^4 and the configuration average is taken over up to 100 realizations of the $\{\phi_i\}$. In Fig. 3 we have plotted our results for several values of the external field E, with $B=1$. These curves allow very high-quality fits to the logarithm of the exponential form in Eq. (6),

$$\ln J(t)=a-bt^{\beta}\ , \tag{10}$$

which requires the simultaneous determination of three parameters. An alternative is to compute the double logarithm, that is,

$$\ln[\ln J(t_1)-\ln J(t)]=\ln b+\beta \ln t\ , \tag{11}$$

where the amplitude $J(t_1)$, or equivalently, the offset t_1 is chosen so as to minimize the deviation from the asymptotic stretched-exponential form. Although error bars of the order of 1% may be achieved for the sets of parameters in Eq. (10) or Eq. (11), the value of the exponent β depends sensitively on the choice of the other two. In Fig. 4, β is plotted as a function of the external field E. The error bars indicate by how much β changes as t_1 is varied, while the quality of the fit is not affected appreciably.

In spite of the large error bars, certain features may be discerned from Fig. 4. For fields well below the threshold, β goes to a minimum value around $\beta=\frac{1}{3}$. Just below the threshold one observes a sharp maximum around $\tilde{E}_{\text{th}}$. Above E_{th} there is no decay in the average current. Right at E_{th} a very slow, powerlike decay seems to hold, but more work needs to be done to determine its functional form. We have computed the power spectrum of the current at E_{th} and find it be fitted by a $1/f$ law. Away from the threshold, the noise in the current becomes Gaussian. We were able to fit the power spectrum with $f^{-\phi}$, $1.8<\phi<2.0$.

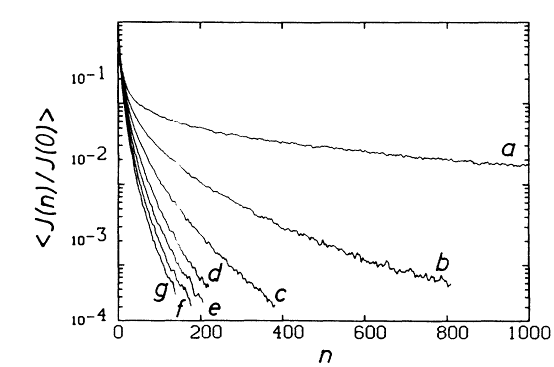

FIG. 3. The polarization current in the pinned phase, against n, the number of time steps, for different values of the external field, with curves a–g corresponding to $E=0.70$, 0.65, 0.60, 0.55, 0.50, 0.45, and 0.40, respectively. Each curve is averaged over up to 100 realizations of the spatial randomness: $N=10^4$, $B=1$, and $\delta t=0.1$. The initial values were chosen randomly.

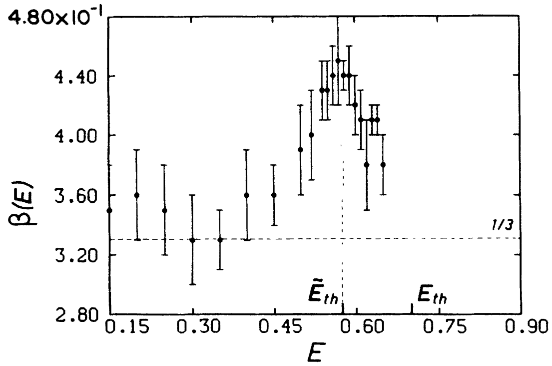

FIG. 4. Fits to the stretched-exponential index β, as a function of the external field. $E_{\rm th}$ indicates the threshold field and $\tilde{E}_{\rm th}$, the field around which the walk departs significantly from a hypercubical surface in phase space.

We have already remarked that $E_{\rm th}$ shows sample-to-sample dependence. For a given sample it may be unambiguously identified as the smallest value of the field for which the largest Lyapunov exponent of the N-dimensional iterative map in Eq. (4) is non-negative.[24]

For comparison, we have computed the time dependence of the autocorrelation function

$$q(t) \sim \langle r^2(\infty)\rangle - \langle r^2(t)\rangle \ , \qquad (12)$$

in terms of the Hamming distance,

$$\langle r^2(t)\rangle = \frac{1}{N}\sum_i^N [\psi_i(t)-\psi_i(0)]^2 \ . \qquad (13)$$

It is easy to show that on a unit N-dimensional hypercube,

$$\langle r^2(\infty)\rangle = \lim_{t\to\infty} \langle r^2(t)\rangle = \tfrac{1}{2} \ .$$

In Fig. 5 we have plotted $\langle r^2(\infty)\rangle$ against E. One sees that for low values of the field where many points on the chain remain permanently pinned, a comparatively small portion of the whole phase space is explored, while as one nears the threshold, the concentration of available vertices goes up and more and more of the phase space becomes accessible. As one raises the field even further, however, the limiting value of the Hamming distance exceeds $\frac{1}{2}$, indicating that the true phase space of the system is no longer confined to the surface of the hypercube, as discussed in Sec. II. Indeed, as can be seen by examining consecutive "snapshots" of the coordinates ψ_i, some sites perform several transitions in this region so that our model of a hypercubic phase space breaks down. The value of the field around which this happens, $\tilde{E}_{\rm th}$, corresponds to the peak in the exponent β observed in Fig. 4. Above the threshold, the ψ_i increase without bound, so that the phase space ceases to be closed altogether.

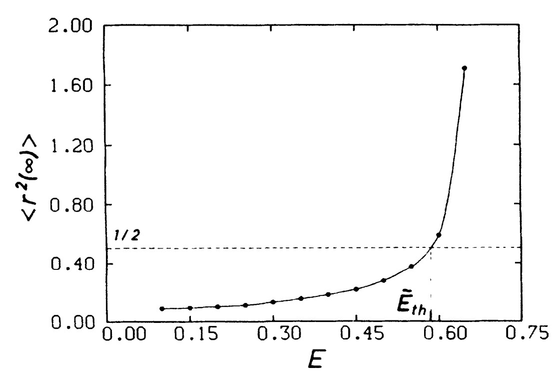

FIG. 5. The normalized Hamming distance [see Eq. (13)] in phase space, in the limit of a very large number of time steps, as a function of external field. Random initial conditions were chosen.

The autocorrelation function has the characteristic stretched-exponential behavior,

$$q(t) \sim \exp(-t^{\gamma}) \ . \qquad (14)$$

In the range $0.3 < E < 0.65$ we find $0.58 < \gamma < 0.73$, which suggests $\gamma \sim 2\beta$, as we would expect from Eqs. (5)–(8), (12) and (13). For values of the field not too close to $E_{\rm th}$, γ is consistent with the value of $\frac{2}{3}$.

Before we proceed to outline our understanding of these findings in terms of scaling laws for random walks, we would like to mention some results that serve to stress the utility of such a scaling approach.

It should be clear from the discussion in Sec. II (also see Fig. 2) that the steplike motion of the ψ_i leads to a total current $J(t)$, which is very nearly a superposition of δ-function-like peaks.[28] Denoting the τ_i the waiting time for ψ_i to make a transition, one may write in a continuum approximation (for $N \gg 1$),

$$J(t) \sim \int \delta(t-\tau)\mathcal{P}(\tau)d\tau \ ,$$

where $\mathcal{P}(t)$ is the distribution function for the τ. The difficulty, as in nearly all such cases, is in the analytical determination of $\mathcal{P}(\tau)$.

For the coupled-map lattice (CML) in Eqs. (3) and (4), we find numerically that for uniform pinning (all ϕ_i=const) $\mathcal{P}(\tau)$ as well as $J(t)$ decays *exponentially*. For random pinning, $\mathcal{P}(t)$ has the same stretched-exponential form as $J(t)$. For comparison, we have investigated a CML with constant pinning but with an added configurational noise term $\delta t \phi_i$, $-0.1 < \phi_i < 0.1$. This yields, once more, a stretched-exponential pattern.

By contrast, the dynamics of an *ensemble* of one-dimensional maps given by Eq. (3) can be deduced analytically[28] from Pomeau-Manneville-type[27] scaling. Expanding $f_i(x)$ around its nearest point of approach to the diagonal (see Fig. 1) one sees that the path length or passage time through this neck, τ_i must scale[27] like $\varepsilon_i^{-1/2}$ where ε_i is the neck width. One finds that ϵ_i is linear in ϕ_i; thus a random distribution of ϕ_i, $-1 < \phi_i < 1$ leads to a scale invariant distribution of relaxation times, $\mathcal{P}(\tau) \sim \tau^{-3}$, resulting in a power-law decay of the total current $J(t) \sim t^{-3}$. We have verified this analytical result with numerical simulations, finding a power of -2.7 ± 0.1. It is interesting to note that the power-law tail found by Hirsch *et al.*[27] for the path length distribution in the presence of δ-correlated time-dependent noise, results in our case from purely "configurational" noise independent of time.

IV. RANDOM WALKS ON CLOSED FRACTAL SURFACES

Making use of the scaling properties[18,29,30] of the distribution function for the position of a random walker, Flesselles and Botet[17] have provided a detailed argument that on an N-dimensional ($N \gg 1$) hypercube at the percolation threshold, one has, for asymptotically large times,

$$\langle r^2(\infty)\rangle - \langle r^2(t)\rangle \sim \exp(-t^{2/d_w}) , \tag{15}$$

where $\langle r^2(t)\rangle$ is the mean-square displacement and d_w the random-walk exponent on a percolation cluster.[18] Together with Eqs. (12) and (13), where one should take Ising spin variables $S_i(t)$ for the $\psi_i(t)$, one obtains

$$q(t) \sim \exp(-t^{1/3}) ,$$

where the mean-field theory (MFT) value,[29] $d_w = 6$ has been substituted.

Here we would like to give a heuristic argument along the lines of Campbell's,[15,16] which has the advantage that it does not depend on the precise form of the scaling function[31] for the probability distribution function $\mathcal{P}(r,t)$ for the random walker. Here, r is the "arc length" or displacement measured within the $(N-1)$-dimensional hypersurface. We assume that for time scales large compared to the stepping times but much shorter than those on which the relaxation patterns are observed walks on the high-dimensional phase space in question are indistin-

gime, $\tilde{\alpha}/d_w = \frac{1}{2}$, in clear contradiction to $\tilde{\alpha} = 2$.

One way to sidestep the issue of the precise form of the operator in Eq. (18) is to make the ansatz such that, at least for large times, we may perform a separation of variables. Then, with $\lim_{t\to\infty}\mathcal{P}(\mathbf{r},t) \to \rho(\mathbf{r})$, where $\rho(\mathbf{r})$ is the density function of the fractal embedded in the closed, unit $(N-1)$-dimensional hypersurface,

$$\mathcal{P}(\mathbf{r},t) = \rho(\mathbf{r})[1 + T(t)\Theta(\mathbf{r})] , \tag{19}$$

where $T(t)$ is a decaying function of time and $\Theta(\mathbf{r})$ must be periodic. Now $T(t)$ satisfies $dT(t)/dt = \text{const}D(t)T(t)$ with the solution $T(t) \propto \exp(-t^{2/d_w})$. Substitution in (19) immediately resutls in Eq. (15). An analogous argument could be extended to the case where there is a bias; then $\rho(\mathbf{r})$ has to be replaced with an appropriately modified asymptotic distribution, which is again independent of time.

As pointed out in Sec. II, we have to take into account the fact that, in our model, in the presence of an external field, the "edges" connecting the vertices of the coarse-grained phase space act like diodes, allowing passage only in the positive direction. The phase point now executes a random walk on this directed percolation network.[22,34,35] The analog of the Einstein relation for conductivity now yields

$$d_w^{\parallel} = 2 + (\mu^{\parallel} - \beta_p)/\nu_p^{\parallel} , \tag{20}$$

where $d_w^{\parallel}$ is defined as the walk exponent associated with the displacement in the "forward" direction, via $\langle r_{\parallel}(t)\rangle \sim t^{1/d_w^{\parallel}}$, $\mu^{\parallel}$ is the conductivity exponent for a random-resistor–diode network,[34–36] and β_p and $\nu_p^{\parallel}$ are the order parameter and correlation length exponents for directed percolation.[22,35] Substituting the MFT such questions as the spatial propagation of noise or spatiotemporal pattern formation. As in previous experiences with nonlinear maps, this endeavor is bound to come up with unexpected new phenomena.

Deterministic or "chaos-induced" diffusion has been discussed by Geisel and co-workers[38–40] and Grossmann and co-workers[41–43] in the context of one-dimensional periodic functions [c.f. Eq. (3)] that affect a decomposition of phase space into cells. Although the trajectory of the phase point is, in detail, completely deterministic, in terms of the scaling properties over time scales large compared to, say, the characteristic time for the accomplishment of an individual "step," it is indistinguishable from a random walk on a discrete grid that has been laid over the "true" phase space. In all the cases they have considered, with the exception of Ref. 43, these authors have found *normal* scaling behavior, with $d_w=2$; the diffusion constant depends upon the parameters of the map (it should be chaotic) and the amplitude of the external noise term, if any.

In our case, the onset of diffusion is made possible by the inclusion of a constant bias, the external field E, and the coupling of a large number ($N \gg 1$) of degrees of freedom. Note that below threshold there is no true chaos, the system eventually reaches equilibrium at a fixed point of the N-dimensional map. The inclusion of configurational randomness in the form of the random pinning potentials, $\{\phi_i\}$, and the external field E, for $E<E_{\rm th}$, push the deterministic diffusion over into the universality class of random walks on dilute resistor-diode networks. The high dimensionality of the phase space considered leads to the MFT value for the *anomalous* walk exponent, $d_w^{\parallel}=3$, for the forward displacement. Since the walk is confined to a closed portion of phase space,[15,16]

$$\langle r^2(\infty)\rangle - \langle r^2(t)\rangle \sim \exp[-r_{\parallel}^2(\text{short times})] \ . \tag{22}$$

For $E>E_{\rm th}$, all the ψ_i are "unpinned," so that the "walk" is no longer over a fractal nor is it confined; however, it is still directed. The polarization [Eq. (5)] in principle increases without bound, according to

$$P(t) \sim r_{\parallel}(t) \sim t^{1/d_w^{\parallel}} \tag{23}$$

where now[22,37] $d_w^{\parallel}=1$. This immediately leads to an average current $\bar{J}(t)$ which is constant over time.

It should be noted that uniform pinning (setting all ϕ_i equal to a constant) leads, below threshold, to an *exponential* decay ($\beta=1$) of the polarization current with time. This is indeed what we would expect on the basis of biased diffusion on a nonfractal, closed space, with $d_w^{\parallel}=1$.

In conclusion, we have shown that the dynamics of CDW may be understood in terms of *anomalous* deterministic diffusion arising from an N-dimensional coupled-map lattice. A dynamic phase transition takes place at the threshold, where there is a crossover between two scaling regimes belonging to different universality classes. The details of this crossover region deserve further study.

ACKNOWLEDGMENTS

It is a pleasure to acknowledge many interesting conversations we have had with A. P. Siebesma. One of us (A.E.) would like to thank the Stichting voor Fundamenteel Onderzoek der Materie, which is financially supported by the Nederlandse Organisatie voor Zuiver Wetenschappelijk Onderzoek for support.

---

*Present address: International Center for Theoretical Physics, Post Office Box 586, 34100 Trieste, Italy.

[1]For a review, see A. Blumen, J. Klafter, and G. Zumofen, *Optical Spectroscopy of Glasses,* edited by I. Zschokke-Granacher, (Riedel, Dordrecht, 1986), p. 199.
[2]G. Kriza and G. Mihaly, Phys. Rev. Lett. **56**, 2529 (1986); also see P. B. Littlewood and R. Rammal, Phys. Rev. B **38**, 2675 (1988), and references therein.
[3]R. G. Palmer, D. L. Stein, A. Abrahams, and P. W. Anderson, Phys. Rev. Lett. **53**, 958 (1984).
[4]E. W. Montroll and M. F. Schlesinger, *Non-Equilibrium Phenomena II. From Stochastics to Hydrodynamics,* edited by J. L. Lebowitz and E. W. Montroll (North-Holland, Amsterdam, 1984).
[5]A. Blumen, J. Klafter, and G. Zumofen, J. Phys. A **19**, L77 (1986).
[6]J. Klafter and M. F. Schlesinger, Proc. Natl. Acad. Sci. USA **83**, 848 (1986), and references therein.
[7]M. Mezard, G. Parisi, N. Sourlas, G. Toulouse, and M. Virasoro, Phys. Rev. Lett. **52**, 1156 (1984); J. Phys. (Paris) **45**, 843 (1984).
[8]C. De Dominicis, H. Orland, and F. Lainée, J. Phys. (Paris) Lett. **46**, L463 (1985); G. Paladin, M. Mezard, and C. De Dominicis, *ibid.* **46**, L985 (1985).
[9]B. A. Huberman and M. Kerszberg, J. Phys. A **18**, L331 (1985).
[10]S. Teitel and E. Domany, Phys. Rev. Lett. **55**, 2176 (1985).
[11]M. Schreckenberg, Z. Phys. B **60**, 483 (1985).
[12]A. Erzan, S. Grossmann, and A. Hernandez-Machado, J. Phys. A **20**, 3913 (1987), and references therein; also, G. J. M. Koper and H. J. Hilhorst, Physica A **160**, 1 (1989).
[13]I. A. Campbell, J. M. Flesselles, R. Jullien, and R. Botet, Phys. Rev. B **37**, 3825 (1988).
[14]A. T. Ogielski, Phys. Rev. B **32**, 7384 (1985).
[15]I. A. Campbell, J. Phys. (Paris) Lett. **46**, L1159 (1985).
[16]I. A. Campbell, Phys. Rev. B **33**, 3587 (1986); I. A. Campbell, J. M. Flesselles, R. Jullien, and R. Botet, J. Phys. C **20**, L47 (1987).
[17]J. M. Flesselles and R. Botet, J. Phys. A **22**, 903 (1989).
[18]Y. Gefen, A. Aharony, and S. Alexander, Phys. Rev. Lett. **50**, 77 (1983).
[19]R. Rammal and G. Toulouse, J. Phys. (Paris) Lett. **44**, L13 (1983).
[20]For a review, see S. Havlin and D. Ben Abram, Adv. Phys. **36**, 695 (1987).
[21]L. Pietronero and S. Strässler, Phys. Rev. B **28**, 5683 (1983).
[22]W. Kinzel, Ann. Israel Phys. Soc. **5**, 425 (1983).

[23]K. Kaneko, Physica D **18**, 475 (1986); Prog. Theor. Phys. **74**, 1033 (1985); *Collapse of Tori and Genesis of Chaos in Dissipative Systems* (World Scientific, Singapore, 1985); R. J. Dreissler, Phys. Lett. **100A**, 451 (1984); I. P. Waller and R. Kapral, Phys. Rev. A **30**, 2047 (1984).
[24]E. Veermans, A. Erzan, R. Heijungs, and L. Pietronero, Physica A (to be published); A. Erzan and E. Veermans (unpublshed).
[25]P. B. Littlewood, Phys. Rev. B **33**, 6694 (1986).
[26]D. S. Fisher, Phys. Rev. B **31**, 136 (1985).
[27]Y. Pomeau, and P. Manneville, Commun. Math. Phys. **74**, 327 (1980); J.-P. Eckmann and P. Wittwer, J. Phys. A **14**, 3153 (1981); J. E. Hirsch, B. A. Huberman, and D. J. Scalapino, Phys. Rev. A **25**, 519 (1982).
[28]R. Heijungs, diploma thesis, University of Groningen, 1987.
[29]S. Alexander and R. Orbach, J. Phys. (Paris) Lett. **43**, 2625 (1982).
[30]B. O'Shaughnessy and I. Procaccia, Phys. Rev. A **32**, 3073 (1985).
[31]A. Brooks Harris and A. Aharony, Europhys. Lett. **4**, 1355 (1987); A. Aharony and A. Brooks Harris (unpublished).
[32]B. Mandelbrot, *Fractal Geometry of Nature* (Freeman, New York, 1982).
[33]E. Veermans, diploma thesis, University of Groningen, 1989.
[34]S. Redner, Ann. Israel Phys. Soc. **5**, 447 (1983).
[35]S. Redner, Phys. Rev. B **25**, 3242 (1982).
[36]S. Redner, Phys. Rev. B **25**, 5646 (1982).
[37]S. Redner, in *The Mathematics and Physics of Disordered Media*, edited by B. D. Hughes and B. W. Ninham (Springer-Verlag, Berlin, 1983).
[38]T. Geisel, and J. Nierwertberg, Phys. Rev. Lett. **48**, 7 (1982).
[39]T. Geisel and J. Nierwertberg, Phys. Rev. A **29**, 2305 (1984).
[40]T. Geisel and S. Thomae, Phys. Rev. Lett. **52**, 1936 (1984).
[41]S. Grossmann and H. Fujisaka, Phys. Rev. A **26**, 1179 (1982).
[42]H. Fujisaka and S. Grossmann, Z. Phys. B **48**, 241 (1982).
[43]H. Fujisaka, S. Grossmann, and S. Thomae, Z. Naturforsch. **40A**, 867 (1985). The results of the latter, showing that randomness in the cell sizes does not affect the scaling behavior unless the variance diverges, gives us reason to believe that the variable size of the steps taken by the ψ_i is of no consequence.