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# Stripe fractionalization I: the generation of Ising local symmetry

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## Abstract

This is part one in a series of two papers dedicated to the notion that the destruction of the topological order associated with stripe phases is about the simplest theory controlled by local symmetry: Ising gauge theory. This first part is intended to be a tutorial- we will exploit the simple physics of the stripes to vividly display the mathematical beauty of the gauge theory. Stripes, as they occur in the cuprates, are clearly ‘topological’ in the sense that the lines of charges are at the same time domain walls in the antiferromagnet. Imagine that the stripes quantum melt so that all what seems to be around is a singlet superconductor. What if this domain wall-ness is still around in a delocalized form? This turns out to be exactly the kind of ‘matter’ which is described by the Ising gauge theory. The highlight of the theory is the confinement phenomenon, meaning that when the domain wall-ness gives up it will do so in a meat-and-potato phase transition. We suggest that this transition might be the one responsible for the quantum criticality in the cuprates. In part two [1] we will become more practical, arguing that another phase is possible according to the theory. It might be that this quantum spin-nematic has already been observed in strongly underdoped  $La_{2-x}Sr_xCuO_4$ .

## I. DYNAMICAL GENERATION OF LOCAL SYMMETRY.

It has been a dream for a long time that the profoundness of non-perturbative gauge theory could come alive in the earthly forms of matter which are of interest to condensed matter physicists. This is far from self-evident. Gauge theory is controlled by local symmetry. The high energy physicist will argue that it better be fundamental because local symmetry is infinitely vulnerable towards explicit symmetry breaking influences making it global – we refer to part II [1] for an example. However, gauge theory carries a myriad of meanings and one usually exploits the principle that local symmetry implies a local conservation law which in turn corresponds with a local constraint. This is at the heart of the slave theories of electron fractionalization: one can pretend that the electron (or Cooper pair) is actually a composite particle, paying the price that the pieces of the electrons are minimally coupled to strongly interacting gauge fields. Although theoretically consistent, the drawback is that these gauge fields are highly mathematical entities lacking a material interpretation. How

to measure the  $SU(2)$  gauge field of Lee and coworkers [2]? The  $Z_2$  theory of Cooper pair fractionalization by Senthil and Fisher [3] is doing better in this regard but it leaves one wondering why the vortices in the dual condensate should form pairs – that is their physics.

We [4], and others [5,6], were astonished when it became clear that a feet-on-the-ground physics problem turned out to be in correspondence with the most elementary gauge theory: the Ising- or  $Z_2$  gauge theory, invented by Wegner in the early 1970's [7], which has played an important role in the early history of non-perturbative gauge theories [8,9]. The physics problem is inspired by the strong empirical case, as presented by e.g. Tranquada in this volume [10], that superconductors of the  $La_{2-x}Sr_xCuO_4$  variant have to do at the same time with a quantum disordered stripe phase. What can be said in general about such a stripe quantum liquid? ‘General’ means here Landau’s method: we are only allowed to use symmetry. Stripe order means that symmetries are broken, and quantum liquid means that these symmetries are restored. The key is that stripes break a variety of symmetries and some of these symmetries might remain broken even in the liquid. States carrying this ‘partial order’ might be easily overlooked by the unprepared experimentalist. Typical examples are the quantum smectic- and nematic states introduced by Kivelson, Fradkin, and Emery [11].

The states which will be discussed here are even more radical. Empirically, the charge stripes are at the same time domain walls in the spin system and in section II we will explain that this domain wall-ness can be viewed as a form of long range order, albeit of a *geometrical* nature. It follows that, in principle, this geometrical order can persist when charge and spin become quantum disordered, such that the system is, in the first instance, a singlet superconductor. Such a superconductor carries a truly *topological* order, and the mathematical description of this order is the essence of the gauge theory (section III). Eventually this order also gets destroyed in a normal phase transition: the meaning of confinement in the Ising gauge theory. However, it is impossible to directly observe the topological order and its destruction by existing experimental machines (section IV).

## II. OUR SPACE AND THE SPACE OF THE SPINS.

What is stripe order? It consists, at the least, of three forms of spontaneous symmetry breaking: (i) the electrical charge breaks space translational and -rotational symmetry. It is just a Wigner crystal with an interesting (orthorombic) crystal structure. Assuming that stripes are made from bosons (electron pairs), one can argue on general grounds that upon the restoration of translational symmetry superconducting condensation will follow automatically, although this state might still break rotational invariance (the quantum nematic [10]). In the remainder we will take this superconductivity for granted. (ii) Stripes might also be co-linear antiferromagnets, breaking spin-rotational symmetry. In a weakly coupled BCS superconductor the spin system is characterized by a quantum disordered singlet ground state, but it seems now generally appreciated that this is a specialty of weak coupling. Being different symmetries, nothing argues against a coexistence state where the same electrons condense at the same time in a superconductor and an anti-ferromagnet.

Subsequently, this anti-ferromagnet can disorder in a spin-only quantum phase transition. This spin disordered state is a spin singlet and such a vacuum is indistinguishable from a BCS superconductor, except than that it can be characterized by very strong antiferromagnetic

spin correlations. This generalized view on superconductivity seems to be much closer to the physics of the cuprate superconductors than the conventional weak coupling view, see Sachdev *et al.* [5]. (iii) However, stripe order is more than just spin and charge order. It is also order associated with the fact that the charge stripes are domain walls,  $\pi$ -phase boundaries in the stripe anti-ferromagnet. The nature of this unconventional order is the subject of this paper.

Empirically, this ‘domain wall-ness’ is clearly a form of order. Long range order means that one can predict what happens at  $+\infty$  when one knows what is going on at  $-\infty$ . Taking two points infinitely far apart in a stripe phase, one can count if there are an even or odd number of charge stripes in between, and one can predict if the staggered order parameters in the two domains are parallel- or antiparallel relative to each other. The key observation is that this order can persist in principle when spin- and charge are disordered, so that the system is in any other regard a normal superconductor.

One has to ask first the simple question: we observe stripes to be domain walls, but domain walls *in what?* Domain walls have to do with scalar fields (Ising systems) and surely neither the spin- nor the charge systems have anything to do with Ising fields. Domain walls are topological entities and the topological excitations associated with charge and spin are different (dislocations/disclinations and skyrmions, respectively). The proper answer is instead, *stripes are domain walls in the geometrical quantity sublattice parity*. It is a requirement for the existence of colinear spin-order that the spin system lives in a bipartite space, meaning that the lattice can be subdivided into an  $A$  and  $B$  sublattice, such that every site on the  $A$  sublattice is surrounded by  $B$  sublattice sites and vice versa. This can be done in two ways,  $\dots - A - B - A - B - \dots$  or  $\dots - B - A - B - A - \dots$ . This  $Z_2$  valued quantity we call the sublattice parity  $p = \pm 1$ . Stripe ‘domain wall-ness’ order means that every time one passes a charge stripe  $p$  changes sign: the sublattice parity is ordered.

FIGURES

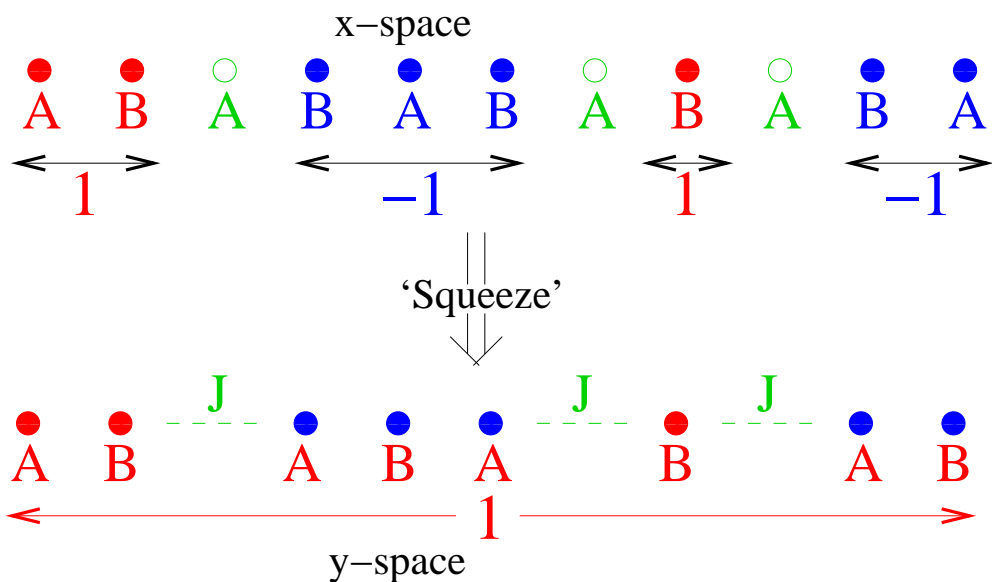


FIG. 1. The Ogata-Shiba mechanism for spin-charge separation in the Luttinger liquid [13,14]. Distribute first the electrons over the Hubbard chain and the amplitude of such a charge configuration will only depend on the positions of the electrons (upper panel). Take such a charge configuration, and take out the sites where the holes are, and substitute these with exchange bonds ( $J$ ) between the spins of the electrons neighboring the holes (lower panel). The spin system lives in this squeezed space. Relative to the real sublattice division of squeezed space, sublattice parity in full space flips every time a hole is passed.

How do we know? A proof of principle can be delivered in the one dimensional context. As we demonstrated recently [13], the Luttinger liquid derived from the 1D Hubbard model turns out to carry also a form of topological sublattice parity order. This rests on the Ogata-Shiba mechanism [14] for charge-spin separation as deduced from the Bethe-Ansatz solution in the large  $U$  limit which we demonstrate to be universal in the scaling limit for all positive  $U$ 's. The remarkable observation by Ogata-Shiba is that the structure of the Bethe-Ansatz wave function is explicitly of a geometric nature, as illustrated in Fig. 1. In a first step, distribute the electrons over the chain and such a charge configuration will acquire an amplitude equal to that of a system of spinless fermions. This configuration of electron charges in turn defines a *new space* in which the spin system lives, obtained by removing the sites where the holes are, substituting these missing sites by antiferromagnetic exchange bonds between the spins neighboring the hole: the ‘squeezed space’. Hence, spin-charge separation means that the spin dynamics is just governed by a spin-only Heisenberg problem defined in squeezed space. However, this spin dynamics cannot be observed directly by doing measurements in the full chain. In the unsqueezing operation, every time one inserts a hole one has also to add a site and this in turn means that the ‘true’ spin dynamics of squeezed space gets modulated in addition with a flip in the sublattice parity attached to the hole: see Fig. 1.

What is the relationship with the 2+1 dimensional case? Just take the Ogata-Shiba mechanism as principle, to see how it should be implemented in higher dimensions. A requirement is that the lattice in the higher dimensional space is bipartite, which is fortu-

nately the case for the square lattice of the cuprates. Insist that the spin system lives on this unfrustrated square lattice. Reinsert the charges and unsqueeze this lattice: this can only be accomplished when the charges form lines which are uninterrupted domain walls in the sublattice parity. In other words, stripes are at least symmetry-wise quite like the 1+1D Luttinger liquids. This should be appreciated as a mere phenomenological argument: given that long-range ‘domain wall-ness’ order is observed, it has to be that squeezed space exists and that the domain walls are carried by sublattice parity. The microscopic origin of the domain wall-ness is a different matter. A fair understanding exists where it is coming from and we refer to the stripe literature [15]: it is about avoiding the frustrations caused by the quantum motions of a single hole in the antiferromagnet by organizing the holes in stripes. At the same time, it does not have to happen in 2+1D. Hence, the domain wall-ness can be destroyed in favor of a conventional, Fermi-liquid/BCS state and this is the subject of the remainder of this paper.

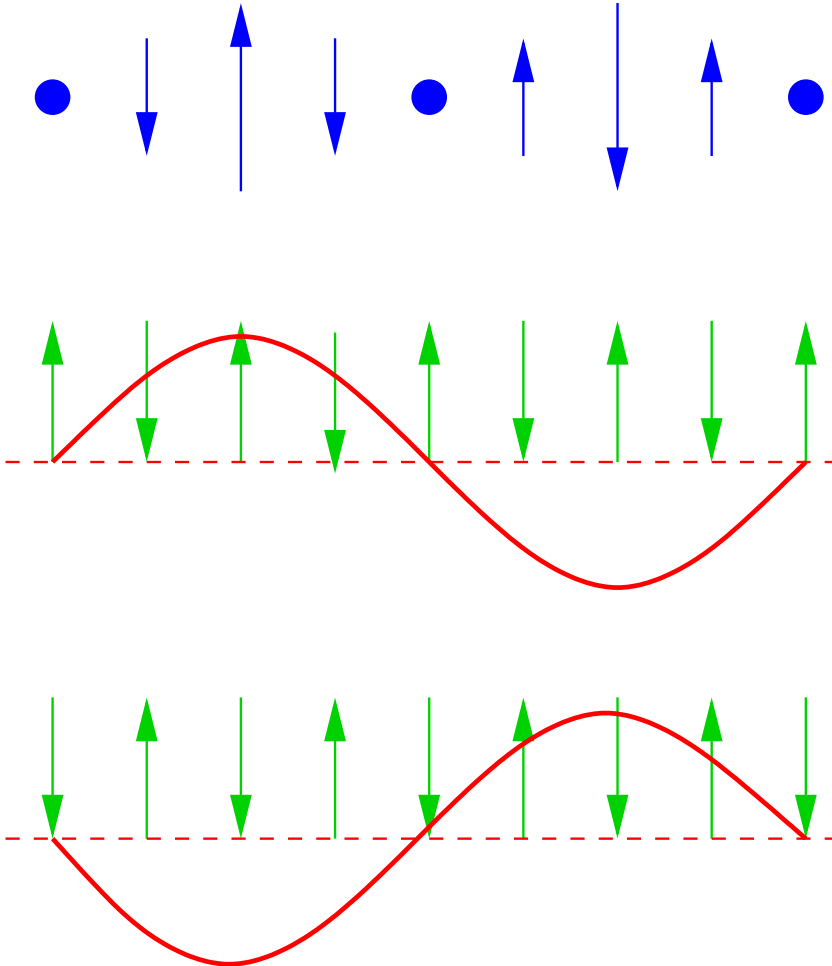


FIG. 2. Insofar as basic symmetries are concerned, spin density waves are just the same as stripes. Implicit in the description of the spin density wave order parameter is a  $Z_2$  gauge redundancy [5,6]. The spin density wave order parameter (upper panel) is a product of staggered spin and amplitude, and the shift in sublattice parity in the former can always be compensated by shifting the phase of the amplitude envelope by  $\pi$ .

On a side, we notice that with regard to symmetry stripes cannot be distinguished from incommensurate co-linear spin density waves. Let us consider the sublattice parity order in this context, using an argument due to Sachdev [5,6]. The order parameter is  $O_{SDW}(\vec{r}) = \Phi(\vec{R})\vec{M}(\vec{r})$ ;  $\vec{M}(\vec{r}) = \langle (-1)^{\vec{r}} \vec{S} \rangle$  is the expectation value for staggered spin, while  $\Phi(\vec{r}) = \cos(\vec{q} \cdot \vec{r}) \Phi$  is the *amplitude* of the spin density wave. Implicit to this description is a  $Z_2$  gauge redundancy:  $\Phi \rightarrow -\Phi, \vec{M} \rightarrow -\vec{M}$  leaves the physical  $O_{SDW}$  unchanged, see fig.3. For a static spin density wave this does not carry any consequence. However, spin amplitude and charge are fundamentally indistinguishable and the SDW should be considered as a special, weakly coupled limit of the static stripes. When charge is disordered, spin amplitude is disordered as well. Assuming that staggered spin  $\vec{M}$  stays ordered the  $Z_2$  gauge becomes alive:  $\vec{M} \leftrightarrow -\vec{M}$  which is the same as the sublattice parity gauge transformations  $\dots - A - B - \dots \leftrightarrow \dots - B - A - \dots$ . The state characterized by  $\langle \vec{M} \rangle \equiv -\langle \vec{M} \rangle$  is the spin nematic which is discussed in paper II.

### III. GEOMETRY, TOPOLOGY AND ISING GAUGE THEORY

Gauge theory has interesting connections with geometry and topology and a vivid, simple example follows from the stripe interpretation of  $Z_2$  gauge theory: it is at least a tutorial tool to explain to students some of the basics.

Obviously, static stripes have nothing to do with Ising local symmetry. This comes alive only when the stripes are truly delocalized. The key is that the topological excitations associated with the destruction of the charge order and the sublattice parity order are different, with the consequence that charge disordered (superconducting) states can exist carrying a truly topological (non-local) domain wall order. Ising gauge theory is the theory describing this topological order, and its destruction.

This is easy to see: the restoration of translation symmetry is uniquely associated with the spontaneous proliferation of dislocations, Fig. 3. When the crystal is bosonic (e.g., build from Cooper pairs) a theorem due to Feynman insists that nothing can prevent the system to become a superconductor when translational symmetry is restored. The elementary dislocation carrying the unit of Burger vector is the ‘stripe coming to an end’, Fig. 3b. However, this is at the same time the topological excitation of the sublattice parity order (‘stripe dislocation’). Besides destroying the charge order, it also causes a seam of sublattice parity mismatch extending to infinity. This injects frustration in the spin system and as long as the correlation length in the spin system is large compared to the lattice constant, the system might want to avoid this penalty by binding two such stripe dislocations into a single ‘charge’ dislocation carrying twice the Burger vector, Fig. 3a. If this happens is a matter of microscopic numbers. However, when it happens the  $Z_2$  local symmetry becomes alive.

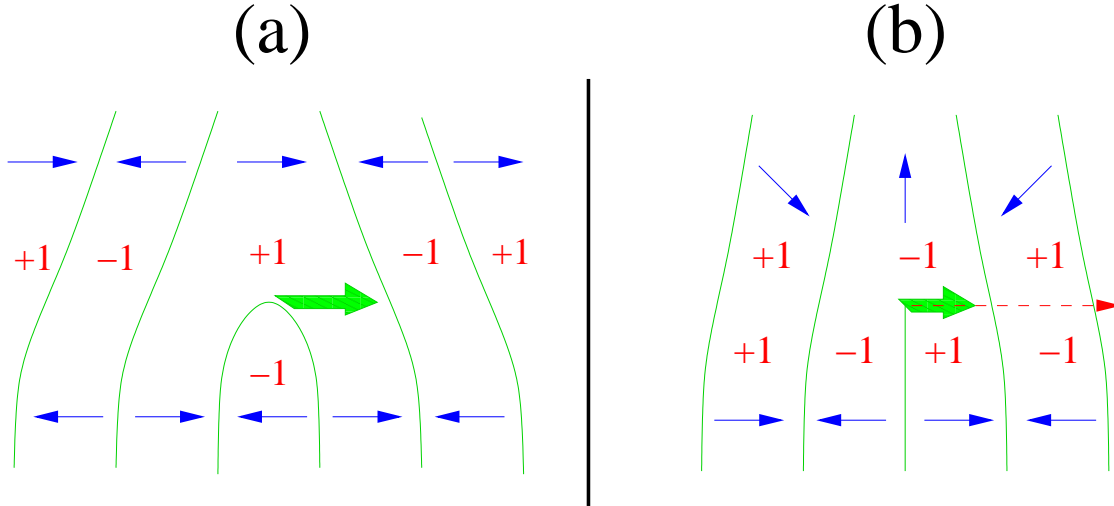


FIG. 3. Two types of topological defects can be distinguished in the stripe phase: the charge- (a) and stripe (b) dislocations. The stripe dislocation carries the minimal topological charge (Burger vector, thick arrow) relative to the charge order. However, it is at the same time destroying the sublattice parity order, causing frustration in the spin system (thin arrows). Although the charge dislocation carries a doubled Burger's vector, it does not affect the sublattice parity order. When charge dislocations quantum-proliferate, sublattice parity turns into the Ising gauge field, and the unbinding of charge dislocations into stripe dislocations corresponds with the confinement phenomenon.

Let us first introduce the Ising gauge theory in its minimal form [8]. Imagine a square lattice characterized by Ising degrees of freedom  $(1, 0), (0, 1)$  defined on every bond  $\langle ij \rangle$  on the lattice, see fig. 4a. The Hamiltonian defining the  $Z_2$  theory is,

$$H_{gauge} = -K \sum_{\square} \sigma^3 \sigma^3 \sigma^3 \sigma^3 - \sum_{\langle ij \rangle} \sigma_{ij}^1. \quad (1)$$

Here,  $\sigma^{1,3}$  are just the Pauli-matrices residing on bond variables, while  $\sum_{\square} \sigma^3 \sigma^3 \sigma^3 \sigma^3$  is a short hand for: 'pick the midpoint of a plaquette (dual lattice,  $\square$ ) and multiply the eigenvalue of the  $\sigma^3$  operators of all 4 bonds surrounding this plaquette and sum the outcomes over the dual lattice.' The plaquette ('flux') operator  $\sigma^3 \sigma^3 \sigma^3 \sigma^3$  is also Ising valued: it has eigenvalue +1 and -1 when the number of +1 bonds is even or odd, respectively (Fig. 4). The *local* symmetry generator  $P_i = \prod_j \sigma_{ij}^1$  flips all bonds leaving an arbitrary site  $i$  and since this operation does not change the (un)evenness probed by the flux operator while it surely commutes with the 'kinetic' energy  $\sim \sum \sigma^1$  it commutes with the Hamiltonian.  $P_i$  is the generator of the Ising gauge transformations.

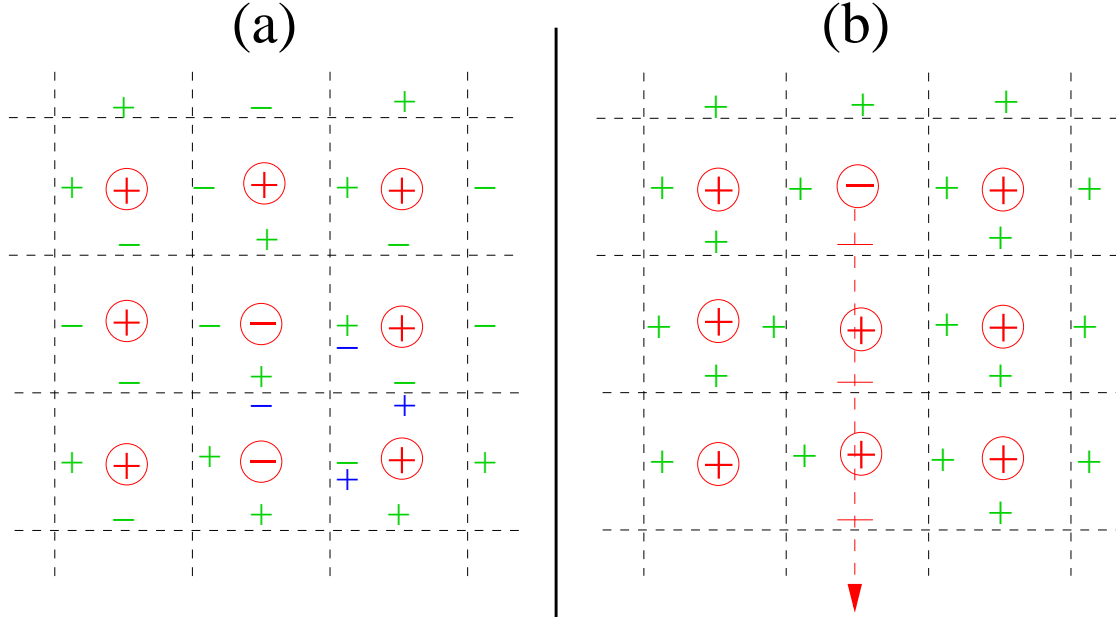


FIG. 4. The bond ( $\pm$ )- and flux ( $\pm$  in the circles) variables of the Ising gauge theory. A gauge transformation is indicated centered on the site shared by the lower-right four plaquettes in (a). It is left as an exercise to demonstrate that by repeated gauge transformations all bonds can be made positive (unitary gauge) except for the bond shared by the two negative fluxes (bound vison pair). The incipient order of the deconfining state is destroyed when visons, isolated minus fluxes, proliferate and these carry half-infinite lines of minus bonds as can be seen in unitary gauge (b).

The pure gauge theory Eq. (1) has two phases in 2+1D as function of  $K$ , separated by a continuous phase transition at a critical coupling  $K_c$ , called the deconfining (large  $K$ ) and confining (small  $K$ ) phases. How to understand these in terms of the stripe physics? Imagine that every site on the gauge theory lattice corresponds with a patch of size  $\xi_c$  (charge correlation length) in the real cuprate lattice. Interpret now the gauge bonds as being responsible for the relative orientations of the sublattice parity on neighboring patches: pick for instance  $A - B - 0 - A - B$  on patch  $i$  ('0' is a stripe) to find  $A - B - 0 - A - B$  on a neighboring patch  $j$  when the bond variable has value  $+1$  and  $B - A - 0 - B - A$  when the bond is  $-1$ . Consider  $K \rightarrow \infty$  so that all fluxes are  $+1$  and consider as representative gauge fix the unitary gauge where all bonds are positive. This implies that all stripe patches have the same sense of sublattice parity, like the ordered stripes. However, act once with  $P_i$  at site  $i$ : this will flip the sublattice parity at patch  $i$  from  $\dots - A - B - \dots$  into  $\dots - B - A - \dots$  and repeating these gauge transformations everywhere else one obtains a state characterized by the fact that sublattice parity is either  $\dots - A - B - \dots$  or  $\dots - B - A - \dots$  although it is impossible to say which choice is actually made. The reader should recognize that this 'deconfining' state is identical to a state characterized by intact but delocalized domain walls: the charge fluctuations have turned into the  $Z_2$  gauge transformations  $P_i$ !

How to understand the confining state? The mathematical beauty of the gauge theory lies in Wegner's discovery [7] of an exact duality transformation. In the Hamiltonian language [8] this is easy to understand. Define the following operators living on the sites  $i^*$  of the dual (midpoints of plaquettes) lattice:  $\mu_{i^*}^1 = \sigma^3 \sigma^3 \sigma^3 \sigma^3$  and  $\mu_{i^*}^3 = \prod_{n=-\infty}^{i^*} \sigma_n^1$ . The operator  $\mu^1$  just measures the gauge flux but it is in  $\mu^3$  that we encounter the novelty: start at the middle of

plaquette  $i^*$  and draw a line to infinity of arbitrary shape except that it crosses two bonds of every plaquette different from  $i^*$ , and flip the spin on every bond which is crossed by the line (Fig. 4b). It is easy to check that  $\mu^1$  and  $\mu^3$  commute like Pauli matrices and using simple operator identities the Hamiltonian becomes in terms of these operators,

$$H_{dual} = -K \sum_{i^*} \mu_{i^*}^1 - \sum_{\langle i^* j^* \rangle} \mu_{i^*}^3 \mu_{j^*}^3. \quad (2)$$

This is just the simple global Ising model in 2+1D in the presence of a transversal field! For  $K \rightarrow 0$  the original gauge theory would be strongly disordered (confinement) and we discover that this disordered state actually corresponds with simple Ising order in terms of the topological excitations, the ‘visons’, created by  $\mu^3$  (Fig. 4b)! Local-global dualities of this kind are at the heart of Abelian gauge theories in 2+1D and it is believed that similar concepts are behind the confinement in 3+1D quantum chromodynamics. In the present context, it is particularly significant that the confinement transition is just a meat-and-potato 3D Ising universality class quantum phase transition.

What is the meaning of confinement in the stripe interpretation? It is simple: take the unitary gauge and let the gauge seam of minuses consequently created by  $\mu^3$  be along a straight line starting in the middle of the plain to disappear to  $y \rightarrow -\infty$  (Fig. 4b). Remembering that these bond values are about the relative orientation of sublattice parity this is just identical to the stripe dislocation, Fig. 3b! Hence, the confining state corresponds with the stripe-fractionalized state and its geometrical meaning is that space is neither  $\dots - A - B - \dots$  nor  $\dots - B - A - \dots$ . Sublattice parity is destroyed by the visons/stripe dislocations.

#### IV. TOPOLOGICAL ORDER AND CUPRATE CRITICALITY.

Let us now turn to the cuprate superconductors. It is widely believed that the regularities observed in the normal state of the optimally doped superconductors reveals the presence of a zero temperature quantum phase transition. It is most natural to expect that this has to do with a spontaneous symmetry breaking in the quantum vacuum, implying some form of ‘order’ is likely found on the underdoped side. However, this order has not been seen experimentally and is therefore called ‘hidden order’.

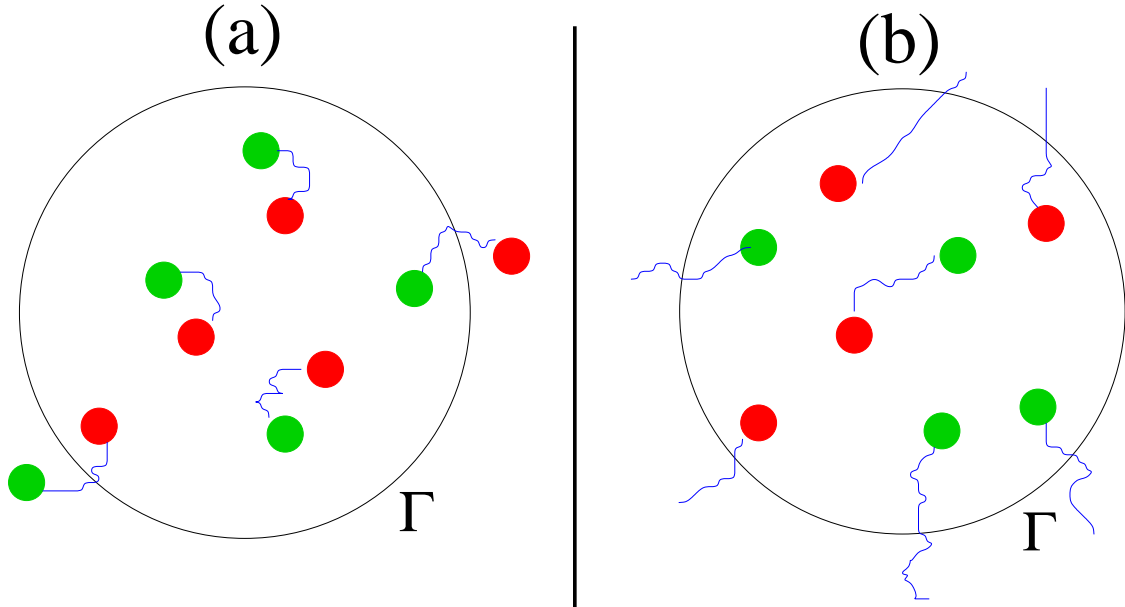


FIG. 5. The only device which can measure the stripe-topological order, and its disappearance at the confinement transition, is the non-local correlation function well known in Gauge theory: the Wilson loop. In the deconfining state, the loop is fluctuated by the gauge seams connecting bound stripe dislocations, and these cause the Wilson loop to decay with its parameter. In the confining state, the probability of a seam crossing a loop is proportional to the enclosed area because stripe dislocations occur freely.

It is a fascinating possibility that this quantum phase transition is about the confinement transition discussed in the above. The key is that stripe-deconfinement order is a topological order which can only be detected by truly topological means, and experimental physicists cannot build the machines to measure it. This order, and its destruction, can only be directly measured by the Wilson loop. This measuring machine looks as follows. First an operator  $\tilde{\sigma}_i^3$  has to be designed measuring if at a particular site (or bond) in the cuprate planes a domain wall is absent (+1) or present (-1) (see references [4], [13]). Define a large closed contour on the plane  $\Gamma$  and calculate the expectation value of the product of all  $\tilde{\sigma}^3$  on the sites/bonds lying on the perimeter of the contour,

$$O_{Wilson}(\Gamma) = \langle \prod_{\Gamma} \tilde{\sigma}_{i \in \Gamma}^3 \rangle. \quad (3)$$

When the stripes would be perfectly connected, every stripe entering the loop would have to come out again meaning that the number of stripes crossing the loop would always be even, with the consequence that upon sending the perimeter  $l$  of the loop to infinity  $O_{Wilson} \rightarrow 1$ . However, in reality, local (virtual) stripe unbindings will always occur (Fig. 5a). A stripe dislocation-antidislocation pair might be created right at the perimeter of the loop adding an unevenness fluctuation which will cause the Wilson loop to decay exponentially. However, the probability for this to happen scales with the perimeter and accordingly  $O_{Wilson} \sim \exp(-\alpha l)$ , the ‘perimeter law’ [8]. On the other hand, in the confining state stripe end points occur as real excitations and accordingly unevenness fluctuations are now proportional to the number of stripe end points occurring anywhere within the loop (fig. 5b). As a consequence,  $O_{Wilson} \sim \exp(-\alpha l^2)$ , the famous area law.

The bad news is that the Wilson-loop machine seems beyond the capacities of present day condensed matter experimentation and all what remains are *indirect* ways to look for signatures of the transition – in fact the same problem is faced by the high energy physicists working on QCD confinement. One key question in this regard is how the gauge fields which become massless at the confinement transition communicate with the fermionic degrees of freedom presumably also present at low energies (nodal fermions). This is a difficult subject which is beyond the scope of the present tutorial discussion and we refer to an interesting recent discussion by Sachdev and Zhang [6].

Yet another way to look for circumstantial evidence is the subject of paper II [1]. Fully connected stripes protect the spin system against frustrations. Although there are other sources of spin fluctuations, squeezed space is still bipartite as long as the stripes are intact. On the other hand, a free vison/stripe dislocation corresponds with a hard frustration in the spin system, destroying colinear spin order globally (Fig. 3b). If the conditions are right, it is imaginable that even when the stripes are delocalized the spin order persists. In squeezed space this is just an antiferromagnet. However, in full space the antiferromagnet is fluctuated by the sublattice parity flips, turning the staggered order parameter in one which is minus itself. This is the quantum spin-nematic order, which is at the center of attention in paper II. As we will argue, also in the case of the spin-nematic it is a bit of a game of hide and seek, but now the experimental physicist has a fair chance to find the suspect.

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