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KIRCHHOFF'S LAW FOR ARBITRARILY POLARIZED LIGHT

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Kirchhoff's law expressing the relation between thermal emission and thermal absorption of a body is extended in terms of Stokes parameters to arbitrary polarizations of the emitted and incident light. Use is made of the principles of reciprocity and of detailed balance.

1. The problem

Kirchhoff's law relating the absorption and thermal emission of a body dates from 1860. Stokes' formulation of the intensity and state of polarization of an arbitrarily polarized wave in terms of four parameters, now called Stokes parameters, dates from 1852. The optical reciprocity principle was formulated by Helmholtz, probably about the same time. Hence, all elements for the correct formulation of Kirchhoff's law for arbitrarily polarized light were present a century ago. It is possible that this formulation has been given somewhere, but it seems not to be contained in modern texts. The present derivation arose as a by-product in studies on the reflection properties of planetary atmospheres with clouds.

2. State of polarization of a light beam

We employ the notation of VAN DE HULST (1957, hereafter referred to as I), which does not differ much from those in other books, e.g. CHANDRASEKHAR (1950), BORN and WOLF (1959). Let an almost-parallel, almost-monochromatic beam travel in the direction $\boldsymbol{n}$. Choose the unit vectors $\boldsymbol{l}$ and $\boldsymbol{l}\times\boldsymbol{n}$ at right angles to $\boldsymbol{n}$; the plane through $\boldsymbol{l}$ and $\boldsymbol{n}$ is called the plane of reference. The electric field in a coherent (= phase-preserving) beam may then be written as

$$\boldsymbol{E} = \mathrm{Re}\left\{[E_l\boldsymbol{l}+E_r(\boldsymbol{l}\times\boldsymbol{n})]\,\mathrm{e}^{-\mathrm{i}k\boldsymbol{n}\boldsymbol{r}+\mathrm{i}\omega t}\right\}$$

with $\boldsymbol{r}$ = radius vector, k = propagation constant, ω = frequency, t = time.

The two complex amplitudes (E_l, E_r) specify the amplitude, state of polarization, and phase of the beam. In most practical situations phase relations are lost by incoherence in the source or by erratic displacements in other parts of the optical system. The four Stokes parameters defined by

$$\begin{aligned} I &= c\ \langle E_lE_l^*+E_rE_r^*\rangle \\ Q &= c\ \langle E_lE_l^*-E_rE_r^*\rangle \\ U &= c\ \langle E_lE_r^*+E_rE_l^*\rangle \\ V &= \mathrm{i}c\,\langle E_lE_r^*-E_rE_l^*\rangle \end{aligned}$$

then form a set of four real quantities specifying the intensity and state of polarization of the beam. Here * means complex conjugate, $\langle\ \rangle$ means a time average, c is a normalizing factor which we shall take for convenience to be such that I is the specific intensity in the beam. This implies that we assume a small spread in directions as well as frequency but it is not necessary to introduce the coherency matrix (BORN and WOLF, 1959) which precisely defines this spread. Replacing $\boldsymbol{l}$ by $-\boldsymbol{l}$ does not change the Stokes parameters.

3. Polarizing filter; reciprocity

Consider a filter separating two regions of (free) space, regions a and b. We assume that a beam as described above maintains its direction $\boldsymbol{n}$, solid angle dΩ, and cross-section O in passing from region a to b through the filter, but that its intensity, phase and state of polarization may be changed. In each region we choose a vector $\boldsymbol{l}_a$ and $\boldsymbol{l}_b$, perpendicular to $\boldsymbol{n}$. The transformation of a coherent beam by the filter is expressed by the matrix multiplication

$$\begin{pmatrix} E_l \\ E_r \end{pmatrix}_{\mathrm{b,\,out}} = \begin{pmatrix} A_2 & A_3 \\ A_4 & A_1 \end{pmatrix} \cdot \begin{pmatrix} E_l \\ E_r \end{pmatrix}_{\mathrm{a,\,in}} .$$

As a consequence, the four-vector S_i formed by the Stokes parameters describing an incoherent beam is transformed by a 4×4-matrix as follows

$$S_i^{\mathrm{b,\,out}} = \sum_k F_{ik} S_k^{\mathrm{a,\,in}}$$

where $i, k = 1, 2, 3, 4$.

In the reverse experiment the direction of propagation is $-\boldsymbol{n}$ and we maintain the choice of $\boldsymbol{l}_{\mathrm{a}}$ and $\boldsymbol{l}_{\mathrm{b}}$. By Helmholtz' reciprocity principle (I, p. 49) the transformation of a coherent beam by the filter then is

$$\begin{pmatrix} E_l \\ E_r \end{pmatrix}_{\mathrm{a,\,out}} = \begin{pmatrix} A_2 & -A_4 \\ -A_3 & A_1 \end{pmatrix} \cdot \begin{pmatrix} E_l \\ E_r \end{pmatrix}_{\mathrm{b,\,in}}$$

and the Stokes parameters are transformed by

$$S_i^{\mathrm{a,\,out}} = \sum_k G_{ik} S_k^{\mathrm{b,\,in}} .$$

After writing the elements of the F and G matrices explicitly in terms of the A-matrix (I, p. 44) and upon introducing

$$f_1 = f_2 = f_4 = 1, \qquad f_3 = -1 ,$$

we find that

$$G_{ik} = f_i f_k F_{ki} .$$

This equation expresses the reciprocity property of the filter for arbitrarily polarized light. In words: the matrix is transposed and minus signs are added to the non-diagonal elements in the third row and third column.

4. Kirchhoff's law for a special body

Kirchhoff's law may with slight modifications be expressed for an entire body, for a volume element or for the surface of a body. We shall work with the formulation in terms of a surface.

Take a black body having a flat surface area

$$O' = O' \sum_j s_j$$

perpendicular to $\boldsymbol{n}$, of which each section $s_j O'$ is covered by a different polarizing filter, characterized for outgoing radiation by the 4×4-matrix F^j_{ik}. The reference direction $\boldsymbol{l}$ is chosen the same for the entire area. (Instead of juxtaposition of areas we may also imagine the successive placing of different filters in the beam.) Let B be the black-body specific intensity given by Planck's formula. The incident radiation on the filters from the body is $(B, 0, 0, 0)$. So the specific intensity emerging from one filter is $F^j_{i1} B$ and at a large distance, after merging together by diffraction, the specific intensity is

$$S_{i,\,\mathrm{out}} = \sum_j s_j F^j_{i1} B = e'_i B$$

where

$$e'_i = \sum_j s_j F^j_{i1}$$

is a four-vector taking the place of the scalar emission coefficient in the familiar Kirchhoff theory. In the reverse experiment the same body is exposed to arbitrary light with specific intensity $S_{i,\,\mathrm{in}}$ incident from outside. The power absorbed per unit area per unit solid angle by the black body behind the filters then is

$$H' = \sum_j s_j \sum_i G^j_{1i} S_{i,\,\mathrm{in}}\, \mathrm{d}\nu$$

which may be written as

$$H' = \sum_i a'_i S_{i,\,\mathrm{in}}\, \mathrm{d}\nu$$

where

$$a'_i = \sum_j s_j G^j_{1i}$$

takes the place of the absorption coefficient in the scalar theory. Employing the reciprocity law for each filter we now simply find

$$a'_i = f_i e'_i .$$

A black body without filters has

$$a'_i = e'_i = (1, 0, 0, 0) .$$

5. Kirchhoff's law for a general body

The preceding derivation was valid for a very artificial body consisting of black body plus filters. The result cannot even be interpreted as Kirchhoff's law for this special body unless we make the additional assumption that the filters have no absorption (and emission) of their own. However, the results may be generalized as follows. Consider an arbitrary body, thermally radiating in the direction $\boldsymbol{n}$ with a projected surface area O. Choose a plane of reference through $\boldsymbol{n}$. The specific intensity emitted in this direction at a large distance r may be written as $e_i B$ and the power absorbed per unit solid angle per unit area from incident radiation with specific intensity S_i may be written as

$$H = \sum_i a_i S_i \,\mathrm{d}\nu .$$

In order to find the general relation between a_i and e_i we place at a large distance r in this direction the test body just described and specified by the quantities O', e'_i, a'_i. The principle of detailed balance requires that the two bodies in thermal equilibrium supply each other equal amounts of energy. We obtain:
from test body to arbitrary body

$$H = O\frac{O'}{r^2}\sum_i a_i e'_i B \,\mathrm{d}\nu ,$$

from arbitrary body to test body

$$H' = O'\frac{O}{r^2}\sum_i a'_i e_i B \,\mathrm{d}\nu .$$

These results are identical only if also the arbitrary body has

$$a_i = f_i e_i ,$$

which is the general expression of Kirchhoff's law.

6. Interpretation; application to an antenna

The familiar interpretation of the Stokes parameters for any beam is

$$Q/I = p\cos 2\beta \cos 2\chi$$
$$U/I = p\cos 2\beta \sin 2\chi$$
$$V/I = p\sin 2\beta$$

where p is the degree of polarization and β and χ specify the form and orientation of the polarization ellipse as shown in figure 1. The factor f_i in our formulae inverts the sign of U only. This arises from the trivial fact that χ is measured counterclockwise as seen in the direction of propagation, so that its sign has to be inverted if we look the opposite way. The orientation of the main axis (for linear light the plane of polarization) does *not* change.

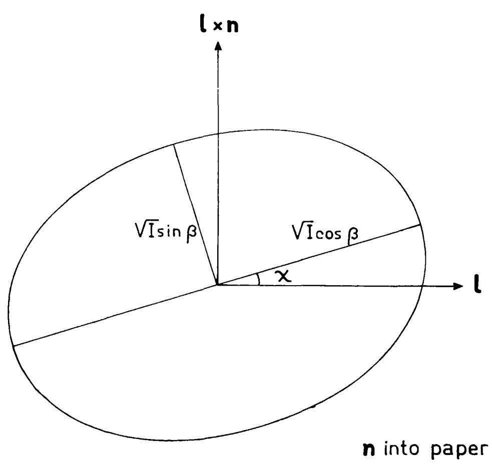

Figure 1. The polarization ellipse.

The quantities

$$Q/pI, \quad U/pI, \quad V/pI$$

plotted in rectangular coordinates, define a point on the Poincaré sphere. The formula for the absorbed power may now be interpreted as follows. Let a (≤ 1) be the absorptivity of the body and hence aB its emissivity. Let the thermally emitted light have the degree of polarization p' and the polarization ellipse ($\beta', -\chi'$). Then the absorption properties of the surface are characterized by the four-vector

$$a_i = (a,\ ap'\cos 2\beta'\cos 2\chi',\ ap'\cos 2\beta'\sin 2\chi',\ ap'\sin 2\beta') .$$

The incident light is similarly characterized by its specific intensity S, its degree of polarization p and its polarization ellipse (β, χ) together forming the four-vector

$$S_i = (S,\ Sp\cos 2\beta\cos 2\chi,\ Sp\cos 2\beta\sin 2\chi,\ Sp\sin 2\beta) .$$

Multiplying, and introducing δ, the angle between the points on the Poincaré sphere corresponding to these two four-vectors, we simply find

$$H = aS(1+pp'\cos\delta)\,\mathrm{d}\nu .$$

This reduces to the familiar expression $aS\,\mathrm{d}\nu$ in two cases: 1) if $p' = 0$, i.e. for a surface which equally well absorbs light of any polarization; 2) if $p = 0$, i.e. for incident unpolarized light. Generally, the strongest absorption occurs if the polarization of the incident light matches the properties of the surface, i.e. if the points on the Poincaré sphere coincide, $\delta = 0$. In the extreme case in which $p = p' = 1$ the factor in parentheses is $1+\cos\delta = 2\cos^2\frac{1}{2}\delta$, which varies between 2 for perfect match ($\delta = 0°$) and 0 for complete mismatch ($\delta = 180°$).

This interpretation is strongly reminiscent of antenna theory. A loss-less antenna terminated by a matched resistance forms indeed a very special body, to which Kirchhoff's law *must* apply. The formula

$$W = \tfrac{1}{2}A_e P(1+p\cos\delta)$$

given by Ko (1961) is a direct consequence of the formula just given. Both sides refer to the entire body, not to a surface element. Further, p' is always assumed to be 1. The factor $\frac{1}{2}$ has been introduced (by the definition of the effective area A_e) because the usual assumption is that the polarization of the incident wave is matched. We may refer to two papers for further explanation of the radio applications. PEAKE (1959) gives a formulation of Kirchhoff's law complete, except for elliptical polarization. Ko (1962) presents the antenna theory including the spread in frequency and direction, employing the coherency matrix.

References

M. BORN and E. WOLF, 1959, *Principles of Optics* (Pergamon Press, London)

S. CHANDRASEKHAR, 1950, *Radiative Transfer* (Oxford Univ. Press, London)

H. C. VAN DE HULST, 1957, *Light Scattering by Small Particles* (John Wiley and Sons, New York)

H. C. KO, 1961, *Proc. Inst. Radio Eng.* **49** 1446

H. C. KO, 1962, *Proc. Inst. Radio Eng.* **50** 1950

W. H. PEAKE, 1959, *Trans. Inst. Radio Eng.* **AP7** S324