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The resistance experienced by the outward moving chromospheric Ca^+ ions and its effect on the density, by *J. Woltjer Jr.*

The outflow of the Ca^+ ions in the chromosphere is opposed by the resistance offered by the gases of the solar atmosphere. In a former discussion of the outward motion [1]) a rough value had been used based on an atomic diameter of 10^{-8} cm. As an analysis [2]) of UNSÖLD's measures of the K line contour in emission has made possible a determination of the chromospheric density from observation, it has become necessary to compute a more accurate value of the resistance force on account of its preponderating influence on the calculation of the density. The subject is well known from the kinetic theory of gases and the usual procedure may be followed. As the number of electrons per ion is small, deviations from the inverse square law need not be taken into account. I have restricted the encounters between two ions to the case that the distance from the one to the asymptotic relative velocity of the other is less than one quarter of the distance between the ions of the solar atmosphere.

1. Denote the temperature by T, the atomic weight by m, the mass of an atom by m_{at} and the components of atomic velocity by the letters

$$(1.1) \qquad \xi\sqrt{\frac{2kT}{m_{at}}},\quad \eta\sqrt{\frac{2kT}{m_{at}}},\quad \zeta\sqrt{\frac{2kT}{m_{at}}};$$

k has its usual meaning. The quantities referring to the Ca^+ ions are distinguished by accents.

The number of ions of the solar atmosphere between the specified limits of velocity per unit of volume is equal to

$$(1.2) \qquad \frac{\rho}{m_{at}}\,\pi^{-\frac{3}{2}}\,e^{-(\xi^2+\eta^2+\zeta^2)}\,d\xi\,d\eta\,d\zeta\,;$$

ρ is the density of the solar atmosphere.

The fraction of Ca^+ ions between the specified limits of velocity is equal to

$$(1.3) \qquad \pi^{-\frac{3}{2}}\,e^{-[\xi'^2+\eta'^2+(\zeta'-\zeta_0')^2]}\,d\xi'\,d\eta'\,d\zeta'\,;$$

the appearance of ζ_0' denotes the macroscopic motion in the ζ direction.

The components of the relative velocity of a Ca^+ ion are denoted by $(a, b, c)\sqrt{\frac{2kT'}{m'_{at}}}$.

Hence, abbreviating by putting $\frac{2kT}{m_{at}} = \varepsilon^2\, 2\,\frac{kT'}{m'_{at}}$, we have

$$(1.4) \qquad \begin{aligned} a &= \xi' - \varepsilon\xi \\ b &= \eta' - \varepsilon\eta \\ c &= \zeta' - \varepsilon\zeta \end{aligned} \qquad s = \sqrt{(a^2+b^2+c^2)}$$

At each encounter a Ca^+ ion describes a hyperbola relatively to an ion of the solar atmosphere. Denote the distance from this centre of motion to the asymptote by p. The repulsive force is determined by the quantity $\mu = -e^2[1/m_{at} + 1/m'_{at}]$ if we suppose each ion to be singly ionised and hence to have the positive charge e.

The increase of ζ velocity of the Ca^+ ion at an encounter consists of two terms; the one equals

$$(1.5) \qquad -\frac{m}{m'+m}\,\frac{2}{1+\frac{p^2}{\mu^2}s^4\left(\frac{2kT'}{m'_{at}}\right)^2}\,c\sqrt{\frac{2kT'}{m'_{at}}}\,;$$

the second term need not to be taken into account as it disappears in the summation over all encounters.

Averaged over all Ca^+ ions the increase of ζ velocity per unit of time per ion equals

$$(1.6) \qquad -\int\!\dots\!\int \frac{m}{m'+m}\,\frac{2c}{1+\frac{p^2}{\mu^2}s^4\left(\frac{2kT'}{m'_{at}}\right)^2}\,\frac{2kT'}{m'_{at}}\,2\pi p s\,\frac{\rho}{\pi^3 m_{at}}\,e^{-[\xi^2+\eta^2+\zeta^2+\xi'^2+\eta'^2+(\zeta'-\zeta'_0)^2]}\,dp\,d\xi\,d\eta\,d\zeta\,d\xi'\,d\eta'\,d\zeta'.$$

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[1]) *B.A.N.* 167. [2]) *B.A.N.* 182.

The limits of integration are from $-\infty$ to $+\infty$ for the velocity components. The limit of p is determined by the consideration that half way between two unaccented ions the repulsive force is zero; hence the limit has been taken equal to $\frac{1}{4}(m_{at}/\rho)^{1/3}$. This substitution will only be made at the end of the computation; provisionally the limit will be simply termed p.

Introduce the variables a, b, c by (4) and integrate with regard to p, ξ, η, ζ; the resulting value of (6) is

$$(1.6') \quad -(1+\varepsilon^2)^{-\frac{3}{2}} \frac{m}{m'+m} \frac{\pi^{-\frac{3}{2}}\rho}{m_{at}} \mu^2 \frac{m'_{at}}{2kT'} \iiint \log\left[1+\frac{p^2}{\mu^2}s^4\left(\frac{2kT'}{m'_{at}}\right)^2\right] \frac{2\pi}{s^4} sce^{-\frac{a^2+b^2+(c-\zeta_0')^2}{1+\varepsilon^2}} da\,db\,dc.$$

If the exponential function is reduced by taking out the terms $-\frac{a^2+b^2+c^2+\zeta_0'^2}{1+\varepsilon^2}$ from the argument and developing the remaining part in powers of ζ_0', then only odd powers survive the integration. Restricting the result to the first power only we find for (6):

$$(1.6'') \quad -\frac{16}{3}(1+\varepsilon^2)^{-\frac{5}{2}} \frac{m}{m'+m} \pi^{\frac{1}{2}} \frac{\rho}{m_{at}} \mu^2 \frac{m'_{at}}{2kT'} \zeta'_0 e^{-\frac{\zeta_0'^2}{1+\varepsilon^2}} \int_0^\infty s \log\left[1+\frac{p^2}{\mu^2}s^4\left(\frac{2kT'}{m'_{at}}\right)^2\right] e^{-\frac{s^2}{1+\varepsilon^2}} ds.$$

The integral induces us to investigate the function

$$(1.7) \qquad G(x) = \int_0^\infty u \log(1+xu^4)\, e^{-u^2} du.$$

It is easily shown that for large values of x it approximates to

$$(1.8) \qquad \tfrac{1}{2}\log_e x.$$

If $x > 1$, the difference between the integral (7) and the approximation (8) is less than unity; hence if $\log_e x$ is a few units the approximation suffices. I adopt the value (8).

It seems unnecessary to continue distinguishing between the two temperatures T and T', though actually a difference may exist.

Hence I get the final result for the force acting on a Ca^+ ion:

$$(1.9) \qquad \text{resistance force per } Ca^+ \text{ ion} = -\frac{\sqrt{(8\pi)}}{3}\left(\frac{m'}{m+m'}\right)^{1/2} \frac{e^4}{\sqrt{m_{at}}} \frac{\rho}{(kT)^{3/2}} v \log_e\left[\frac{1}{4}\frac{m_{at}^{2/3}}{\rho^{2/3}}\frac{k^2T^2}{e^4}\right];$$

v is the macroscopic outward velocity. I have dropped the exponential factor with the argument $\frac{m}{m+m'}\frac{m'_{at}}{2kT}v^2$, for, though it controls the result for very large values of v, it belongs in the final result to the third order terms in v, that have already been omitted in the transition from (6′) to (6″)

2. I compute numerical values with $m'=40$, $m=20$, $T=5000°$.[1]) The argument of the logarithm in (1.9) then equals

$$(2.1) \qquad 0{\cdot}00233\,\rho^{-2/3}.$$

Hence:

$$(2.2) \quad \text{resistance force per } Ca^+ \text{ ion} = -2{\cdot}17\times10^{-8}\rho v \log_e[0{\cdot}00233\,\rho^{-\frac{2}{3}}] \text{ c. g. s.}$$

The value of the resistance force used in my former investigations[2]) corresponded to an atomic diameter of 10^{-8} and was equal to

$$(2.3) \qquad -1{\cdot}58\times10^{-10}\rho v \quad \text{c. g. s.}$$

Hence the more detailed computation of this note has increased the magnitude of the resistance by the factor

$$(2.4) \qquad 3{\cdot}16\times10^2\,[-2{\cdot}63-\tfrac{2}{3}\log_{10}\rho].$$

[1]) *B. A. N.* 167. [2]) *l. c.*

The computation of the chromospheric density has been given in a previous note[1]). It made use of the density at the critical point S in which the outward systematic Ca^+ velocity equalled $\sqrt{RT/m'}$, where R is the absolute gas-constant, and the resistance force exactly balanced the outward force due to the excess of radiation pressure over gravity. The value (2·2) gives:

$$(2.5) \qquad 2{\cdot}17\times10^8\rho_S\sqrt{\frac{RT}{m'}}\log_e[0{\cdot}00233\,\rho_S^{-\frac{2}{3}}] = \tfrac{1}{2}m_{Ca}g,$$

if the outward radiation acceleration is taken equal to $\frac{3}{2}g$, g being the acceleration by gravitation[2]).

Hence $\rho_S = 0{\cdot}2\times10^{-16}$; the value formerly used[3]) was $0{\cdot}5\times10^{-13}$. The corresponding reduction of the theoretical estimate of the chromospheric density amounts to division by a factor 2000. This result is satisfactory as the empirical data[4]) reduced the former theoretical estimate by a factor 100, though this number is open to revision.

As to this value of ρ_S corresponds a value of about 10^8 of the argument x in (1·8) we need not fear the approximation of (1·7) by (1.8).

[1]) *B. A. N.* 167. [2]) *l. c.* [3]) *l. c.* [4]) *B. A. N.* 182.