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## Motion in the core of a triaxial potential

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**Summary.** The near-harmonic motion in the core of a general triaxial potential is studied by means of the averaging method. The case of a near 1:1:1 resonance between the fundamental orbital frequencies is investigated in detail. It is shown that all orbits possess a second, asymptotic, isolating integral in addition to the energy. In the spherical limit this integral reduces to the square of the angular momentum. The simple periodic orbits are derived, and their stability properties are calculated. There may be up to 14 distinct families of simple periodic orbits, some of which are not in any of the symmetry planes of the potential.

For potentials that are nearly spherical, which is the case for the gravitational potentials of elliptical galaxies, an approximate third integral of motion exists. In this case the orbital structure becomes much simpler, and is nearly identical to that in a Stäckel potential, for which the equations of motion separate in ellipsoidal coordinates. The effects of figure rotation on these results are discussed briefly.

### 1 Introduction

Schwarzschild (1979, 1981) has shown that insight into the structure of triaxial elliptical galaxies can be gained by a study of the full variety of individual stellar orbits in realistic triaxial potentials. At first sight this might seem to be a nearly impossible task. However, numerical experiments have shown that in the triaxial potentials that are thought to be relevant for elliptical galaxies, most orbits possess three isolating integrals of motion, i.e. two non-classical ones in addition to the energy integral (Schwarzschild 1979, 1982; Goodman & Schwarzschild 1981). This means that phase-space is well ordered, and that most orbits belong to one of a few major families, each connected with a simple periodic orbit (e.g. Arnold 1978; Verhulst 1983). Thus, many properties of the general orbits can be deduced from a study of the simple periodic orbits.

In the absence of figure rotation, the potentials of triaxial elliptical galaxies are well approximated by the special potentials that were first classified by Stäckel (1890), and introduced in stellar dynamics by Eddington (1915). For these Stäckel potentials the equations of motion are separable in ellipsoidal coordinates, so that every orbit enjoys three exact isolating integrals (de

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Zeeuw 1984, 1985; de Zeeuw & Lynden-Bell 1985). Many triaxial mass models with a potential of Stäckel form exist that are nearly indistinguishable from the mass models that have been inferred from observations of the light distribution of elliptical galaxies (de Zeeuw, Peletier & Franx 1985, in preparation). In each of these separable models the orbits can be divided into four families. These are identical to the four major orbit families that exist in Schwarzschild's (1979) non-separable triaxial potential. Apparently, to a high degree of accuracy, motion in non-rotating elliptical galaxies can be considered as motion in a perturbed Stäckel potential.

It is likely that elliptical galaxies have (slowly) rotating figures (Binney 1978; Illingworth 1981). Schwarzschild (1982) found that motion in a slowly rotating version of his potential is similar to that in the non-rotating case. There now is a distinction between retrograde and direct motion, but most orbits still possess three integrals of motion, and belong to one of a few major families. For many purposes the effect of slow rotation can be considered as a perturbation on the non-rotating motion.

A full treatment of perturbed motion in ellipsoidal coordinates is not available. In the present paper we take a limited step in that direction by considering motion in the core. Here the gravitational potential has a minimum and the motion can be treated as perturbed harmonic motion in Cartesian coordinates. Many standard techniques are available for this purpose (e.g. Hagihara 1974). de Zeeuw & Merritt (1983, hereafter ZM) have applied one of these techniques – the method of averaging (Bogoliubov & Mitropolsky 1961; Verhulst 1979) – to the two-dimensional motion in the equatorial plane of Schwarzschild's model. These authors found that the motion is characteristic of a 1 : 1 resonance between the fundamental orbital frequencies. Here we extend their analysis to three dimensions, and a study of the 1 : 1 : 1 resonance by means of the same method. We shall consider the integrals of motion and derive the simple periodic orbits that may exist. A preliminary account of some of the results presented here has been given by de Zeeuw (1982).

Aside from the application to galaxies, an analytic study of perturbed harmonic motion in three dimensions is interesting on its own. The number of investigations of systems with three degrees of freedom is still small, in sharp contrast with the situation in two dimensions (Verhulst 1983).

## 2 Averaging

We restrict ourselves to models which are symmetric with respect to three perpendicular principal planes, i.e. models for which the density can be written as  $\rho = \rho(x_1^2, x_2^2, x_3^2)$ . Inhomogeneous triaxial ellipsoids are an example, but also all models with a Stäckel potential. We first take the models to be non-rotating. Effects of rotation are discussed briefly in Section 5.

### 2.1 BASIC EQUATIONS AND LINEAR ANALYSIS

The equations of motion for a single star are

$$\ddot{x}_i = -\partial V / \partial x_i, \quad (1)$$

where  $V = V(x_1^2, x_2^2, x_3^2)$  is the gravitational potential. We further assume that  $V$  can be expanded in a Taylor series about the origin as

$$V = \frac{1}{2} \sum_{i=1}^3 \kappa_i^2 x_i^2 + \frac{1}{4} \sum_{i,j=1}^3 a_{ij} x_i^2 x_j^2 + \dots, \quad (2)$$

where it is understood that  $0 \leq \kappa_1 \leq \kappa_2 \leq \kappa_3$ . The  $a_{ij}$  are constants, and we have assumed that  $a_{ji} = a_{ij}$ . Here and below  $i$  and  $j$  always take the values 1, 2 and 3.

In zeroth order only the quadratic terms in  $V$  are retained. The equations of motion (1) are simply

$$\ddot{x}_i = -\kappa_i^2 x_i. \quad (3)$$

These equations describe three uncoupled harmonic oscillators:

$$x_i = A_i \cos(\kappa_i t + \phi_i), \quad (4)$$

where the amplitudes  $A_i$  and phases  $\phi_i$  are constants determined by the initial conditions, and  $t$  denotes time.

The non-linear analysis is performed most easily in terms of *action-angle variables* ( $I_i, \theta_i$ ) defined by

$$x_i = \sqrt{2I_i/\kappa_i} \cos \theta_i, \quad \dot{x}_i = -\sqrt{2\kappa_i I_i} \sin \theta_i. \quad (5)$$

In these variables the Hamiltonian  $H_2$  that corresponds to the linearized equations of motion (3) is

$$H_2 = \sum_{i=1}^3 \kappa_i I_i. \quad (6)$$

The corresponding equations of motion are

$$\dot{I}_i = 0, \quad \dot{\theta}_i = \kappa_i, \quad (7)$$

with as solutions

$$I_i = \frac{1}{2} \kappa_i A_i^2 = E_i, \quad \theta_i = \kappa_i t + \phi_i, \quad (8)$$

where the  $E_i$  are constants, and  $\kappa_i E_i$  is the energy of oscillator  $i$ . In zeroth order (very near the centre) there are thus three independent isolating integrals of motion for each star.

We also use *normal variables* ( $Q_i, P_i$ ) defined by

$$Q_i = \sqrt{2I_i} \cos \theta_i, \quad P_i = -\sqrt{2I_i} \sin \theta_i, \quad (9)$$

and *comoving variables* ( $q_i, p_i$ ) defined by

$$q_i = Q_i \cos \kappa_i t - P_i \sin \kappa_i t, \quad p_i = Q_i \sin \kappa_i t + P_i \cos \kappa_i t. \quad (10)$$

For a given potential  $V$  the quantities  $\kappa_i$  are fixed, so all orbits have the same fundamental frequencies in zeroth order. If the  $\kappa_i$  are incommensurable then the orbit completely fills a rectangular box-shaped region in configuration space ( $-A_i \leq x_i \leq A_i$ ). If there is one relation of the form  $l_i \kappa_i = 0$ , where the  $l_i$  are integers, then an appropriate phase difference is an extra isolating integral, and the orbit fills only a two-dimensional region within the box. If two such relations exist between the frequencies then two independent phase differences are conserved. The orbit is now periodic, and is a three-dimensional Lissajous figure enclosed by the box.

## 2.2 NON-LINEAR ANALYSIS

In first order the quartic terms in the potential (2) should be included. We rescale the coordinates by writing  $x_i = \varepsilon^2 x_i^*$ , where  $\varepsilon$  is a small positive parameter. Dividing  $V$  by  $\varepsilon^2$  and omitting the asterisks after the transformation we find

$$V = \frac{1}{2} \sum_{i=1}^3 \kappa_i^2 x_i^2 + \frac{1}{4} \varepsilon^2 \sum_{i,j=1}^3 a_{ij} x_i^2 x_j^2 + \dots, \quad (11)$$

By means of (5) the corresponding Hamiltonian  $H$  can be written as

$$H = H_2(I_i) + \varepsilon^2 H_4(I_{ij}, \theta_i) + O(\varepsilon^4), \quad (12)$$

where  $H_2$  is given in equation (6).  $H_4$  is a lengthy expression, and the resulting equations of motion show that both actions and angles now depend on time. These equations can generally not be solved exactly, even when terms of  $O(\varepsilon^4)$  are omitted.

In first order it is sufficient to consider  $H_2 + \varepsilon^4 H_4$  averaged over a dynamical time-scale (Bogoliubov & Mitropolsky 1961; Jaeger & Lichtenberg 1972). ZM describe the procedure in two dimensions. The three-dimensional case can be treated in the same manner.

First, we represent the frequency ratios  $\kappa_1/\kappa_2$  and  $\kappa_3/\kappa_2$  by rational numbers and we admit the irrational numbers by a small perturbation of the rationals (*cf.* Verhulst 1979):

$$\frac{\kappa_1}{\kappa_2} = \frac{m}{n} + \delta_1, \quad \frac{\kappa_3}{\kappa_2} = \frac{l}{n} + \delta_2, \quad (13)$$

where  $l$ ,  $m$  and  $n$  are relatively prime integers, and the *detuning parameters*  $\delta_1$  and  $\delta_2$  are of  $O(\varepsilon^2)$ . Next we substitute for the actions and angles that occur in  $H$  the expressions

$$\begin{aligned} I_i &= I_i(t) \\ \theta_1(t) &= (m/n)\kappa_2 t + \phi_1(t), \\ \theta_2(t) &= \kappa_2 t + \phi_2(t), \\ \theta_3(t) &= (l/n)\kappa_2 t + \phi_3(t), \end{aligned} \quad (14)$$

For fixed  $I_i$  and  $\phi_i$  the Hamiltonian is now a periodic function of time, since  $m/n$  and  $l/n$  are rational. The time average of  $H$  over its period  $T$  is the first order averaged Hamiltonian. We denote it by  $\langle H(\bar{I}_i, \bar{\theta}_i) \rangle$ . The bar is introduced to indicate that averaging has taken place and terms of  $O(\varepsilon^4)$  have been omitted.

The solutions  $\bar{I}_i(t)$ ,  $\bar{\theta}_i(t)$  of the averaged equations of motion, i.e. Hamilton's equations for  $\langle H \rangle$ , are first order asymptotic approximations of the solutions  $I_i(t)$ ,  $\theta_i(t)$  of the exact equations of motion (1):

$$\begin{aligned} |I_i(t) - \bar{I}_i(t)| &\leq C\varepsilon^2, \\ |\theta_i(t) - \bar{\theta}_i(t)| &\leq C\varepsilon^2, \end{aligned} \quad \text{for } 0 \leq t \leq LT/\varepsilon^2, \quad (15)$$

where  $C$  and  $L$  are constants of order unity (Bogoliubov & Mitropolsky 1961; van der Burgh 1974, 1976; Verhulst 1979).

The result of averaging  $H$  depends on the values of  $l$ ,  $m$  and  $n$ , i.e. on the frequency ratios. For a potential  $V$  of the form (2) three cases have to be distinguished, depending on whether  $l$ ,  $m$  and  $n$  are all unequal, two of them are equal, or all three are equal. We consider each case in turn.

### 2.3 NO FIRST-ORDER RESONANCE

When  $l \neq m \neq n$  the first-order averaged Hamiltonian is a function of the actions only. It is given by

$$\langle H \rangle = H_0 = \sum_{i=1}^3 \kappa_i \bar{I}_i + \frac{1}{2} \varepsilon^2 \sum_{i,j=1}^3 \mu_{ij} \bar{I}_i \bar{I}_j, \quad (16)$$

where

$$\mu_{ii} = \frac{3a_{ii}}{4\kappa_i^2}, \quad \mu_{ij} = \frac{a_{ij}}{2\kappa_i \kappa_j}, \quad i \neq j. \quad (17)$$

Hamilton's equations can be solved immediately. We find

$$\begin{aligned}\bar{I}_i &= E_i, \\ \bar{\theta}_i &= [\kappa_i + \varepsilon^2 \sum_{j=1}^3 \mu_{ij} E_j] t + \bar{\theta}_i(0),\end{aligned}\quad (18)$$

where  $E_i$  and  $\bar{\theta}_i(0)$  are constants determined by the initial conditions.

As in the linear analysis, there are three independent isolating integrals of motion. The energy in each fundamental oscillation, or *normal mode*, is conserved, within an error of  $O(\varepsilon^2)$  on a time-scale of  $O(1/\varepsilon^2)$ . The fundamental oscillations are uncoupled on this time-scale.

#### 2.4 THE 1 : 1 RESONANCE

When  $m=n$  the averaged Hamiltonian is

$$\langle H \rangle = H_0(\bar{I}_1, \bar{I}_2, \bar{I}_3) + \frac{1}{2} \varepsilon^2 \mu_{12} \bar{I}_1 \bar{I}_2 \cos(2\bar{\theta}_1 - 2\bar{\theta}_2), \quad (19)$$

where  $H_0$  is given in equation (16). The other cases,  $l=n$  and  $m=l$ , are equivalent and can be obtained from (19) by a rotation of the variables.

The term that is added to  $H_0$  couples the two oscillations that are resonant. The third oscillation is uncoupled. As a result, the study of the motion in this Hamiltonian is a two-dimensional problem to this order of approximation. The analysis of this 1 : 1 resonance is described in ZM, so there is no need to investigate it further. We only remark that for the Hamiltonian (19) three integrals of motion exist: the energy in the  $x_3$ -oscillation, and the two integrals found by ZM for the  $(x_1, x_2)$ -motion. In the  $(x_1, x_2)$ -plane two families of simple periodic orbits may occur in addition to the normal modes.

#### 2.5 THE 1 : 1 : 1 RESONANCE

When  $l=m=n$  the first-order averaged Hamiltonian  $\langle H \rangle$  is equal to

$$\begin{aligned}H_{1:1:1} &= H_0(\bar{I}_1, \bar{I}_2, \bar{I}_3) + \frac{1}{2} \varepsilon^2 \{ \mu_{12} \bar{I}_1 \bar{I}_2 \cos(2\bar{\theta}_1 - 2\bar{\theta}_2) \\ &\quad + \mu_{13} \bar{I}_1 \bar{I}_3 \cos(2\bar{\theta}_1 - 2\bar{\theta}_3) + \mu_{23} \bar{I}_2 \bar{I}_3 \cos(2\bar{\theta}_2 - 2\bar{\theta}_3) \}.\end{aligned}\quad (20)$$

The averaged equations of motion that follow from  $H_{1:1:1}$  are

$$\begin{aligned}\dot{\bar{I}}_1 &= \varepsilon^2 [ \mu_{12} \bar{I}_1 \bar{I}_2 \sin(2\bar{\theta}_1 - 2\bar{\theta}_2) + \mu_{13} \bar{I}_1 \bar{I}_3 \sin(2\bar{\theta}_1 - 2\bar{\theta}_3) ], \\ \dot{\bar{\theta}}_1 &= \kappa_1 + \varepsilon^2 [ \mu_{11} \bar{I}_1 + \mu_{12} \bar{I}_2 + \mu_{13} \bar{I}_3 \\ &\quad + \frac{1}{2} \mu_{12} \bar{I}_2 \cos(2\bar{\theta}_1 - 2\bar{\theta}_2) + \frac{1}{2} \mu_{13} \bar{I}_3 \cos(2\bar{\theta}_1 - 2\bar{\theta}_3) ], \\ \dot{\bar{I}}_2 &= \varepsilon^2 [ -\mu_{12} \bar{I}_1 \bar{I}_2 \sin(2\bar{\theta}_1 - 2\bar{\theta}_2) + \mu_{23} \bar{I}_2 \bar{I}_3 \sin(2\bar{\theta}_2 - 2\bar{\theta}_3) ], \\ \dot{\bar{\theta}}_2 &= \kappa_2 + \varepsilon^2 [ \mu_{12} \bar{I}_1 + \mu_{22} \bar{I}_2 + \mu_{23} \bar{I}_3 \\ &\quad + \frac{1}{2} \mu_{12} \bar{I}_1 \cos(2\bar{\theta}_1 - 2\bar{\theta}_2) + \frac{1}{2} \mu_{23} \bar{I}_3 \cos(2\bar{\theta}_2 - 2\bar{\theta}_3) ], \\ \dot{\bar{I}}_3 &= \varepsilon^2 [ -\mu_{13} \bar{I}_1 \bar{I}_3 \sin(2\bar{\theta}_1 - 2\bar{\theta}_3) - \mu_{23} \bar{I}_2 \bar{I}_3 \sin(2\bar{\theta}_2 - 2\bar{\theta}_3) ], \\ \dot{\bar{\theta}}_3 &= \kappa_3 + \varepsilon^2 [ \mu_{13} \bar{I}_1 + \mu_{23} \bar{I}_2 + \mu_{33} \bar{I}_3 \\ &\quad + \frac{1}{2} \mu_{13} \bar{I}_1 \cos(2\bar{\theta}_1 - 2\bar{\theta}_3) + \frac{1}{2} \mu_{23} \bar{I}_2 \cos(2\bar{\theta}_2 - 2\bar{\theta}_3) ].\end{aligned}\quad (21)$$

It is clear that the terms in curly brackets in (20) couple all three oscillations. Analysis of the motion in  $H_{1:1:1}$  is a genuine three-dimensional problem. In the next section we consider the

integrals of motion admitted by the averaged equations of motion (21), and we find all simple periodic solutions. An incomplete analysis of the 1:1:1 resonance has been given by Lake & Norman (1983). All the results presented in Section 3 will be valid for the exact equations of motion (1) within an error of  $O(\varepsilon^2)$  on a time-scale of  $O(1/\varepsilon^2)$ . In some cases these estimates can be improved.

### 3 The 1:1:1 resonance: integrals and simple periodic orbits

#### 3.1 INTEGRALS OF MOTION

It is evident from equations (21) that the actions  $\bar{I}_1$ ,  $\bar{I}_2$  and  $\bar{I}_3$  are not conserved separately. However, it can be seen immediately that

$$\bar{I}_1 + \bar{I}_2 + \bar{I}_3 = K_1, \quad (22)$$

is an integral of motion. The existence of this integral is related to the fact that only two independent angle combinations,  $\bar{\theta}_1 - \bar{\theta}_2$  and  $\bar{\theta}_2 - \bar{\theta}_3$ , say, occur in  $H_{1:1:1}$ . As a result, there is an ignorable coordinate and a corresponding conserved quantity. We can write

$$H_{1:1:1} = \kappa_2 K_1 + \varepsilon^2 K_2. \quad (23)$$

Since  $H_{1:1:1}$  is conserved,  $K_2$  is a second isolating integral of equations (21), independent of  $K_1$ . It is given by

$$K_2 = \frac{1}{2} \sum_{i,j=1}^3 \mu_{ij} \bar{I}_i \bar{I}_j^2 + \frac{\kappa_2 \delta_1}{\varepsilon^2} \bar{I}_1 + \frac{\kappa_2 \delta_2}{\varepsilon^2} \bar{I}_3 + \frac{1}{2} \mu_{12} \bar{I}_1 \bar{I}_2 \cos(2\bar{\theta}_1 - 2\bar{\theta}_2) \\ + \frac{1}{2} \mu_{13} \bar{I}_1 \bar{I}_3 \cos(2\bar{\theta}_1 - 2\bar{\theta}_3) + \frac{1}{2} \mu_{23} \bar{I}_2 \bar{I}_3 \cos(2\bar{\theta}_2 - 2\bar{\theta}_3). \quad (24)$$

$K_1$  and  $K_2$  are exact integrals of the averaged equations of motion, and they are approximate integrals of the exact equations of motion. By (15) we find the approximation to be of  $O(\varepsilon^2)$  on the time-scale  $O(1/\varepsilon^2)$ . From (23) it follows, however, that  $\kappa_2 K_1$  is an  $O(\varepsilon^2)$  approximation of the total energy valid for all times. Using the argument given by ZM we find that  $K_2$  is an  $O(\varepsilon^2)$  approximation of a second asymptotic isolating integral that is also valid for all times.

An alternative way of writing the second integral is

$$K_2 = \frac{1}{2} \mu_{11} \bar{I}_1^2 + \frac{3}{2} \mu_{12} \bar{I}_1 \bar{I}_2 + \frac{3}{2} \mu_{13} \bar{I}_1 \bar{I}_3 + \frac{1}{2} \mu_{22} \bar{I}_2^2 + \frac{3}{2} \mu_{23} \bar{I}_2 \bar{I}_3 + \frac{1}{2} \mu_{33} \bar{I}_3^2 \\ + \frac{\kappa_2 \delta_1}{\varepsilon^2} \bar{I}_1 + \frac{\kappa_2 \delta_2}{\varepsilon^2} \bar{I}_3 - \frac{1}{4} \mu_{12} \bar{L}_z^2 - \frac{1}{4} \mu_{13} \bar{L}_y^2 - \frac{1}{4} \mu_{23} \bar{L}_x^2, \quad (25)$$

where  $L_x$ ,  $L_y$  and  $L_z$  are the three components of the angular momentum vector  $\mathbf{L} = (L_x, L_y, L_z) = (x_2 \dot{x}_3 - x_3 \dot{x}_2, x_3 \dot{x}_1 - x_1 \dot{x}_3, x_1 \dot{x}_2 - x_2 \dot{x}_1)$ . In the spherical limit, equation (25) reduces to  $\frac{1}{2} \mu_{11} (K_1^2 - \frac{1}{3} L^2)$ , showing that  $K_2$  can be considered as a generalization of the square of the total angular momentum.

In axisymmetric potentials which are nearly spherical there is, in addition to the total energy and  $L_z$ , a third isolating asymptotic integral which is a generalization of  $L^2$  (Saaf 1968; Innanen & Papp 1977). Evidently,  $K_2$  is the triaxial counterpart of this integral.

Whether the averaged equations of motion (21) admit a third independent isolating integral, for arbitrary values of the  $\mu_{ij}$ , is not obvious. We shall return to this question in Section 4.5.

#### 3.2 SIMPLE PERIODIC ORBITS

For  $l=m=n$  all orbits are periodic in the linear approximation. In first order, the quartic terms in the Hamiltonian cause the fundamental orbital frequencies to depend on energy. As a result, only a few periodic orbits remain at each value of the energy.

The *simple periodic orbits* are those orbits of the linear approximation that are still periodic solutions in first order. For these orbits the quartic perturbation averages to zero around the orbit, so that it remains periodic. Thus, the simple periodic orbits are the solutions of the averaged equations of motion (21) that have constant values of the actions ( $\dot{I}_1 = \dot{I}_2 = \dot{I}_3 = 0$ ).

The stability of a simple periodic orbit can be investigated by considering the averaged Hamiltonian in its neighbourhood.  $H_{1:1:1}$  depends on two independent angle combinations and three actions. By means of (22) we can eliminate one action, so that at fixed energy  $K_1$  the Hamiltonian is a function of only four independent variables. On each simple periodic orbit the first order partial derivatives of  $H_{1:1:1}$  with respect to these variables are zero. The orbit is stable if, and only if,  $H_{1:1:1}$  has an extremum on it (e.g. Kurth 1976). As a result, the (in)stability can be deduced from a calculation of the *characteristic matrix*, the  $4 \times 4$  symmetric matrix of second-order derivatives of  $H_{1:1:1}$ , evaluated on the periodic orbit. The orbit is stable if and only if all four eigenvalues of the characteristic matrix have the same sign.

It turns out that there are a total of 14 families of simple periodic solutions of equations (21); we have numbered them consecutively. They divide naturally in four groups, depending on whether one, two, or all three actions are non-zero, and depending on the values of the phase differences  $\bar{\theta}_1 - \bar{\theta}_2$  and  $\bar{\theta}_2 - \bar{\theta}_3$ . In the following sections we discuss each group separately. For all cases one can find explicitly the values of the actions and angles as function of energy, as well as the eigenvalues of the characteristic matrix. For one orbit family in each group we discuss the derivation of these quantities in some detail. The results for all 14 families are given in Tables 1, 2 and 3, and equations (45).

**Table 1.** Properties of the three axial orbits.

ORBIT FAMILY	ACTIONS & ANGLES	EIGENVALUES OF CHARACTERISTIC MATRIX	BIFURCATIONS WITH FAMILY
1. x-axis orbits	$\bar{I}_1 = K_1$	$\frac{3}{2} \mu_{12}^{-\mu_{11}} - \frac{\kappa_2 \delta_1}{\epsilon^2 K_1}$	4
	$\bar{I}_2 = 0$	$\frac{1}{2} \mu_{12}^{-\mu_{11}} - \frac{\kappa_2 \delta_1}{\epsilon^2 K_1}$	5
	$\bar{I}_3 = 0$	$\frac{3}{2} \mu_{13}^{-\mu_{11}} + \frac{\kappa_2 (\delta_2 - \delta_1)}{\epsilon^2 K_1}$	6
	$\bar{\theta}_1 = (\kappa_1 + \epsilon^2 \mu_{11} K_1) t + \bar{\theta}_1(0)$	$\frac{1}{2} \mu_{13}^{-\mu_{11}} + \frac{\kappa_2 (\delta_2 - \delta_1)}{\epsilon^2 K_1}$	7
2. y-axis orbits	$\bar{I}_1 = 0$	$\frac{3}{2} \mu_{12}^{-\mu_{22}} + \frac{\kappa_2 \delta_1}{\epsilon^2 K_1}$	4
	$\bar{I}_2 = K_1$	$\frac{1}{2} \mu_{12}^{-\mu_{22}} + \frac{\kappa_2 \delta_1}{\epsilon^2 K_1}$	5
	$\bar{I}_3 = 0$	$\frac{3}{2} \mu_{23}^{-\mu_{22}} + \frac{\kappa_2 \delta_2}{\epsilon^2 K_1}$	8
	$\bar{\theta}_2 = (\kappa_2 + \epsilon^2 \mu_{22} K_1) t + \bar{\theta}_2(0)$	$\frac{1}{2} \mu_{23}^{-\mu_{22}} + \frac{\kappa_2 \delta_2}{\epsilon^2 K_1}$	9
3. z-axis orbits	$\bar{I}_1 = 0$	$\frac{3}{2} \mu_{13}^{-\mu_{33}} + \frac{\kappa_2 (\delta_1 - \delta_2)}{\epsilon^2 K_1}$	6
	$\bar{I}_2 = 0$	$\frac{1}{2} \mu_{13}^{-\mu_{33}} + \frac{\kappa_2 (\delta_1 - \delta_2)}{\epsilon^2 K_1}$	7
	$\bar{I}_3 = K_1$	$\frac{3}{2} \mu_{23}^{-\mu_{33}} - \frac{\kappa_2 \delta_2}{\epsilon^2 K_1}$	8
	$\bar{\theta}_3 = (\kappa_3 + \epsilon^2 \mu_{33} K_1) t + \bar{\theta}_3(0)$	$\frac{1}{2} \mu_{23}^{-\mu_{33}} - \frac{\kappa_2 \delta_2}{\epsilon^2 K_1}$	9

Table 2. Properties of the periodic orbits in the principal planes.

ORBIT FAMILY	ACTIONS & ANGLES	EIGENVALUES OF CHARACTERISTIC MATRIX	BIFURCATIONS WITH FAMILY
4. inclined linear orbits in (x,y)-plane	$\bar{I}_1 = \frac{3}{2}\mu_{12}^{-\mu_{22}} + \frac{\kappa_2 \delta_1}{\epsilon^2 K_1} K_1$	$\mu_{11}^{-3\mu_{12} + \mu_{22}}$	-
	$\bar{I}_2 = K_1 - \bar{I}_1, \quad \bar{I}_3 = 0$	$-2\mu_{12} \bar{I}_1 \bar{I}_2$	1, 2
	$\bar{\theta}_1 = \{ \kappa_1 + \epsilon^2 [ \frac{3}{2}\mu_{12} K_1 + (\mu_{11} - \frac{3}{2}\mu_{12}) \bar{I}_1 ] \} t + \bar{\theta}_1(0)$	$\frac{(\sigma_1 + \epsilon_1) \xi_1 - (\tau_1 + \eta_1) \zeta_1}{\mu_{11}^{-3\mu_{12} + \mu_{22}}} K_1$	10
in-phase	$\bar{\theta}_2 = \bar{\theta}_1 \quad \text{or} \quad \bar{\theta}_2 = \bar{\theta}_1 + \pi$	$\frac{(\sigma_2 + \epsilon_2) \xi_2 - (\tau_2 + \eta_2) \zeta_2}{\mu_{11}^{-3\mu_{12} + \mu_{22}}} K_1$	11
5. elliptic orbits in (x,y)-plane	$\bar{I}_1 = \frac{1}{2}\mu_{12}^{-\mu_{22}} + \frac{\kappa_2 \delta_1}{\epsilon^2 K_1} K_1$	$\mu_{11}^{-\mu_{12} + \mu_{22}}$	-
	$\bar{I}_2 = K_1 - \bar{I}_1, \quad \bar{I}_3 = 0$	$2\mu_{12} \bar{I}_1 \bar{I}_2$	1, 2
	$\bar{\theta}_1 = \{ \kappa_1 + \epsilon^2 [ \frac{1}{2}\mu_{12} K_1 + (\mu_{11} - \frac{1}{2}\mu_{12}) \bar{I}_1 ] \} t + \bar{\theta}_1(0)$	$\frac{(\sigma_3 + \epsilon_3) \xi_3 - (\tau_3 + \eta_3) \zeta_3}{\mu_{11}^{-\mu_{12} + \mu_{22}}} K_1$	12
out-phase	$\bar{\theta}_2 = \bar{\theta}_1 + \frac{\pi}{2} \quad \text{or} \quad \bar{\theta}_2 = \bar{\theta}_1 - \frac{\pi}{2}$	$\frac{(\sigma_4 + \epsilon_4) \xi_4 - (\tau_4 + \eta_4) \zeta_4}{\mu_{11}^{-\mu_{12} + \mu_{22}}} K_1$	13
6. inclined linear orbits in (x,z)-plane	$\bar{I}_1 = \frac{3}{2}\mu_{13}^{-\mu_{33}} + \frac{\kappa_2 (\delta_1 - \delta_2)}{\epsilon^2 K_1} K_1$	$\mu_{11}^{-3\mu_{13} + \mu_{33}}$	-
	$\bar{I}_2 = 0, \quad \bar{I}_3 = K_1 - \bar{I}_1$	$-2\mu_{13} \bar{I}_1 \bar{I}_3$	1, 3
	$\bar{\theta}_1 = \{ \kappa_1 + \epsilon^2 [ \frac{3}{2}\mu_{13} K_1 + (\mu_{11} - \frac{3}{2}\mu_{13}) \bar{I}_1 ] \} t + \bar{\theta}_1(0)$	$\frac{\tau_1 (\zeta_1 + \epsilon_1) - \sigma_1 (\eta_1 + \xi_1)}{\mu_{11}^{-3\mu_{13} + \mu_{33}}} K_1$	10
in-phase	$\bar{\theta}_3 = \bar{\theta}_1 \quad \text{or} \quad \bar{\theta}_3 = \bar{\theta}_1 + \pi$	$\frac{\tau_4 (\zeta_4 + \epsilon_4) - \sigma_4 (\eta_4 + \xi_4)}{\mu_{11}^{-3\mu_{13} + \mu_{33}}} K_1$	13
7. elliptic orbits in (x,z)-plane	$\bar{I}_1 = \frac{1}{2}\mu_{13}^{-\mu_{33}} + \frac{\kappa_2 (\delta_1 - \delta_2)}{\epsilon^2 K_1} K_1$	$\mu_{11}^{-\mu_{13} + \mu_{33}}$	-
	$\bar{I}_2 = 0, \quad \bar{I}_3 = K_1 - \bar{I}_1$	$2\mu_{13} \bar{I}_1 \bar{I}_3$	1, 3
	$\bar{\theta}_1 = \{ \kappa_1 + \epsilon^2 [ \frac{1}{2}\mu_{13} K_1 + (\mu_{11} - \frac{1}{2}\mu_{13}) \bar{I}_1 ] \} t + \bar{\theta}_1(0)$	$\frac{\tau_2 (\zeta_2 + \epsilon_2) - \sigma_2 (\eta_2 + \xi_2)}{\mu_{11}^{-\mu_{13} + \mu_{33}}} K_1$	11
out-phase	$\bar{\theta}_3 = \bar{\theta}_1 + \frac{\pi}{2} \quad \text{or} \quad \bar{\theta}_3 = \bar{\theta}_1 - \frac{\pi}{2}$	$\frac{\tau_3 (\zeta_3 + \epsilon_3) - \sigma_3 (\eta_3 + \xi_3)}{\mu_{11}^{-\mu_{13} + \mu_{33}}} K_1$	12
8. inclined linear orbits in (v,z)-plane	$\bar{I}_2 = \frac{3}{2}\mu_{23}^{-\mu_{33}} - \frac{\kappa_2 \delta_2}{\epsilon^2 K_1} K_1$	$\mu_{22}^{-3\mu_{23} + \mu_{33}}$	-
	$\bar{I}_1 = 0, \quad \bar{I}_3 = K_1 - \bar{I}_2$	$-2\mu_{23} \bar{I}_2 \bar{I}_3$	2, 3
	$\bar{\theta}_2 = \{ \kappa_2 + \epsilon^2 [ \frac{3}{2}\mu_{23} K_1 + (\mu_{22} - \frac{3}{2}\mu_{23}) \bar{I}_2 ] \} t + \bar{\theta}_2(0)$	$\frac{\sigma_1 \eta_1 - \tau_1 \xi_1}{\mu_{22}^{-3\mu_{23} + \mu_{33}}} K_1$	10
in-phase	$\bar{\theta}_3 = \bar{\theta}_2 \quad \text{or} \quad \bar{\theta}_3 = \bar{\theta}_2 + \pi$	$\frac{\sigma_3 \eta_3 - \tau_3 \xi_3}{\mu_{22}^{-3\mu_{23} + \mu_{33}}} K_1$	12
9. elliptic orbits in (y,z)-plane	$\bar{I}_2 = \frac{1}{2}\mu_{23}^{-\mu_{33}} - \frac{\kappa_2 \delta_2}{\epsilon^2 K_1} K_1$	$\mu_{22}^{-\mu_{23} + \mu_{33}}$	-
	$\bar{I}_1 = 0, \quad \bar{I}_3 = K_1 - \bar{I}_2$	$2\mu_{23} \bar{I}_2 \bar{I}_3$	2, 3
	$\bar{\theta}_2 = \{ \kappa_2 + \epsilon^2 [ \frac{1}{2}\mu_{23} K_1 + (\mu_{22} - \frac{1}{2}\mu_{23}) \bar{I}_2 ] \} t + \bar{\theta}_2(0)$	$\frac{\sigma_2 \eta_2 - \tau_2 \xi_2}{\mu_{22}^{-\mu_{23} + \mu_{33}}} K_1$	11
out-phase	$\bar{\theta}_3 = \bar{\theta}_2 + \frac{\pi}{2} \quad \text{or} \quad \bar{\theta}_3 = \bar{\theta}_2 - \frac{\pi}{2}$	$\frac{\sigma_4 \eta_4 - \tau_4 \xi_4}{\mu_{22}^{-\mu_{23} + \mu_{33}}} K_1$	13

Table 3. Properties of the three-dimensional periodic orbits that cross a principal axis.

ORBIT FAMILY	ACTIONS & ANGLES	EIGENVALUES OF CHARACTERISTIC MATRIX	BIFURCATIONS WITH FAMILY
10. linear orbits inclined to all principal axes	$\bar{I}_1 = \frac{\tau_1 \xi_1^{-\sigma_1} n_1}{\zeta_1 n_1 - \xi_1^2} K_1$ $\bar{I}_2 = \frac{\sigma_1 (n_1 + \xi_1) - \tau_1 (\zeta_1 + \xi_1)}{\zeta_1 n_1 - \xi_1^2} K_1$ $\bar{I}_3 = K_1 - \bar{I}_1 - \bar{I}_2$ $\bar{\theta}_1 = \{ \kappa_1 + \epsilon^2 [ (\mu_{11} - \frac{3}{2} \mu_{13}) \bar{I}_1 + \frac{3}{2} (\mu_{12} - \mu_{13}) \bar{I}_2 + \frac{3}{2} \mu_{13} K_1 ] \} t + \bar{\theta}_1(0)$	$\frac{1}{2} [\zeta_1 + n_1] \pm \frac{1}{2} \sqrt{(\zeta_1 - n_1)^2 + 4\xi_1^2}$ $\mu_{12} \bar{I}_1 \bar{I}_2 + 2\mu_{13} \bar{I}_1 \bar{I}_3 + \mu_{23} \bar{I}_2 \bar{I}_3$ $\pm \sqrt{(\mu_{12} \bar{I}_1 - \mu_{23} \bar{I}_3)^2 \bar{I}_2^2 + 4\mu_{13}^2 \bar{I}_1^2 \bar{I}_3^2}$	-
in-phase	$\bar{\theta}_2 = \bar{\theta}_1$ or $\bar{\theta}_2 = \bar{\theta}_1 + \pi$		4, 6, 8, 14
out-phase	$\bar{\theta}_3 = \bar{\theta}_2$ or $\bar{\theta}_3 = \bar{\theta}_2 + \pi$		
11. elliptic plane tilted orbits crossing z-axis	$\bar{I}_1 = \frac{\tau_2 \xi_2^{-\sigma_2} n_2}{\zeta_2 n_2 - \xi_2^2} K_1$ $\bar{I}_2 = \frac{\sigma_2 (n_2 + \xi_2) - \tau_2 (\zeta_2 + \xi_2)}{\zeta_2 n_2 - \xi_2^2} K_1$ $\bar{I}_3 = K_1 - \bar{I}_1 - \bar{I}_2$ $\bar{\theta}_1 = \{ \kappa_1 + \epsilon^2 [ (\mu_{11} - \frac{1}{2} \mu_{13}) \bar{I}_1 + (\frac{3}{2} \mu_{12} - \frac{1}{2} \mu_{13}) \bar{I}_2 + \frac{1}{2} \mu_{13} K_1 ] \} t + \bar{\theta}_1(0)$	$\frac{1}{2} [\zeta_2 + n_2] \pm \frac{1}{2} \sqrt{(\zeta_2 - n_2)^2 + 4\xi_2^2}$ $\mu_{12} \bar{I}_1 \bar{I}_2 - 2\mu_{13} \bar{I}_1 \bar{I}_3 - \mu_{23} \bar{I}_2 \bar{I}_3$ $\pm \sqrt{(\mu_{12} \bar{I}_1 + \mu_{23} \bar{I}_3)^2 \bar{I}_2^2 + 4\mu_{13}^2 \bar{I}_1^2 \bar{I}_3^2}$	-
in-phase	$\bar{\theta}_2 = \bar{\theta}_1$ or $\bar{\theta}_2 = \bar{\theta}_1 + \pi$		4, 7, 9, 14
out-phase	$\bar{\theta}_3 = \bar{\theta}_2 + \frac{\pi}{2}$ or $\bar{\theta}_3 = \bar{\theta}_2 - \frac{\pi}{2}$		
12. elliptic plane tilted orbits crossing x-axis	$\bar{I}_1 = \frac{\tau_3 \xi_3^{-\sigma_3} n_3}{\zeta_3 n_3 - \xi_3^2} K_1$ $\bar{I}_2 = \frac{\sigma_3 (n_3 + \xi_3) - \tau_3 (\zeta_3 + \xi_3)}{\zeta_3 n_3 - \xi_3^2} K_1$ $\bar{I}_3 = K_1 - \bar{I}_1 - \bar{I}_2$ $\bar{\theta}_1 = \{ \kappa_1 + \epsilon^2 [ (\mu_{11} - \frac{1}{2} \mu_{13}) \bar{I}_1 + \frac{1}{2} (\mu_{12} - \mu_{13}) \bar{I}_2 + \frac{1}{2} \mu_{13} K_1 ] \} t + \bar{\theta}_1(0)$	$\frac{1}{2} [\zeta_3 + n_3] \pm \sqrt{(\zeta_3 - n_3)^2 + 4\xi_3^2}$ $-\mu_{12} \bar{I}_1 \bar{I}_2 - 2\mu_{13} \bar{I}_1 \bar{I}_3 + \mu_{23} \bar{I}_2 \bar{I}_3$ $\pm \sqrt{(\mu_{12} \bar{I}_1 + \mu_{23} \bar{I}_3)^2 \bar{I}_2^2 + 4\mu_{13}^2 \bar{I}_1^2 \bar{I}_3^2}$	-
out-phase	$\bar{\theta}_2 = \bar{\theta}_1 + \frac{\pi}{2}$ or $\bar{\theta}_2 = \bar{\theta}_1 - \frac{\pi}{2}$		5, 7, 8, 14
in-phase	$\bar{\theta}_3 = \bar{\theta}_2$ or $\bar{\theta}_3 = \bar{\theta}_2 + \pi$		
13. elliptic plane tilted orbits crossing y-axis	$\bar{I}_1 = \frac{\tau_4 \xi_4^{-\sigma_4} n_4}{\zeta_4 n_4 - \xi_4^2} K_1$ $\bar{I}_2 = \frac{\sigma_4 (n_4 + \xi_4) - \tau_4 (\zeta_4 + \xi_4)}{\zeta_4 n_4 - \xi_4^2} K_1$ $\bar{I}_3 = K_1 - \bar{I}_1 - \bar{I}_2$ $\bar{\theta}_1 = \{ \kappa_1 + \epsilon^2 [ (\mu_{11} - \frac{3}{2} \mu_{13}) \bar{I}_1 + (\frac{1}{2} \mu_{12} - \frac{3}{2} \mu_{13}) \bar{I}_2 + \frac{3}{2} \mu_{13} K_1 ] \} t + \bar{\theta}_1(0)$	$\frac{1}{2} [\zeta_4 + n_4] \pm \sqrt{(\zeta_4 - n_4)^2 + 4\xi_4^2}$ $-\mu_{12} \bar{I}_1 \bar{I}_2 + 2\mu_{13} \bar{I}_1 \bar{I}_3 - \mu_{23} \bar{I}_2 \bar{I}_3$ $\pm \sqrt{(\mu_{12} \bar{I}_1 - \mu_{23} \bar{I}_3)^2 \bar{I}_2^2 + 4\mu_{13}^2 \bar{I}_1^2 \bar{I}_3^2}$	-
out-phase	$\bar{\theta}_2 = \bar{\theta}_1 + \frac{\pi}{2}$ or $\bar{\theta}_2 = \bar{\theta}_1 - \frac{\pi}{2}$		5, 6, 9, 14
out-phase	$\bar{\theta}_3 = \bar{\theta}_2 + \frac{\pi}{2}$ or $\bar{\theta}_3 = \bar{\theta}_2 - \frac{\pi}{2}$		



eigenvalue in column 3 of Table 1 is zero for  $K_1 = \kappa_2 \delta_1 / \varepsilon^2 (3/2 \mu_{12} - \mu_{11})$ . At this energy orbit family 4, to be discussed in Section 3.4, branches off the  $x_1$ -axis.

Next to each eigenvalue in Table 1 we give the number of the orbit family with which there is a bifurcation at the energy where this eigenvalue is zero. It can be seen that the axial oscillations may have bifurcations with the orbits 4, 5, 6, 7, 8 and 9. These are the simple periodic orbits that remain in the principal planes (Section 3.4). It should be noted, however, that in a given potential a bifurcation will occur only if the energy of bifurcation is non-negative; if it is so large that the first-order approximation is inadequate, the bifurcation need not occur either.

### 3.4 PERIODIC ORBITS IN A PRINCIPAL PLANE (4–9)

From the fact that  $V = V(x_1^2, x_2^2, x_3^2)$  it follows that a star moving in one of the principal planes will remain in it, so that motion is two-dimensional. This is confirmed by the averaged equations of motion: for  $\bar{I}_k = 0$  also  $\dot{\bar{I}}_k = 0$  ( $k = 1, 2, 3$ ). We consider the  $(x_1, x_2)$ -plane in some detail. The other principal planes can be treated in a similar manner.

For  $\bar{I}_3 = 0$  the angle  $\bar{\theta}_3$  is irrelevant. The four remaining equations of motion are identical to the averaged equations of motion belonging to the two-dimensional Hamiltonian  $H_{1:1}$  equal to

$$H_{1:1} = \kappa_1 \bar{I}_1 + \kappa_2 \bar{I}_2 + 1/2 \varepsilon^2 [\mu_{11} \bar{I}_1^2 + 2\mu_{12} \bar{I}_1 \bar{I}_2 + \mu_{22} \bar{I}_2^2 + \mu_{12} \bar{I}_1 \bar{I}_2 \cos(2\bar{\theta}_1 - 2\bar{\theta}_2)]. \quad (29)$$

$H_{1:1}$  describes the 1:1 resonance in the  $(x_1, x_2)$ -plane, which has been discussed by ZM. We briefly show how the simple periodic orbits are derived, and then consider their three-dimensional stability.

It is evident from the averaged equations of motion that  $\dot{\bar{I}}_1 = \dot{\bar{I}}_2 = 0$  not only when  $\bar{I}_1 = 0$  or  $\bar{I}_2 = 0$ , which leads to the normal modes already discussed, but also in case  $\sin(2\bar{\theta}_1 - 2\bar{\theta}_2) = 0$ , hence  $2\bar{\theta}_1 - 2\bar{\theta}_2 = 0$  or  $\pi$ . This leads to additional simple periodic orbits if  $\bar{\theta}_1 - \bar{\theta}_2 = 0$ , or

$$1/2 \mu_{12} (\bar{I}_2 - \bar{I}_1) \cos(2\bar{\theta}_1 - 2\bar{\theta}_2) + (\mu_{11} - \mu_{12}) \bar{I}_1 + (\mu_{12} - \mu_{22}) \bar{I}_2 + \frac{\kappa_2 \delta_1}{\varepsilon^2} = 0, \quad (30)$$

where  $\cos(2\bar{\theta}_1 - 2\bar{\theta}_2)$  is either +1 or -1. By means of  $\bar{I}_1 + \bar{I}_2 = K_1$  we eliminate  $\bar{I}_2$  and are left with a linear equation for  $\bar{I}_1$  as function of  $K_1$ .

For  $\cos(2\bar{\theta}_1 - 2\bar{\theta}_2) = 1$  equation (30) becomes

$$(\mu_{11} - 3\mu_{12} + \mu_{22}) \bar{I}_1 + (3/2 \mu_{12} - \mu_{22}) K_1 + \frac{\kappa_2 \delta_1}{\varepsilon^2} = 0. \quad (31)$$

If  $\mu_{11} - 3\mu_{12} + \mu_{22} \neq 0$  then (31) immediately gives  $\bar{I}_1$  in terms of  $K_1$  and the coefficients of the potential. The equation for  $\bar{\theta}_1$  can then also be integrated. The complete solution is given in Table 2, and is denoted as orbit family 4. The orbits are straight lines through the origin inclined to the principal axes. Orbits with  $\bar{\theta}_1 - \bar{\theta}_2 = 0$  and those with  $\bar{\theta}_1 - \bar{\theta}_2 = \pi$  are mirror images of each other with respect to the principal axes (Fig. 1). The angle of inclination depends on energy. The orbits exist for all energies for which  $0 \leq \bar{I}_1 \leq K_1$ , i.e. for all  $K_1$  for which

$$0 \leq \frac{3/2 \mu_{12} - \mu_{22} + \kappa_2 \delta_1 / \varepsilon^2 K_1}{-\mu_{11} + 3\mu_{12} - \mu_{22}} K_1 \leq 1. \quad (32)$$

If, in (31),  $\mu_{11} - 3\mu_{12} + \mu_{22}$  happens to be zero for the given potential  $V$ , then there is no solution, except for  $K_1$  such that also  $(3/2 \mu_{12} - \mu_{22}) K_1 + \kappa_2 \delta_1 / \varepsilon^2 = 0$ . For this value of  $K_1$  all values of  $\bar{I}_1$  between 0 and  $K_1$  are permitted, assuming that  $K_1 > 0$ , so that every straight line through the centre with this energy is an orbit. Verhulst (1979) calls this a *global bifurcation*.

When we take  $\cos(2\bar{\theta}_1 - 2\bar{\theta}_2) = -1$  in equation (30), so that  $\bar{\theta}_1 - \bar{\theta}_2 = \pm\pi/2$ , we find the solution

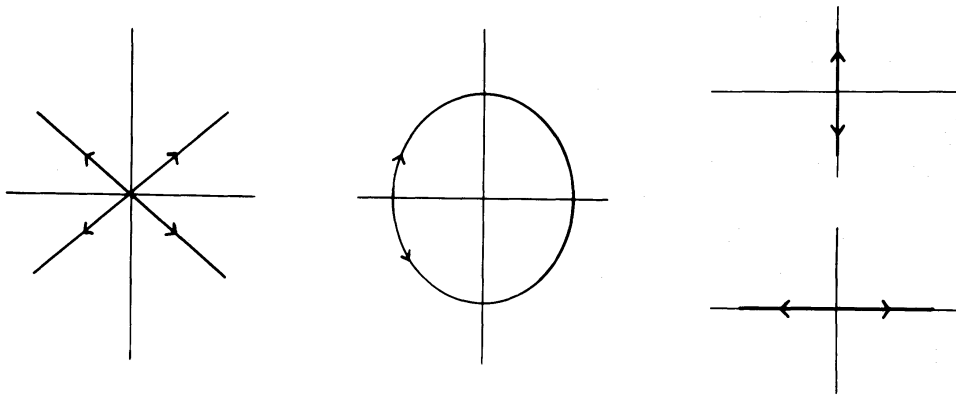


Figure 1. Simple periodic orbits in a principal plane.

labelled as orbit family 5 in Table 2. The orbits are identical ellipses centred on the origin and traversed in opposite directions for the two values of  $\bar{\theta}_1 - \bar{\theta}_2$  (Fig. 1). They exist for all energies  $K_1$  such that  $0 \leq \bar{I}_1 \leq K_1$ . If  $\mu_{11} - \mu_{12} + \mu_{22}$  turns out to be zero for the given potential then there is a global bifurcation at  $K_1 = \kappa_2 \delta_1 / (\mu_{22} - \frac{1}{2} \mu_{12}) \varepsilon^2$ .

Both orbit families 4 and 5 can be thought of as combinations of the axial orbits for which the amplitudes have exactly the values for which the frequencies of these oscillations are equal at the given energy. For the inclined linear orbits the frequencies are matched in phase, and for the elliptic closed orbits they are matched out of phase.

Although motion in the  $(x_1, x_2)$ -plane is two-dimensional, the orbits may be stable or unstable not only to perturbations in this plane, but also to perturbations perpendicular to it. For the stability analysis we transform  $(\bar{I}_3, \bar{\theta}_3)$  to comoving variables  $(\bar{q}_3, \bar{p}_3)$  by means of (9) and (10). Furthermore, in the  $(x_1, x_2)$ -plane we transform to new action-angle variables  $(\bar{J}_1, \bar{\chi}_1, \bar{J}_2, \bar{\chi}_2)$  defined by the generating function  $F_2(\bar{J}_i, \bar{\theta}_i)$  (Goldstein 1980):

$$F_2(\bar{J}_i, \bar{\theta}_i) = \bar{J}_1(\bar{\theta}_1 - \bar{\theta}_2) + \bar{J}_2 \bar{\theta}_2, \quad (33)$$

so that  $\bar{\chi}_i = \partial F_2 / \partial \bar{J}_i$  and  $\bar{I}_i = \partial F_2 / \partial \bar{\theta}_i$ . Thus,

$$\begin{aligned} \bar{\chi}_1 &= \bar{\theta}_1 - \bar{\theta}_2, & \bar{J}_1 &= \bar{I}_1, \\ \bar{\chi}_2 &= \bar{\theta}_2, & \bar{J}_2 &= \bar{I}_1 + \bar{I}_2. \end{aligned} \quad (34)$$

At fixed energy  $K_1 = \bar{J}_2 + \frac{1}{2}(\bar{q}_3^2 + \bar{p}_3^2)$  the Hamiltonian  $H_{1:1:1}$  depends on the four variables  $\bar{J}_1, \bar{\chi}_1, \bar{q}_3$ , and  $\bar{p}_3$ .

The characteristic matrix for orbit family 4 is

$$\begin{bmatrix} \mu_{11} - 3\mu_{12} + \mu_{22} & 0 & 0 & 0 \\ 0 & -2\mu_{12}\bar{I}_1\bar{I}_2 & 0 & 0 \\ 0 & 0 & \alpha(1 + \frac{1}{2}\cos 2\bar{\theta}_2) + \beta & -\frac{1}{2}\alpha \sin 2\bar{\theta}_2 \\ 0 & 0 & -\frac{1}{2}\alpha \sin 2\bar{\theta}_2 & \alpha(1 - \frac{1}{2}\cos 2\bar{\theta}_2) + \beta \end{bmatrix} \quad (35)$$

where

$$\begin{aligned} \alpha &= (\mu_{13} - \mu_{23})\bar{I}_1 + \mu_{23}K_1, \\ \beta &= (\mu_{22} - \frac{3}{2}\mu_{12})\bar{I}_1 - \mu_{22}K_1 + \frac{\kappa_2 \delta_2}{\varepsilon^2}, \end{aligned} \quad (36)$$

and  $\bar{\chi}_2 = \bar{\theta}_2$  can be expressed in terms of  $K_1$  by means of the expressions given in Table 2, although this is not necessary for the calculation of the eigenvalues. Column 3 of Table 2 gives the eigenvalues.

The first eigenvalue depends only on the  $\mu_{ij}$ , not on the energy. It is zero when there is a global bifurcation, so that in that case all orbits are marginally unstable. The second eigenvalue depends on the values of  $\bar{I}_1$  and  $\bar{I}_2 = K_1 - \bar{I}_1$  on the orbit. If  $\bar{I}_2 = 0$  the eigenvalue is zero and the orbit is an  $x_1$ -normal mode. Hence there is a bifurcation with orbit family 1, in agreement with the results of Section 3.3. Similarly, there is a bifurcation with family 2 when  $\bar{I}_1 = 0$ . The remaining two eigenvalues are related to the perpendicular (in)stability. They are given in Table 2 as a factor times  $K_1$  with the factor a function of coefficients  $\zeta_k, \eta_k, \xi_k, \sigma_k$  and  $\tau_k$  ( $k=1, 2, 3, 4$ ). These are abbreviations for linear expressions in the  $\mu_{ij}$  and are given in the Appendix. Only  $\sigma_k$  and  $\tau_k$  depend also on  $\delta_1, \delta_2$  and  $K_1$ . When  $\lambda_3$  or  $\lambda_4$  equals zero there is a bifurcation with a three-dimensional periodic orbit family. These are indicated in Table 2 by their number.

The (in)stability of orbit family 5 can be analysed in the same way. The result for this orbit and also for the linear inclined orbits (6, 8) and elliptic closed orbits (7, 9) in the other two principal planes are all presented in Table 2. It is evident that these six orbit families have bifurcations with the three axial orbits and with four distinct three-dimensional orbits (10, 11, 12 and 13).

### 3.5 THREE-DIMENSIONAL PERIODIC ORBITS THAT INTERSECT A PRINCIPAL AXIS (10–13)

Now we turn to the solutions of (21) that have all three actions constant but non-zero. Clearly,  $\dot{\bar{I}}_1 = \dot{\bar{I}}_2 = 0$  (and hence  $\dot{\bar{I}}_3 = 0$ ) is possible if simultaneously  $\sin(2\bar{\theta}_1 - 2\bar{\theta}_2) = 0$  and  $\sin(2\bar{\theta}_2 - 2\bar{\theta}_3) = 0$ . This requires both  $2\bar{\theta}_1 - 2\bar{\theta}_2$  and  $2\bar{\theta}_2 - 2\bar{\theta}_3$  to be equal to 0 or  $\pi$ . Thus, we have to solve  $\bar{\theta}_1 - \bar{\theta}_2 = 0$  and  $\bar{\theta}_2 - \bar{\theta}_3 = 0$  where we have to distinguish four cases:  $(2\bar{\theta}_1 - 2\bar{\theta}_2, 2\bar{\theta}_2 - 2\bar{\theta}_3) = (0, 0), (0, \pi), (\pi, 0)$  and  $(\pi, \pi)$ .

Substitution of these values of the phase-differences in the equations for  $\dot{\bar{\theta}}_1 - \dot{\bar{\theta}}_2$  and  $\dot{\bar{\theta}}_2 - \dot{\bar{\theta}}_3$ , and elimination of  $\bar{I}_3$  by means of (22), produces in all four cases two linear equations:

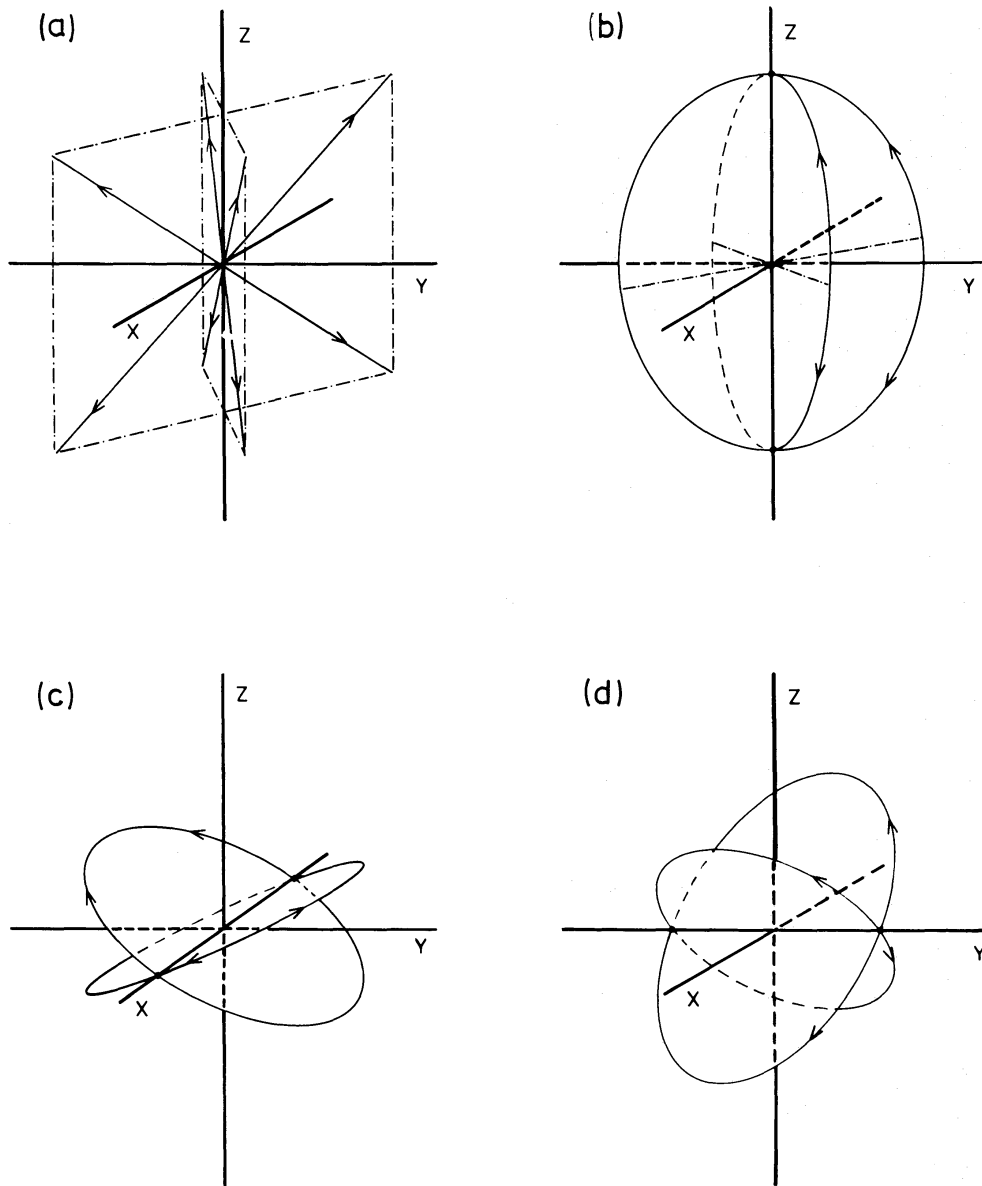
$$\begin{aligned} (\zeta_k + \xi_k)\bar{I}_1 + \xi_k\bar{I}_2 &= -\sigma_k K_1, \\ (\eta_k + \xi_k)\bar{I}_1 + \eta_k\bar{I}_2 &= -\tau_k K_1, \quad k=1, 2, 3, 4. \end{aligned} \tag{37}$$

Here  $\zeta_k, \eta_k, \xi_k, \sigma_k$  and  $\tau_k$  are the expressions given in the Appendix, and the four cases are labelled by  $k$ .

If  $\zeta_k \eta_k - \xi_k^2 \neq 0$  the two equations (37) are independent and may be solved to give  $\bar{I}_1$  and  $\bar{I}_2$  as functions of  $K_1$ . The angles can then be found immediately from the averaged equations of motion. The solutions for the four cases are presented in Table 3 as orbit families 10, 11, 12 and 13. They exist for all energies for which  $0 \leq \bar{I}_1 \leq \bar{I}_1 + \bar{I}_2 \leq K_1$ .

The orbits of family 10 are straight lines through the centre, inclined to all principal axes (Fig. 2a). The angles of inclination depend on the energy. For these orbits  $\bar{\theta}_1 - \bar{\theta}_2 = 0$  or  $\pi$  and  $\bar{\theta}_2 - \bar{\theta}_3 = 0$  or  $\pi$  hence  $\bar{\theta}_1 - \bar{\theta}_3 = 0$  or  $\pi$ . Thus, all three normal modes are matched in-phase. At a given energy four mirror images occur, depending on the combination of the values of the phase-differences.

The orbits of families 11, 12 and 13 are identical ellipses, lying in a plane containing one of the principal axes and tilted relative to two principal planes (Fig. 2b, c and d). The inclination angle of the orbital plane and the eccentricity of the orbits change with energy. The mirror image of each ellipse with respect to a principal plane also occurs. The ellipses are traversed in both directions, depending on the values of the phase differences  $\bar{\theta}_1 - \bar{\theta}_2$  and  $\bar{\theta}_2 - \bar{\theta}_3$ . Family 11 crosses the  $x_3$ -axis, Family 12 crosses the  $x_1$ -axis and Family 13 crosses the  $x_2$ -axis. In all three cases the orbits can be thought of as combinations of the three normal modes in which two oscillations are matched in-phase and the third is out of phase.



**Figure 2.** Three-dimensional simple periodic orbits that cross a principal axis. (a) Family 10, (b) family 11, (c) family 12, (d) family 13.

The quantities  $\zeta_k$ ,  $\eta_k$  and  $\xi_k$  depend only on the parameters in the expansion (2) of the potential. If it turns out that  $\zeta_k \eta_k - \xi_k^2 = 0$ , then a global bifurcation may occur.

For the stability analysis we transform to new action-angle variables  $(\bar{J}_i, \bar{\psi}_i)$  by means of the generating function

$$F_2(\bar{J}_i, \bar{\theta}_i) = \bar{J}_1(\bar{\theta}_1 - \bar{\theta}_2) + \bar{J}_2(\bar{\theta}_2 - \bar{\theta}_3) + \bar{J}_3 \bar{\theta}_3. \quad (38)$$

Then  $\bar{\psi}_i = \partial F_2 / \partial \bar{J}_i$  and  $\bar{I}_i = \partial F_2 / \partial \bar{\theta}_i$  so that we find

$$\begin{aligned} \bar{\psi}_1 &= \bar{\theta}_1 - \bar{\theta}_2, & \bar{J}_1 &= \bar{I}_1, \\ \bar{\psi}_2 &= \bar{\theta}_2 - \bar{\theta}_3, & \bar{J}_2 &= \bar{I}_1 + \bar{I}_2, \\ \bar{\psi}_3 &= \bar{\theta}_3, & \bar{J}_3 &= \bar{I}_1 + \bar{I}_2 + \bar{I}_3. \end{aligned} \quad (39)$$

Evidently,  $\bar{J}_3 = K_1$ . The characteristic matrix of orbit family 10 is

$$\begin{bmatrix} \xi_1 & \xi_1 & 0 & 0 \\ \xi_1 & \eta_1 & 0 & 0 \\ 0 & 0 & -2\bar{I}_1(\mu_{12}\bar{I}_2 + \mu_{13}\bar{I}_3) & -2\mu_{13}\bar{I}_1\bar{I}_3 \\ 0 & 0 & -2\mu_{13}\bar{I}_1\bar{I}_3 & -2\bar{I}_3(\mu_{13}\bar{I}_1 + \mu_{23}\bar{I}_2) \end{bmatrix} \quad (40)$$

The eigenvalues are given in Table 3. The first two, corresponding to the matrix in the upper left corner of (40), depend only on the  $\mu_{ij}$ . Their product is  $\xi_1\eta_1 - \xi_1^2$ , showing that in case of a global bifurcation the orbits are marginally unstable. The product of the other two eigenvalues is

$$\lambda_3\lambda_4 = 4\bar{I}_1\bar{I}_2\bar{I}_3(\mu_{12}\mu_{13}\bar{I}_1 + \mu_{12}\mu_{23}\bar{I}_2 + \mu_{13}\mu_{23}\bar{I}_3). \quad (41)$$

This is zero whenever one action is zero. In that case the orbit collapses on one of the principal planes and becomes one of the inclined linear orbits we encountered in Section 3.4. Thus, orbit family 10 may have bifurcations with families 4, 6 and 8. However, the product (41) is also zero when the term in brackets is zero. At the corresponding energy there is a bifurcation with family 14.

The (in)stability of orbit families 11, 12 and 13 can be investigated in the same way. Each of these orbits may have three bifurcations with orbits in the principal planes, and one with orbit family 14. The results are summarized in Table 3.

### 3.6 THREE-DIMENSIONAL PERIODIC ORBITS THAT INTERSECT NO PRINCIPAL AXIS (14)

From the results obtained in Section 3.5 we conclude that we have not yet found all simple periodic solutions of equations (21). Although it may not be obvious, these transcendental equations may be solved explicitly.

We assume  $\bar{I}_1 \neq 0$ ,  $\bar{I}_2 \neq 0$ ,  $\bar{I}_3 \neq 0$ ,  $\sin(2\bar{\theta}_1 - 2\bar{\theta}_2) \neq 0$  and  $\sin(2\bar{\theta}_2 - 2\bar{\theta}_3) \neq 0$ . If one of these quantities is zero the equations simplify and we obtain the solutions 1–13 already presented. By equation (22) only two of the three equations  $\dot{\bar{I}}_1 = 0$ ,  $\dot{\bar{I}}_2 = 0$  and  $\dot{\bar{I}}_3 = 0$  are independent. We use these to eliminate the angles. From  $\dot{\bar{I}}_1 + \dot{\bar{I}}_3 = 0$  we find

$$\sin(2\bar{\theta}_2 - 2\bar{\theta}_3) = (\mu_{12}\bar{I}_1 / \mu_{23}\bar{I}_3) \sin(2\bar{\theta}_1 - 2\bar{\theta}_2). \quad (42)$$

Substitution of this result in the equation  $\dot{\bar{I}}_1 = 0$ , and use of elementary goniometric relations yields, after some straightforward algebra,

$$\begin{aligned} \cos(2\bar{\theta}_1 - 2\bar{\theta}_2) &= \frac{-\mu_{12}^2\mu_{13}^2\bar{I}_1^2 - \mu_{12}^2\mu_{23}^2\bar{I}_2^2 + \mu_{13}^2\mu_{23}^2\bar{I}_3^2}{2\mu_{12}^2\mu_{13}\mu_{23}\bar{I}_1\bar{I}_2}, \\ \cos(2\bar{\theta}_1 - 2\bar{\theta}_3) &= \frac{-\mu_{12}^2\mu_{13}^2\bar{I}_1 + \mu_{12}^2\mu_{23}^2\bar{I}_2^2 - \mu_{13}^2\mu_{23}^2\bar{I}_3^2}{2\mu_{12}\mu_{13}^2\mu_{23}\bar{I}_1\bar{I}_3}, \\ \cos(2\bar{\theta}_2 - 2\bar{\theta}_3) &= \frac{+\mu_{12}^2\mu_{13}^2\bar{I}_1^2 - \mu_{12}^2\mu_{23}^2\bar{I}_2^2 - \mu_{13}^2\mu_{23}^2\bar{I}_3^2}{2\mu_{12}\mu_{13}\mu_{23}^2\bar{I}_2\bar{I}_3}. \end{aligned} \quad (43)$$

Evidently, an additional solution of (21) with constant values of the action may exist, but this requires (43) to be valid and hence  $\dot{\bar{\theta}}_1 - \dot{\bar{\theta}}_2 = 0$  and  $\dot{\bar{\theta}}_2 - \dot{\bar{\theta}}_3 = 0$ . Substitution of (43) in the resulting equations, and elimination of  $\bar{I}_3$  by means of (22), produces two linear equations for  $\bar{I}_1$  and  $\bar{I}_2$ :

$$\begin{aligned} (\zeta_5 + \xi_5)\bar{I}_1 + \xi_5\bar{I}_2 &= -\sigma_5 K_1, \\ (\eta_5 + \xi_5)\bar{I}_1 + \eta_5\bar{I}_2 &= -\tau_5 K_1, \end{aligned} \quad (44)$$

where the coefficients  $\zeta_5$ ,  $\eta_5$ ,  $\xi_5$ ,  $\sigma_5$  and  $\tau_5$  are functions of the  $\mu_{ij}$ ;  $\sigma_5$  and  $\tau_5$  also depend on  $\delta_1$ ,  $\delta_2$  and  $K_1$ . The explicit expressions are given in the Appendix. If  $\zeta_5\eta_5 - \xi_5^2 \neq 0$  the two equations (44) are independent and may be solved for each value of  $K_1$ . The equations for the angles can then also be integrated. The result is

$$\begin{aligned}\bar{I}_1 &= \frac{\tau_5 \xi_5 - \sigma_5 \eta_5}{\zeta_5 \eta_5 - \xi_5^2} K_1, \\ \bar{I}_2 &= \frac{(\eta_5 + \xi_5) \sigma_5 - (\zeta_5 + \xi_5) \tau_5}{\zeta_5 \eta_5 - \xi_5^2} K_1, \\ \bar{I}_3 &= K_1 - \bar{I}_1 - \bar{I}_2, \\ \bar{\theta}_1 &= \left[ \kappa_1 + \varepsilon^2 \left\{ \left( \mu_{11} - \frac{\mu_{12} \mu_{13}}{2\mu_{23}} \right) \bar{I}_1 + \mu_{12} \bar{I}_2 + \mu_{13} \bar{I}_3 \right\} \right] t + \bar{\theta}_1(0), \\ \bar{\theta}_2 &= \left[ \kappa_2 + \varepsilon^2 \left\{ \mu_{12} \bar{I}_1 + \left( \mu_{22} - \frac{\mu_{12} \mu_{23}}{2\mu_{13}} \right) \bar{I}_2 + \mu_{23} \bar{I}_3 \right\} \right] t + \bar{\theta}_2(0), \\ \bar{\theta}_3 &= \left[ \kappa_3 + \varepsilon^2 \left\{ \mu_{13} \bar{I}_1 + \mu_{23} \bar{I}_2 + \left( \mu_{33} - \frac{\mu_{13} \mu_{23}}{2\mu_{12}} \right) \bar{I}_3 \right\} \right] t + \bar{\theta}_3(0).\end{aligned}\tag{45}$$

The solution (45) exists for all values of  $K_1$  such that  $0 \leq \bar{I}_1 \leq \bar{I}_1 + \bar{I}_2 \leq K_1$ , and such that the right-hand sides of (43) lie between  $-1$  and  $1$ . The values of the integration constants  $\bar{\theta}_1(0)$ ,  $\bar{\theta}_2(0)$  and  $\bar{\theta}_3(0)$  depend on the initial conditions. Once  $\bar{\theta}_1(0)$  is fixed, the other two can take only a limited number of values. For given  $\bar{I}_1$ ,  $\bar{I}_2$  and  $\bar{I}_3$  it follows from (43) that eight combinations of the phase differences can occur.

It should be noted that  $\zeta_5$ ,  $\eta_5$  and  $\xi_5$  depend only on the  $\mu_{ij}$ . If the potential  $V$  happens to be such that  $\zeta_5\eta_5 - \xi_5^2 = 0$  a global bifurcation may occur.

The orbit family 14 consists of four ellipses in tilted planes, each traversed in both directions, but crossing none of the three principal axes. The phase differences are unequal to  $0$ ,  $\pi/2$ ,  $\pi$  or

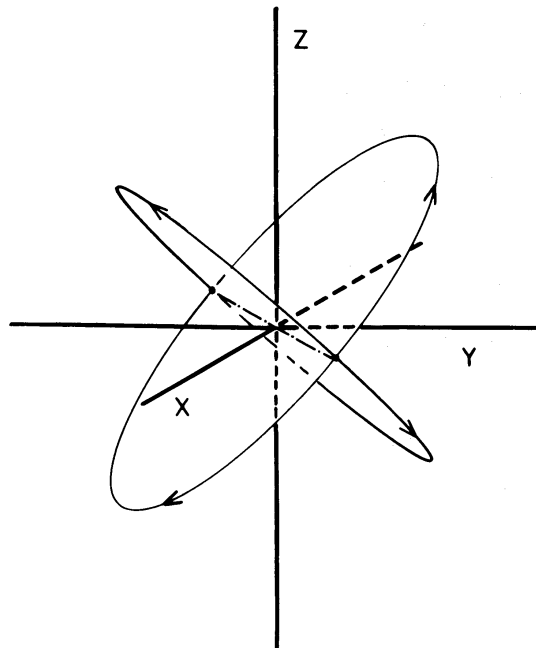


Figure 3. Elliptic plane tilted orbits crossing no principal axis (family 14).

$3\pi/2$ , and depend on energy, so that the inclination of the orbital plane and the orbital eccentricity are functions of energy. Fig. 3 shows two of these ellipses. The other two are their mirror images with respect to the principal planes.

The stability of family 14 can be investigated in the same way as was done in Section 3.5 for the families 10, 11, 12 and 13. In the present case all elements of the characteristic matrix are non-zero. The calculation of the eigenvalues, although straightforward, is very lengthy, and we have not carried it out. However, it is not difficult to show that the product of the four eigenvalues contains factors  $\sin^2(2\bar{\theta}_1-2\bar{\theta}_2)$  and  $\zeta_5\eta_5-\xi_5^2$ . When  $\sin(2\bar{\theta}_1-2\bar{\theta}_2)=0$  then, by (42) also  $\sin(2\bar{\theta}_2-2\bar{\theta}_3)=0$ , so that both  $2\bar{\theta}_1-2\bar{\theta}_2$  and  $2\bar{\theta}_2-2\bar{\theta}_3$  are 0 or  $\pi$ . As a result, at the energy for which  $\sin(2\bar{\theta}_1-2\bar{\theta}_2)=0$ , family 14 has a bifurcation with one of the orbits 10–13, depending on the values of  $2\bar{\theta}_1-2\bar{\theta}_2$  and  $2\bar{\theta}_2-2\bar{\theta}_3$ . There are no other zeros of  $\lambda_1\lambda_2\lambda_3\lambda_4$ , so that in a given potential the orbits 14 are either stable or unstable for all energies for which they exist. When there is a global bifurcation the orbits are marginally (un)stable.

### 3.7 SOME SPECIAL CASES

If  $\delta_1$  and  $\delta_2$  are not of  $O(\varepsilon^2)$  but are much larger, of  $O(1)$ , say, then it follows from the expressions for the simple periodic orbits given in Tables 1, 2 and 3, and in equation (45), that only the axial oscillations (1–3) exist. For the solutions 4–14 we find that  $\bar{I}_1/K_1$  and  $\bar{I}_2/K_1$  are of  $O(1/\varepsilon^2)$ , so that the condition for existence,  $0 \leq \bar{I}_1 \leq \bar{I}_1 + \bar{I}_2 \leq K_1$ , is not satisfied. This is in agreement with the results obtained in Section 2.3 for the case of no resonance.

The 1:1 resonances  $m=n$ ,  $l=n$  or  $m=l$  (Section 2.4) are recovered in a similar way. If  $\delta_1=O(\varepsilon^2)$  but  $\delta_2=O(1)$  there is a 1:1 resonance between the  $x_1$  and  $x_2$  oscillation ( $m=n$ ). We find that, in addition to the normal modes, only orbit families 4 and 5 may occur, in agreement with the results obtained by ZM. For the 1:1 resonance between the  $x_2$  and  $x_3$  oscillations ( $l=n$ ) we have  $\delta_2=O(\varepsilon^2)$  and  $\delta_1=O(1)$ , and only families 8 and 9 may occur in addition to the normal modes. Finally, if  $\delta_1$  and  $\delta_2$  are both of  $O(1)$ , but their difference is of  $O(\varepsilon^2)$ , there is a 1:1 resonance between the  $x_1$  and  $x_3$  oscillations ( $l=m$ ) and orbit families 6 and 7 may exist.

When there is an exact 1:1:1 resonance ( $\delta_1=\delta_2=0$ ), the expressions for all simple periodic orbit families have the form  $\bar{I}_1=f_k(\mu_{ij})K_1$ ,  $\bar{I}_2=g_k(\mu_{ij})K_1$  and  $\bar{I}_3$  follows from (22), where  $f_k$  and  $g_k$  ( $k=1, \dots, 14$ ) are functions of the parameters  $\mu_{ij}$  that define the potential. As a result, whether or not an orbit family occurs in a given potential depends only on the  $\mu_{ij}$ . When the family exists, it exists for all energies (within the range of validity of the first-order approximation). The potential studied by Martinet & de Zeeuw (1983) is an example with  $\delta_1=\delta_2=0$  in which all 14 families exist at all energies.

For spherical potentials not only  $\delta_1=\delta_2=0$ , but also

$$\mu_{11}=\mu_{22}=\mu_{33}=\frac{3}{2}\mu_{12}=\frac{3}{2}\mu_{13}=\frac{3}{2}\mu_{23}. \quad (46)$$

We have already seen that in this case  $K_2$  reduces to  $\frac{1}{2}\mu_{11}(K_1^2-\frac{1}{3}L^2)$ . We find from the Tables that the linear inclined orbits (4, 6, 8, 10) are all global bifurcations. Together with families 1, 2 and 3 they now are the radial zero angular momentum orbits that exist at every energy in a spherical potential. The remaining families 5, 7, 9, 11, 12, 13 and 14 reduce to the circular orbits that exist in each plane through the origin, at every energy. This case illustrates that a triaxial perturbation of a spherical potential destroys most of the infinite number of radial and circular orbits. At every energy only a few simple periodic orbits remain.

For the application to elliptical galaxies it is of interest to consider the case where  $V$  is nearly spherical (see also Section 4). Then  $\delta_1 \neq 0$ ,  $\delta_2 \neq 0$  and

$$\begin{aligned} \mu_{22}-\mu_{11} &= O(\varepsilon^2), & \mu_{33}-\mu_{11} &= O(\varepsilon^2), \\ \mu_{13}-\mu_{12} &= O(\varepsilon^2), & \mu_{23}-\mu_{12} &= O(\varepsilon^2), & \mu_{12}-\frac{2}{3}\mu_{11} &= O(\varepsilon^2). \end{aligned} \quad (47)$$

Inspection of the expressions given in Tables 1, 2 and 3, and in equation (45), reveals that the denominators of the solutions 4, 6, 8, 10–14 are all of  $O(\varepsilon^2)$ . Since the numerators are of  $O(1)$ , the condition for existence is satisfied for none of these. The only simple periodic orbits that exist in this case are the three normal modes, and, above a certain energy, the elliptic closed orbits 5, 7 and 9 in the three principal planes. We shall show in Section 4.5 that in this case it is possible to find a third integral.

#### 4 Application to Stäckel potentials

If the potential  $V$  is of Stäckel form, the equations of motion (1) can be separated, and hence can be solved by quadratures. We apply the results obtained in Sections 2 and 3, which are valid for a general triaxial potential  $V$  that has an expansion of the form (2), to this special case.

##### 4.1 STÄCKEL POTENTIALS

For a Stäckel potential  $V_S$  the equations of motion separate in ellipsoidal coordinates  $(\lambda, \mu, \nu)$ . These are defined as the roots for  $\tau$  of

$$\frac{x_1^2}{\tau+\alpha} + \frac{x_2^2}{\tau+\beta} + \frac{x_3^2}{\tau+\gamma} = 1, \quad (48)$$

where  $\alpha$ ,  $\beta$  and  $\gamma$  are constants, and  $-\gamma \leq \nu \leq -\beta \leq \mu \leq -\alpha \leq \lambda$ . The coordinate surfaces are ellipsoids ( $\lambda$ ), hyperboloids of one sheet ( $\mu$ ), and hyperboloids of two sheets ( $\nu$ ). The coordinates have foci at  $(x_1, x_2, x_3) = (0, \pm\sqrt{\beta-\alpha}, 0)$ ,  $(0, 0, \pm\sqrt{\gamma-\beta})$  and  $(0, 0, \pm\sqrt{\gamma-\alpha})$ . The Stäckel potentials are

$$V_S = -\frac{\zeta^*(\lambda)}{(\lambda-\mu)(\lambda-\nu)} - \frac{\eta^*(\mu)}{(\mu-\nu)(\mu-\lambda)} - \frac{\theta^*(\nu)}{(\nu-\lambda)(\nu-\mu)}, \quad (49)$$

where  $\zeta^*(\lambda)$ ,  $\eta^*(\mu)$  and  $\theta^*(\nu)$  are arbitrary functions. All other known non-rotating separable potentials can be derived from (48) and (49) by taking appropriate limits (e.g. Morse & Feshbach 1953; Lynden-Bell 1962).

Every orbit in a Stäckel potential possesses three exact independent isolating integrals of motion,  $H$ ,  $J$  and  $K$ , say. These can be written as (de Zeeuw & Lynden-Bell 1985)

$$\begin{aligned} H &= E_1 + E_2 + E_3, \\ J &= \frac{1}{2}L^2 - (\beta+\gamma)E_1 - (\gamma+\alpha)E_2 - (\alpha+\beta)E_3, \\ K &= -\frac{1}{2}\alpha L_x^2 - \frac{1}{2}\beta L_y^2 - \frac{1}{2}\gamma L_z^2 + \beta\gamma E_1 + \gamma\alpha E_2 + \alpha\beta E_3, \end{aligned} \quad (50)$$

where  $E_i = \frac{1}{2}\dot{x}_i^2 + V_i$ , and the  $V_i$  are expressions in  $\lambda$ ,  $\mu$  and  $\nu$  that also contain  $\zeta^*(\lambda)$ ,  $\eta^*(\mu)$  and  $\theta^*(\nu)$ .  $H$  is the Hamiltonian, and is equal to the total energy  $E$ . All three integrals are quadratic in the velocities.

Any function of  $H$ ,  $J$  and  $K$  is also an integral. Three useful combinations are

$$\begin{aligned} I_\alpha &= \alpha^2 H + \alpha J + K = (\alpha-\beta)(\alpha-\gamma)E_1 + \frac{1}{2}(\alpha-\beta)L_y^2 + \frac{1}{2}(\alpha-\gamma)L_z^2, \\ I_\beta &= \beta^2 H + \beta J + K = (\beta-\gamma)(\beta-\alpha)E_2 + \frac{1}{2}(\beta-\alpha)L_x^2 + \frac{1}{2}(\beta-\gamma)L_z^2, \\ I_\gamma &= \gamma^2 H + \gamma J + K = (\gamma-\beta)(\gamma-\alpha)E_3 + \frac{1}{2}(\gamma-\alpha)L_x^2 + \frac{1}{2}(\gamma-\beta)L_y^2. \end{aligned} \quad (51)$$

Since the equations of motion for  $V_S$  are separable, each orbit is the sum of three oscillations or rotations, one in each coordinate  $\lambda$ ,  $\mu$  and  $\nu$ . As a result, the orbits are confined – by the values of

the integrals of motion – between coordinate surfaces, i.e. they lie in volumes bounded by parts of ellipsoids and of hyperboloids of one and two sheets. The simple periodic orbits are the orbits for which motion is in one coordinate only.

Motion in smooth Stäckel potentials that increase monotonically with increasing radius – the ones relevant for elliptical galaxies – has been analysed by de Zeeuw (1984, 1985). There are six families of simple periodic orbits: three axial oscillations and three elliptic closed orbits, one family in each principal plane. The general orbits fill volumes of four distinct shapes.

#### 4.2 LOCAL FITTING

A Stäckel potential that is regular has an expansion around the origin of the form (2), in which, however, not all coefficients are independent. Following two-dimensional work by van de Hulst (1962), de Zeeuw & Lynden-Bell (1985) investigated to what order an arbitrary expansion (2), or (11), may be matched term by term with a similar expansion of a Stäckel potential  $V_S$ . They found that the terms proportional to  $x_1^{2n}$ ,  $x_2^{2n}$  and  $x_3^{2n}$  can be matched exactly to all orders, but constraints appear on the coefficients of the mixed terms. By properly choosing  $\zeta^*(\lambda)$ ,  $\eta^*(\mu)$  and  $\theta^*(\nu)$  in (49), and taking the ellipsoidal coordinate system defined by (48) such that [cf. (17)],

$$\beta - \alpha = \frac{\kappa_1^2 - \kappa_2^2}{\varepsilon^2 \mu_{12} \kappa_1 \kappa_2}, \quad \gamma - \beta = \frac{\kappa_2^2 - \kappa_3^2}{\varepsilon^2 \mu_{23} \kappa_2 \kappa_3}, \quad \alpha - \gamma = \frac{\kappa_3^2 - \kappa_1^2}{\varepsilon^2 \mu_{13} \kappa_3 \kappa_1}, \quad (52)$$

an exact match is possible up to and including quartic order, provided one condition is satisfied. Since  $\beta - \alpha + \gamma - \beta = \gamma - \alpha$ , it follows from (52) that the relation

$$\frac{\kappa_1^2 - \kappa_2^2}{\mu_{12} \kappa_1 \kappa_2} + \frac{\kappa_2^2 - \kappa_3^2}{\mu_{23} \kappa_2 \kappa_3} + \frac{\kappa_3^2 - \kappa_1^2}{\mu_{13} \kappa_3 \kappa_1} = 0, \quad (53)$$

must be valid. Thus, an expansion of  $V_S$  around the origin has the form (2), but with (53) as a relation between the coefficients of the quadratic and quartic terms. At higher order other relations occur.

The functions  $V_i$  that appear in the expressions for the integrals of motion admitted by  $V_S$  (Section 4.1) can also be expanded. de Zeeuw & Lynden-Bell (1985) find, for  $V_S$  of the form (11),

$$V_i = \frac{1}{2} \kappa_i^2 x_i^2 + \frac{1}{4} \varepsilon^2 \sum_{j=1}^3 a_{ij} x_i^2 x_j^2 + O(\varepsilon^4), \quad (54)$$

where, naturally, the coefficients satisfy (53), with the  $\mu_{ij}$  given in (17).

From these results it follows that, if for the expansion (11) of a general triaxial potential  $V$  the condition (53) is satisfied, we can so choose the higher order terms – departing from  $V$  at  $O(\varepsilon^4)$  – that the expansion converges to a separable one. The integrals of motion in  $V$  may then be approximated by the quadratic integrals of  $V_S$ , where the  $V_i$  are evaluated by means of (54). The orbits in  $V_S$  can be used as approximate orbits in  $V$ .

In the next sections we compare this *local fitting* with the results of the averaging method. This is useful for at least two reasons:

(i) If (53) is satisfied, local fitting produces at once *three* integrals. Their form is of help in the search for an additional integral of the averaged equations of motion (21) for the 1:1:1 resonance (Section 4.5).

(ii) We have no estimate of the accuracy of local fitting. The averaging method is a rigorous asymptotic method. It can be used to delineate the circumstances under which local fitting is applicable (Sections 4.3 and 4.4).

## 4.3 NO FIRST-ORDER RESONANCE

Consider a potential  $V$  with expansion (11) that satisfies the condition (53).  $V$  may be equal to some Stäckel potential  $V_S$ , but it may also differ from  $V_S$  by terms of sextic or higher order.

If  $\kappa_1$ ,  $\kappa_2$  and  $\kappa_3$  are such that there is no first-order resonance, then, according to the averaging method, the harmonic energies  $\kappa_1\bar{I}_1$ ,  $\kappa_2\bar{I}_2$  and  $\kappa_3\bar{I}_3$  are conserved separately, with an error of  $O(\varepsilon^2)$  on a time-scale of  $O(1/\varepsilon^2)$ . In this approximation all orbits are combinations of three uncoupled slightly anharmonic oscillations, and are bounded by rectangular boxes.

Local fitting produces the same results. Since  $\kappa_1/\kappa_2$  and  $\kappa_3/\kappa_2$  by assumption differ by amounts of  $O(1)$ , it follows from (52) that the quantities  $\beta-\alpha$ ,  $\gamma-\beta$  and  $\gamma-\alpha$  are all three of  $O(1/\varepsilon^2)$ . We find for the integrals  $I_\alpha$ ,  $I_\beta$  and  $I_\gamma$ :

$$\begin{aligned}\frac{I_\alpha}{(\alpha-\beta)(\alpha-\gamma)} &= E_1 + O(\varepsilon^2), \\ \frac{I_\beta}{(\beta-\gamma)(\beta-\alpha)} &= E_2 + O(\varepsilon^2), \\ \frac{I_\gamma}{(\gamma-\alpha)(\gamma-\beta)} &= E_3 + O(\varepsilon^2).\end{aligned}\tag{55}$$

Thus, close to the centre, the energies  $E_1$ ,  $E_2$  and  $E_3$  are approximately conserved separately. By using (5) and (54) we find

$$E_i = \kappa_i I_i + O(\varepsilon^2).\tag{56}$$

This shows that the integrals given by local fitting agree, within the expected errors, with those given by the method of averaging.

Since  $\beta-\alpha$ ,  $\gamma-\beta$  and  $\gamma-\alpha$  are large, the foci of the ellipsoidal coordinate system in which  $V_S$  is separable lie at a large distance from the centre. As a result, in the region close to the centre where the expansion (11) of  $V$  is accurate, and the averaging method is applicable, the ellipsoidal coordinates are nearly Cartesian. The orbits in  $V_S$  that remain close to the centre are all box orbits (de Zeeuw 1984, 1985). It follows that the boxes are nearly rectangular.

We conclude that, if the harmonic frequencies  $\kappa_1$ ,  $\kappa_2$  and  $\kappa_3$  are all different, then the descriptions of the motion in the centre as non-resonant perturbed harmonic motion or as unperturbed motion in a locally best-fitting Stäckel potential are equivalent to first order.

## 4.4 THE 1:1:1 RESONANCE

In first order a 1:1:1 resonance occurs when [cf. (13)]

$$\kappa_1/\kappa_2 = 1 + \delta_1, \quad \kappa_3/\kappa_2 = 1 + \delta_2,\tag{57}$$

and  $\delta_1$  and  $\delta_2$  are of  $O(\varepsilon^2)$ . In this case the actions  $\bar{I}_1$ ,  $\bar{I}_2$  and  $\bar{I}_3$  are not conserved separately, but we have derived two approximate integrals,  $K_1$  and  $K_2$ , given in equations (22) and (25). A total of 14 simple periodic orbit families may occur.

If (53) is satisfied,  $V$  can be fitted by a Stäckel potential  $V_S$  up to and including quartic terms in ellipsoidal coordinates defined by the expressions (52), which can be written as

$$\beta-\alpha = \frac{2\delta_1}{\varepsilon^2\mu_{12}}, \quad \gamma-\beta = \frac{-2\delta_2}{\varepsilon^2\mu_{23}}, \quad \alpha-\gamma = \frac{2(\delta_1-\delta_2)}{\varepsilon^2\mu_{13}},\tag{58}$$

where we have neglected terms of  $O(\varepsilon^2)$ . It is clear from (58) that the interfocal distances of the

ellipsoidal coordinates are now of  $O(1)$ . The foci may therefore lie inside the region close to the origin where the expansion (11) of  $V$  is accurate, and where the averaging method applies. The condition (53) that  $\beta - \alpha + \gamma - \beta = \gamma - \alpha$  becomes, within an error of  $O(\varepsilon^4)$ ,

$$\frac{\delta_1}{\mu_{12}} - \frac{\delta_2}{\mu_{23}} + \frac{\delta_2 - \delta_1}{\mu_{13}} = 0. \quad (59)$$

Both local fitting and averaging give the total energy as an integral, as it should be. For the other two integrals provided by local fitting we can choose any two independent combinations of  $I_\alpha$ ,  $I_\beta$  and  $I_\gamma$ . We take

$$\frac{\mu_{12}I_\alpha - \mu_{23}I_\gamma}{2(\gamma - \alpha)} = -\frac{1}{4}\mu_{12}L_z^2 - \frac{1}{4}\mu_{13}L_y^2 - \frac{1}{4}\mu_{23}L_x^2 + \frac{\delta_1}{\varepsilon^2}E_1 + \frac{\delta_2}{\varepsilon^2}E_3 + O(\varepsilon^2), \quad (60)$$

$$\frac{\mu_{12}I_\beta}{(\beta - \gamma)} = +\frac{1}{4}\mu_{12}L_z^2 + \frac{1}{4}\mu_{23}\frac{\delta_1}{\delta_2}L_x^2 + \frac{\delta_1}{\varepsilon^2}E_2 + O(\varepsilon^2).$$

Here we have used (59) to transform the coefficient of  $L_y^2$  to  $-\frac{1}{4}\mu_{13}$ . This, and the use of (58), introduces the error of  $O(\varepsilon^2)$ . By means of (5) we express these integrals in terms of action-angle variables  $(I_i, \theta_i)$ . Averaging of the result in the way described in Section 2.2 then gives

$$\frac{\mu_{12}\bar{I}_\alpha - \mu_{23}\bar{I}_\gamma}{2(\gamma - \alpha)} = -\frac{1}{4}\mu_{12}\bar{L}_z^2 - \frac{1}{4}\mu_{13}\bar{L}_y^2 - \frac{1}{4}\mu_{23}\bar{L}_x^2 + \frac{\kappa_2\delta_1}{\varepsilon^2}\bar{I}_1 + \frac{\kappa_2\delta_2}{\varepsilon^2}\bar{I}_3 + O(\varepsilon^2), \quad (61)$$

and

$$\frac{\mu_{12}\bar{I}_\beta}{2(\beta - \gamma)} = \frac{1}{4}\mu_{12}\bar{L}_z^2 + \frac{1}{4}\mu_{23}\frac{\delta_1}{\delta_2}\bar{L}_x^2 + \frac{\kappa_2\delta_1}{\varepsilon^2}\bar{I}_2 + O(\varepsilon^2). \quad (62)$$

A comparison of (61) with the expression (25) for the second integral  $K_2$  given by the averaging method shows that

$$K_2 = \frac{\mu_{12}\bar{I}_\alpha - \mu_{23}\bar{I}_\gamma}{2(\gamma - \alpha)} + \frac{1}{2}\mu_{11}\bar{I}_1^2 + \frac{3}{2}\mu_{12}\bar{I}_1\bar{I}_2 + \frac{3}{2}\mu_{13}\bar{I}_1\bar{I}_3 + \frac{1}{2}\mu_{22}\bar{I}_2^2 + \frac{3}{2}\mu_{23}\bar{I}_2\bar{I}_3 + \frac{1}{2}\mu_{33}\bar{I}_3^2 + O(\varepsilon^2). \quad (63)$$

If the potential  $V$  we are considering is nearly spherical, as defined in (47), we may write this as

$$K_2 = \frac{\mu_{12}\bar{I}_\alpha - \mu_{23}\bar{I}_\gamma}{2(\gamma - \alpha)} + \frac{1}{2}\mu_{11}K_1^2 + O(\varepsilon^2). \quad (64)$$

Since the actions  $\bar{I}_1$ ,  $\bar{I}_2$  and  $\bar{I}_3$  are not conserved separately, we conclude from (63) and (64) that for general  $\mu_{ij}$  and also for the  $\mu_{ij}$  that satisfy (53), the expression (61) is *not* conserved by the averaged equations of motion (21), except when the potential is nearly spherical. For the expression (62) we reach the same conclusion. Thus, *in case of a 1:1:1 resonance, the two extra integrals provided by local fitting are rigorous asymptotic integrals if, and only if, the potential is nearly spherical in the sense defined in (47).*

A similar result is obtained when we consider the approximations to the orbits in the potential  $V$ , as given by both methods. In a Stäckel potential  $V_S$  the simple periodic orbits are the orbits for which motion is in one ellipsoidal coordinate only. For potentials  $V_S$  with expansions around the origin of the form (11) the simple periodic orbits are the three axial oscillations and the elliptic closed orbits in each of the principal planes. The latter branch off the axes at the foci of the ellipsoidal coordinates. Inclined linear orbits through the centre can occur only as box orbits in which all three frequencies are equal. If this happens then the phase differences between the

oscillations in  $\lambda$ ,  $\mu$  and  $\nu$  are constant, and may take all values. The whole resulting collection of periodic orbits fills the box. Thus, inclined linear orbits can occur in  $V_S$  only as global bifurcations. Therefore, local fitting cannot provide all simple periodic orbit families that are possible according to the averaging method. However, if the potential is nearly spherical then both averaging and local fitting give as simple periodic orbits the three axial oscillations and the elliptic closed orbits (cf. Section 3.7).

If  $\kappa_1 = \kappa_2 = \kappa_3$  in (11), then (53) is satisfied but it follows from (52) that  $\beta - \alpha = \gamma - \alpha = \gamma - \beta = 0$ . This means that the foci of the ellipsoidal coordinate system coincide with the origin so that the coordinates are spherical. The general potential separable in spherical coordinates is regular at the origin only if it is a function of  $x_1^2 + x_2^2 + x_3^2$  (e.g. Lynden-Bell 1962). Hence, local fitting is done with a spherical potential. The integrals so obtained agree with the results of averaging only if the potential that is fitted is itself nearly spherical.

If the potential  $V$  that is expanded in the form (11) is exactly equal to a Stäckel potential, the integrals (60) are exact. We have seen in Section 4.3 that in the absence of a first order resonance the averaging method gives the correct first-order approximations to these integrals. It is evident from the analysis in the present section that this is not the case when the frequencies are nearly equal. The reason for this discrepancy is that separability is a property of all orders.

If  $V = V_S$  is expanded about the origin in the form (11) and all terms of  $O(\varepsilon^4)$  are omitted, the result is not separable, although it satisfies the condition (53). Given this truncated expansion, by local fitting we can choose higher order terms such that the complete expansion becomes separable. The integrals (50), with the  $V_i$  given by the complete expansions (54), are then exact in the so-constructed Stäckel potential. The averaging method, on the other hand, assumes only that the higher order terms in (11) are small, and treats the (non-separable) truncated expansion as any other. The results obtained are therefore generic for the expansion (11), and it is not surprising that in general the results do not agree with local fitting. The averaging method, carried out to first order, is apparently not well suited for the detection of separability.

We conclude that, *in the case of nearly equal harmonic frequencies, the results of local fitting are rigorous, and generic, only if the potential  $V$  is nearly spherical.* In other words, for nearly spherical potentials it makes no difference in first order if we describe motion in it as perturbed 1:1:1 resonant harmonic motion, or as unperturbed motion in a potential separable in ellipsoidal coordinates.

#### 4.5 THE THIRD INTEGRAL

When the potential  $V$  is nearly spherical in the sense defined in equation (47), it also satisfies condition (53) or, equivalently, (59). The preceding analysis has shown that in this case the averaged equations of motion (21) for the 1:1:1 resonance admit a third integral  $K_3$ , where

$$K_3 = \frac{1}{4}\mu_{12}\bar{L}_z^2 + \frac{1}{4}\mu_{23}\frac{\delta_1}{\delta_2}\bar{L}_x^2 + \frac{\kappa_2\delta_1}{\varepsilon^2}\bar{I}_2 + O(\varepsilon^2). \quad (65)$$

This integral can be considered as the generalization of  $\bar{L}_z^2$  conserved in potentials axisymmetric with respect to the  $z$ -axis.

Whether or not a third integral of equations (21) exists under more general circumstances is not clear. If another integral exists, it has to be discovered from the equations of motion, which is not a simple task. Equation (65) might suggest the following Ansatz for  $K_3$ :

$$K_3 = c_1\bar{I}_1^2 + c_2\bar{I}_1\bar{I}_2 + c_3\bar{I}_1\bar{I}_3 + c_4\bar{I}_2^2 + c_5\bar{I}_2\bar{I}_3 + c_6\bar{I}_3^2 + c_7\bar{L}_x^2 + c_8\bar{L}_y^2 + c_9\bar{L}_z^2 + c_{10}\bar{I}_1 + c_{11}\bar{I}_2 + c_{12}\bar{I}_3, \quad (66)$$

where the  $c_k$  are constants to be determined ( $k=1, \dots, 12$ ). For (66) to be an integral the Poisson bracket with the Hamiltonian,  $\{K_3, H\}$  must be zero for all values of the actions and the angles.

This requirement leads to 12 linear equations for the 12  $c_k$ . For general  $\mu_{ij}$  the only solutions of these equations are functions of  $K_1$  and  $K_2$ , however. Only if the  $\mu_{ij}$  satisfy (47) is there another solution. Not surprisingly, this is (65). As a result, if the averaged equations of motion (21) admit a third integral in case the potential  $V$  is not nearly spherical, this integral is not of the form (66).

## 5 Rotation

We now consider the case where the mass model, and hence the potential, is rotating uniformly around the  $x_3$ -axis, with angular frequency  $\Omega$ . In corotating Cartesian coordinates  $(x_1, x_2, x_3)$  the potential is then  $V=V(x_1^2, x_2^2, x_3^2)$ , and the equations of motion are

$$\ddot{x}_1 = -2\Omega\dot{x}_2 - \partial V/\partial x_1, \quad \ddot{x}_2 = +2\Omega\dot{x}_1 - \partial V/\partial x_2, \quad \ddot{x}_3 = -\partial V/\partial x_3. \quad (67)$$

Due to the Coriolis force the investigation of the equations of motion is considerably more complicated than in the non-rotating case. Direct and retrograde motion differ from each other.

The reflection symmetry of  $V$  with respect to the  $(x_1, x_2)$ -plane ensures that motion in this plane can be treated as two-dimensional motion. This has been the subject of a considerable number of studies (e.g. Contopoulos & Mertzanides 1977; Contopoulos & Papayannopoulos 1980; Athanassoula *et al.* 1983; Teuben & Sanders 1985). ZM have applied the averaging method in the centre and compared the results with numerical calculations for a rotating version of the triaxial potential used by Schwarzschild (1979). Here we apply the averaging method in three dimensions. Numerical studies in three dimensions have been made by Heisler, Merritt & Schwarzschild (1982), Magnenat (1982) and Mulder & Hooimeyer (1984).

### 5.1 LINEAR ANALYSIS

We again expand the potential  $V$  in the form (2), and in zeroth order retain only the quadratic terms. Then the equations of motion (67) are

$$\ddot{x}_1 = -2\Omega\dot{x}_2 - \kappa_1^2 x_1, \quad \ddot{x}_2 = +2\Omega\dot{x}_1 - \kappa_2^2 x_2, \quad \ddot{x}_3 = -\kappa_3^2 x_3. \quad (68)$$

These equations describe a uniformly rotating three-dimensional harmonic oscillator. The general solution is (Freeman 1966; Hunter 1974; ZM)

$$\begin{aligned} x_1 &= A_1 \cos(\omega_1 t + \phi_1) + \alpha A_2 \sin(\omega_2 t + \phi_2), \\ x_2 &= \beta A_1 \sin(\omega_1 t + \phi_1) + A_2 \cos(\omega_2 t + \phi_2), \\ x_3 &= A_3 \cos(\omega_3 t + \phi_3), \end{aligned} \quad (69)$$

where the frequencies  $\omega_i$  are given by

$$\begin{aligned} \omega_{1,2}^2 &= \frac{1}{2}(\kappa_1^2 + \kappa_2^2) \mp \sqrt{(\kappa_1^2 + \kappa_2^2 + 4\Omega^2) - 4\kappa_1^2 \kappa_2^2}, \\ \omega_3 &= \kappa_3, \end{aligned} \quad (70)$$

and

$$\begin{aligned} \alpha &= 2\Omega\omega_2/(\omega_2^2 - \kappa_1^2) = (\omega_2^2 - \kappa_2^2)/2\Omega\omega_2, \\ \beta &= (\kappa_1^2 - \omega_1^2)/2\Omega\omega_2 = 2\Omega\omega_1/(\kappa_2^2 - \omega_1^2), \end{aligned} \quad (71)$$

so that  $|\alpha| \leq 1$ ,  $|\beta| \leq 1$  and  $\omega_1 \leq \omega_2$ . The amplitudes  $A_i$  and the phases  $\phi_i$  are constants determined by the initial conditions.

Normal variables ( $Q_i, P_i$ ) are defined as

$$\begin{aligned} Q_1 &= \sqrt{\beta/\sigma} A_1 \cos(\omega_1 t + \phi_1), & P_1 &= -\sqrt{\beta/\sigma} A_1 \sin(\omega_1 t + \phi_1), \\ Q_2 &= \sqrt{\alpha/\sigma} A_2 \cos(\omega_2 t + \phi_2), & P_2 &= -\sqrt{\alpha/\sigma} A_2 \sin(\omega_2 t + \phi_2), \\ Q_3 &= \sqrt{\omega_3} A_3 \cos(\omega_3 t + \phi_3), & P_3 &= -\sqrt{\omega_3} A_3 \sin(\omega_3 t + \phi_3), \end{aligned} \quad (72)$$

with  $\sigma = 2\Omega/(\omega_2^2 - \omega_1^2)$ . Action-angle variables ( $I_i, \theta_i$ ) are now defined as in equation (9). The Hamiltonian  $H_2$ , corresponding to the linearized equations (68) is

$$H_2 = \sum_{i=1}^3 \omega_i I_i, \quad (73)$$

and the solution (69) becomes

$$I_i = \frac{1}{2} \omega_i A_i^2, \quad \theta_i = \omega_i t + \phi_i. \quad (74)$$

Each orbit is a combination of three normal modes. The motion has been described in detail by Freeman (1966) and Hunter (1974). The normal mode  $I_2=0$  in the equatorial plane is a direct elliptic orbit elongated along the  $x_1$ -axis with axial ratio  $1:|\beta|$ . The other normal mode in this plane is a retrograde elliptic orbit, elongated along the  $x_2$ -axis and with axial ratio  $|\alpha|:1$ . The third normal mode is the  $x_3$ -axis oscillation.

## 5.2 FIRST-ORDER AVERAGING

We rescale the coordinates as in Section 2.2, retain also the quartic terms in the expansion for  $V$ , and write the Hamiltonian in terms of the action-angle variables that were defined in the above. Furthermore, we write the frequency ratios as in (13):

$$\frac{\omega_1}{\omega_2} = \frac{m}{n} + 1, \quad \frac{\omega_3}{\omega_2} = \frac{l}{n} + \delta_2. \quad (75)$$

The first-order averaged Hamiltonian can then be calculated by the procedure outlined in Section 2.2, for all choices of  $l, m$  and  $n$ .

In their study of motion in the equatorial plane, ZM found that in addition to the 1:1 resonance, also a 1:3 and a 3:1 resonance occurs in first order for  $\Omega \neq 0$ . In three dimensions also more cases have to be distinguished for the averaged Hamiltonian than in Section 2.2. Resonances occur for  $n=l, l=m, m=n, m=n/3, 2l=n-m, 2l=m-n$  and  $2l=m+n$ . When one of these conditions is satisfied there is a resonance between two degrees of freedom. Genuine three-dimensional resonances occur when at least two of these conditions are satisfied simultaneously. There are four distinct possibilities:  $(m:n:l) = (1:1:1), (1:3:1), (1:3:2)$  or  $(1:3:3)$ . Equivalent cases may be obtained by exchanging  $m$  and  $n$  in the last three of these.

In what follows we shall consider briefly the case of no resonance, and then the 1:1:1 resonance. The latter is the three-dimensional resonance most relevant for elliptical galaxies. The other cases can be analysed along similar lines. If the rotation is around the shortest axis then  $\omega_1 \leq \omega_2 \leq \omega_3$  and among the other three-dimensional resonances only the case 1:3:3 may be of importance. When the assumption that  $V$  has three reflection symmetries is relaxed, still other resonances may occur. An enumeration has been given by Verhulst (1983).

## 5.3 NO FIRST-ORDER RESONANCE

When there is no resonance in first order, the averaged Hamiltonian is

$$\langle H \rangle = H_0 = \sum_{i=1}^3 \omega_i \bar{I}_i + \frac{1}{2} \varepsilon^2 \sum_{i,j=1}^3 \mu_{ij} \bar{I}_i \bar{I}_j. \quad (76)$$

The coefficients  $\mu_{ij}$  now depend not only on the coefficients in the expansion of the potential, but also on the frequency of rotation; they are given in the Appendix.

The solutions of the equations of motion that follow from (76) are identical to (18), if we substitute  $\omega_i$  for  $\kappa_i$ . As in the non-rotating case (Section 2.3), the energy in each normal mode is conserved separately, with an error of  $O(\varepsilon^2)$  on a time-scale of  $O(1/\varepsilon^2)$ . The frequencies of the oscillations now depend on energy.

#### 5.4 THE 1:1:1 RESONANCE: INTEGRALS

When  $l=m=n$  the first-order averaged Hamiltonian  $\langle H \rangle$  is

$$\begin{aligned} H_{1:1:1} = & H_0(\bar{I}_1, \bar{I}_2, \bar{I}_3) + \varepsilon^2 \{ \mu_5 \bar{I}_1 \bar{I}_2 \cos(2\bar{\theta}_1 - 2\bar{\theta}_2) \\ & + [\mu_4 \bar{I}_1 + \mu_8 \bar{I}_2] \sqrt{\bar{I}_1 \bar{I}_2} \sin(\bar{\theta}_1 - \bar{\theta}_2) \\ & + \nu_1 \bar{I}_1 \bar{I}_3 \cos(2\bar{\theta}_1 - 2\bar{\theta}_3) + \nu_4 \bar{I}_2 \bar{I}_3 \cos(2\bar{\theta}_2 - 2\bar{\theta}_3) \\ & + [2\nu_2 \sin(\bar{\theta}_1 - \bar{\theta}_2) + \nu_3 \sin(\bar{\theta}_1 + \bar{\theta}_2 - 2\bar{\theta}_3)] \bar{I}_3 \sqrt{\bar{I}_1 \bar{I}_2} \}, \end{aligned} \quad (77)$$

where  $H_0$  is given in equation (76) and the coefficients  $\mu_4, \mu_5, \mu_8, \nu_1, \dots, \nu_4$  are given in the Appendix. For  $\Omega=0$  we find  $\mu_4=\mu_8=\nu_2=\nu_3=0$ ,  $2\mu_5=\mu_{12}$ ,  $2\nu_1=\mu_{13}$  and  $2\nu_4=\mu_{23}$ , and we recover (20), as it should be.

Just as in the non-rotating case, only two independent combinations of angles occur in  $H_{1:1:1}$ . As a result, there is an ignorable coordinate and a corresponding integral of motion. This is easily seen to be

$$K_1 = \bar{I}_1 + \bar{I}_2 + \bar{I}_3. \quad (78)$$

We can write

$$H_{1:1:1} = \omega_2 K_1 + \varepsilon^2 K_2. \quad (79)$$

This shows that  $\omega_2 K_1$  is an  $O(\varepsilon^2)$  approximation of the total energy (in the rotating coordinate system) and also that  $K_2$  is a second isolating integral of motion. Its expression in actions and angles can be deduced easily by use of (77) and (78).  $K_2$  is the rotating generalization of the second asymptotic isolating integral we found in Section 3.1.

It is not clear whether the averaged Hamiltonian (77) admits a third independent isolating integral of motion for arbitrary values of  $\Omega$  and the parameters that occur in the expansion of the potential. In the absence of rotation we were able to derive an approximate third integral for the case where the potential is nearly spherical. On basis of the numerical calculations by Schwarzschild (1982) it is to be expected that a generalization of this integral exists for  $\Omega \geq 0$ .

#### 5.5 THE 1:1:1 RESONANCE: SIMPLE PERIODIC ORBITS

By an analysis similar to that given in Section 3 for the non-rotating case, one can find all simple periodic solutions of the averaged equations of motion that belong to the Hamiltonian (77). The calculations required for the derivation of the solutions, and the investigation of their stability, are lengthy. We briefly mention a few results.

*Normal modes.* Of the three normal modes of the linear approximation, only the one along the  $x_3$ -axis is a solution in first order. This is in agreement with the fact that the  $x_3$ -axis oscillation is an exact solution of the equations of motion (67).

*Equatorial plane.* From the symmetry of the potential with respect to the equatorial plane it follows that a star moving in this plane will remain in it, so that motion is two-dimensional. This is confirmed by the averaged equations of motion: for  $\bar{I}_3=0$  also  $\dot{\bar{I}}_3=0$ . The simple periodic orbits in

the equatorial plane have been described by ZM. The derivation of the various solutions is similar to that given in Section 3.4 for  $\Omega=0$ , but is not quite as simple.

It turns out that  $\bar{I}_1=\bar{I}_2=0$  when  $\bar{\theta}_1-\bar{\theta}_2=\pi/2$  or  $-\pi/2$ , but not when  $\bar{\theta}_1-\bar{\theta}_2=0$  or  $\pi$ . As usual, substitution of the value  $\pi/2$  for the phase difference in the equation  $\dot{\bar{\theta}}_1-\dot{\bar{\theta}}_2=0$ , and elimination of  $\bar{I}_2$  by means of  $\bar{I}_1+\bar{I}_2=K_1$ , results in an equation for  $\bar{I}_1$  in terms of  $K_1$ . Whereas for  $\Omega=0$  this equation is linear, now it is non-linear, and contains terms proportional to  $\sqrt{\bar{I}_1}$ . The product of this equation with the similar one that is obtained for  $\bar{\theta}_1-\bar{\theta}_2=-\pi/2$  is an equation of degree 4 in  $\bar{I}_1$ ; it is given below table 2B of ZM. There are two real roots with  $0\leq\bar{I}_1\leq K_1$  for all values of  $K_1\geq 0$ : a direct elliptic orbit, elongated in the  $x_1$ -direction, and a retrograde elliptic orbit, elongated along the  $x_2$ -axis. For certain energies the quartic equation admits two other real roots. The corresponding orbits are typically direct, one stable and the other unstable. In the limit of no rotation these four orbits reduce to the  $x_1$  and  $x_2$  normal modes and the elliptic closed orbits which in that case may be traversed in both directions (families 1, 2 and 5 of Section 3). The solutions are shown schematically in Fig. 4.

In general, also simple periodic solutions in the equatorial plane exist with  $\bar{\theta}_1-\bar{\theta}_2$  unequal to 0 or  $\pi$ , and a function of the energy. These orbits are ellipses, the symmetry axes of which are inclined to the  $x_1$ - and  $x_2$ -axes (Fig. 4). They are the rotating counterparts of family 4, the inclined elliptic orbits of the non-rotating case.

*( $x_1, x_3$ ) and ( $x_2, x_3$ ) planes.* The averaged equations of motion show that  $\dot{\bar{I}}_1\neq 0$  for  $\bar{I}_1=0$ , and similarly  $\dot{\bar{I}}_2\neq 0$  for  $\bar{I}_2=0$ . Thus, motion strictly confined to the  $(x_1, x_3)$  and  $(x_2, x_3)$  planes is not possible (excepting the  $x_3$ -axis orbit). The orbits that in the absence of rotation lie in one of these planes now tip out of them, due to the Coriolis force.

*Three-dimensional orbits.* Inspection of the averaged equations of motion shows that  $\dot{\bar{I}}_1=0$ ,  $\dot{\bar{I}}_2=0$  (and hence  $\dot{\bar{I}}_3=0$ ) is possible for  $\bar{I}_1$ ,  $\bar{I}_2$  and  $\bar{I}_3$  all three non-zero if  $(\bar{\theta}_1-\bar{\theta}_2, 2\bar{\theta}_2-2\bar{\theta}_3)=(\pi/2, 0)$ ,  $(-\pi/2, 0)$ ,  $(\pi/2, \pi)$  or  $(-\pi/2, \pi)$ . Substitution of  $-\pi/2$  and 0 for the values of the phase differences in the equations  $\dot{\bar{\theta}}_1-\dot{\bar{\theta}}_2=0$  and  $\dot{\bar{\theta}}_2-\dot{\bar{\theta}}_3=0$ , and elimination of  $\bar{I}_3$  by means of (78), then leads to two equations for  $\bar{I}_1$  and  $\bar{I}_2$  in terms of  $K_1$ . Elimination of  $\bar{I}_2$  between these two leaves a non-linear equation for  $\bar{I}_1$ . The product of this equation and the similar one obtained for  $(\bar{\theta}_1-\bar{\theta}_2, 2\bar{\theta}_2-2\bar{\theta}_3)=(\pi/2, 0)$  is an equation of degree 6 in  $\bar{I}_1$ . The real roots of this equation (0, 2, 4 or 6) are simple periodic solutions if they satisfy the condition for existence  $0\leq\bar{I}_1\leq\bar{I}_1+\bar{I}_2\leq K_1$ . They are the counterparts of the orbit families 7 and 13 found in Section 3 for  $\Omega=0$ .

The cases  $(\bar{\theta}_1-\bar{\theta}_2, 2\bar{\theta}_2-2\bar{\theta}_3)=(-\pi/2, 0)$  and  $(\pi/2, 0)$  can be combined similarly, and also yield an equation of degree 6 in  $\bar{I}_1$ . The resulting orbits are the generalizations of the orbit families 9 and 12 of the non-rotating case.

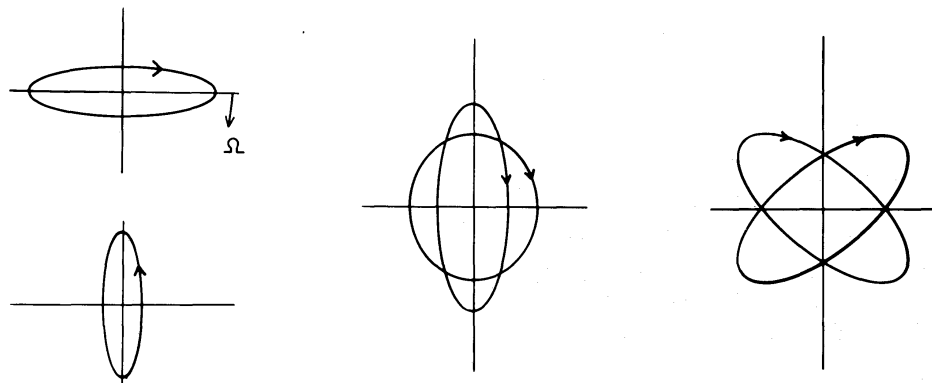


Figure 4. Simple periodic orbits in the equatorial plane of a slowly rotating triaxial potential.

No three-dimensional simple periodic solutions exist with  $\bar{\theta}_1 - \bar{\theta}_2 = 0$  or  $\pi$ . However, just as in Section 3.6, one can use the equations  $\dot{I}_1 = 0$  and  $\dot{I}_2 = 0$  to express the phase differences in terms of the actions. Substitution of the results in the equations  $\dot{\theta}_1 - \dot{\theta}_2 = 0$  and  $\dot{\theta}_2 - \dot{\theta}_3 = 0$ , and elimination of  $\bar{I}_3$  by (78), then yields two equations for  $\bar{I}_1$  and  $\bar{I}_2$  in terms of  $K_1$ , which can be used to find  $\bar{I}_1$  and  $\bar{I}_2$ . This calculation is not easy to carry out in practice. We expect that it will produce the rotating equivalents of the orbit families 6, 8, 10, 11 and 14 of the non-rotating case.

Martinet & de Zeeuw (1983) have numerically calculated the simple periodic orbits in the potential

$$V = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 + \frac{1}{2}x_3^2 + \frac{1}{4}\{x_1^4 + x_1^2x_2^2 + x_1^2x_3^2 + x_2^4 + x_2^2x_3^2 + x_3^4\}. \quad (80)$$

For  $\Omega = 0$  there is an exact 1:1:1 resonance, and all 14 orbit families given by the averaging method were found to exist, at all energies. Their properties agree in detail with the results obtained in Section 3. For  $\Omega \neq 0$  there is a detuned 1:1:1 resonance (since  $\omega_1 < \omega_2$  even if  $\kappa_1 = \kappa_2$ ). Indeed, the numerical calculations showed that at low energies, very close to the centre, only the normal modes of the linear approximation exist. At higher energies counterparts to all other families (4–14) of the non-rotating case appear. The retrograde and direct members of each family have different shapes for  $\Omega \neq 0$ . Thus, all simple periodic orbits found in the absence of rotation may also occur for  $\Omega \neq 0$ .

In case the potential  $V$  is nearly spherical in the sense defined in (47), only the families 1, 2, 3, 5, 7 and 9 exist in the absence of rotation. It is to be expected that, with rotation, only the counterparts – retrograde as well as direct – of these six families exist. This expectation is supported by the numerical calculations of Heisler *et al.* (1982) and Magnenat (1982). These authors find that the orbits 7 and 9 tip out of the  $(x_1, x_3)$ - and  $(x_2, x_3)$ -plane, respectively. They remain approximately elliptic and almost planar. The tilt of the orbital plane decreases with increasing amplitude. Retrograde and direct members of the same family tip in opposite directions. The periodic orbits in the equatorial plane are the counterparts of families 1, 2 and 5 (see ZM).

## 6 Concluding remarks

It is evident that the orbital structure in the core of a moderately triaxial potential is rich. Due to the 1:1:1 resonance between the fundamental orbital frequencies, many families of simple periodic orbits may occur. Even in the absence of rotation many of these are not in any of the principal planes. The stable periodic orbits have families of general (i.e. non-periodic) orbits around them. All orbits have a second asymptotic isolating integral; whether a third integral exists in the general case is not clear.

No elliptical galaxies flatter than about E6 have been observed (Binney & de Vaucouleurs 1981). This means that, although probably triaxial, their gravitational potentials are nearly spherical. In this case the averaging method shows that in the absence of figure rotation the simple periodic orbits are the three oscillations along the principal axes and the elliptic closed orbits in the principal planes. A third integral exists, and the general orbits can be divided in a few major families. Although these results have been derived for the core, the numerical calculations by Schwarzschild (1979) indicate that no changes occur at large radii. In case of slow rotation of the figure, retrograde and direct motion are distinct, but the orbital structure in the central regions is fundamentally the same. It is described by Heisler *et al.* (1982) and Magnenat (1982). All families of the non-rotating case have rotating counterparts. However, near the corotation radius the orbital structure can be quite complicated (Mulder & Hooimeyer 1984).

The potentials of elliptical galaxies are nearly of Stäckel form. We have shown that for nearly spherical Stäckel potentials, i.e. *precisely for the case of elliptical galaxies*, the simple periodic orbits are identical to those given by first order averaging for a general, moderately triaxial, but

nearly spherical potential. This means that a perturbation of the special Stäckel form will not cause drastic changes in the orbital structure. In the core the descriptions as 1:1:1 resonant perturbed harmonic motion or as motion in a Stäckel potential are equivalent. In other words, *only small – second order – errors are introduced if in the core the potential of an elliptical galaxy is taken to be exactly of Stäckel form. This approximation should work equally well at larger radii* also, since the orbits in Stäckel potentials resemble those found in Schwarzschild's (1979) non-rotating model at all radii.

If one uses a Stäckel potential for very flattened, or rapidly rotating systems, a small perturbation may cause many of the in-phase simple periodic orbits to exist. In this case local fitting will therefore not give accurate results. Furthermore, other resonances may occur, which can not be represented by the separable motion.

The results we have obtained do not depend critically on the assumption that  $V = V(x_1^2, x_2^2, x_3^2)$ . Relaxation of the symmetry conditions on the potential introduces more resonances (see Verhulst 1983). For the 1:1:1 resonance, which is of most interest for elliptical galaxies, still a second integral can be found, even if  $V = V(x_1, x_2, x_3)$ . From experience with two-dimensional problems (Verhulst 1979; ZM), and some three-dimensional investigations (van der Aa & Sanders 1979; van der Aa 1983), we expect that lowering of the symmetry will change the shapes of the various simple periodic orbits, but not the total number of families that may exist.

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## Appendix

The coefficients  $\zeta_k$ ,  $\eta_k$ ,  $\xi_k$ ,  $\sigma_k$  and  $\tau_k$  ( $k=1, \dots, 5$ ) that occur in Section 3 [cf. equations (37) and (44)] are the following expressions:

$$\begin{aligned}
 \zeta_1 &= \mu_{11} - 3\mu_{12} + \mu_{22}, & \zeta_2 &= \mu_{11} - 3\mu_{12} + \mu_{22}, \\
 \eta_1 &= \mu_{22} - 3\mu_{23} + \mu_{33}, & \eta_2 &= \mu_{22} - \mu_{23} + \mu_{33}, \\
 \xi_1 &= \frac{3}{2}\mu_{12} - \frac{3}{2}\mu_{13} + \frac{3}{2}\mu_{23} - \mu_{22}, & \xi_2 &= \frac{3}{2}\mu_{12} - \frac{1}{2}\mu_{13} + \frac{1}{2}\mu_{23} - \mu_{22}, \\
 \sigma_1 &= \frac{3}{2}\mu_{13} - \frac{3}{2}\mu_{23} + \kappa_2\delta_1/\varepsilon^2 K_1, & \sigma_2 &= \frac{1}{2}\mu_{13} - \frac{1}{2}\mu_{23} + \kappa_2\delta_1/\varepsilon^2 K_1, \\
 \tau_1 &= \frac{3}{2}\mu_{23} - \mu_{33} - \kappa_2\delta_2/\varepsilon^2 K_1, & \tau_2 &= \frac{1}{2}\mu_{23} - \mu_{33} - \kappa_2\delta_2/\varepsilon^2 K_1, \\
 \zeta_3 &= \mu_{11} - \mu_{12} + \mu_{22}, & \zeta_4 &= \mu_{11} - \mu_{12} + \mu_{22}, \\
 \eta_3 &= \mu_{22} - 3\mu_{23} + \mu_{33}, & \eta_4 &= \mu_{22} - \mu_{23} + \mu_{33}, \\
 \xi_3 &= \frac{1}{2}\mu_{12} - \frac{1}{2}\mu_{13} + \frac{3}{2}\mu_{23} - \mu_{22}, & \xi_4 &= \frac{1}{2}\mu_{12} - \frac{3}{2}\mu_{13} + \frac{1}{2}\mu_{23} - \mu_{22}, \\
 \sigma_3 &= \frac{1}{2}\mu_{13} - \frac{3}{2}\mu_{23} + \kappa_2\delta_1/\varepsilon^2 K_1, & \sigma_4 &= \frac{3}{2}\mu_{13} - \frac{1}{2}\mu_{23} + \kappa_2\delta_1/\varepsilon^2 K_1, \\
 \tau_3 &= \frac{3}{2}\mu_{23} - \mu_{33} - \kappa_2\delta_2/\varepsilon^2 K_1, & \tau_4 &= \frac{1}{2}\mu_{23} - \mu_{33} - \kappa_2\delta_2/\varepsilon^2 K_1, \\
 \zeta_5 &= -\mu_{12}^2\mu_{13}^2 - \mu_{12}^2\mu_{23}^2 + 2\mu_{12}\mu_{13}\mu_{23}(\mu_{11} - 2\mu_{12} + \mu_{22}), \\
 \eta_5 &= \mu_{12}^2\mu_{23}^2 + 2\mu_{12}\mu_{13}\mu_{23}(\mu_{12} - \mu_{13} + \mu_{23} - \mu_{22}), \\
 \xi_5 &= -\mu_{13}^2\mu_{23}^2 - \mu_{12}^2\mu_{23}^2 + 2\mu_{12}\mu_{13}\mu_{23}(\mu_{22} - 2\mu_{23} + \mu_{33}), \\
 \sigma_5 &= 2\mu_{12}\mu_{13}\mu_{23}(\mu_{13} - \mu_{23} + \kappa_2\delta_1/\varepsilon^2 K_1), \\
 \tau_5 &= \mu_{13}^2\mu_{23}^2 + 2\mu_{12}\mu_{13}\mu_{23}(\mu_{23} - \mu_{33} - \kappa_2\delta_2/\varepsilon^2 K_1),
 \end{aligned}$$

where the  $\mu_{ij}$  were defined in terms of the coefficients in the Taylor expansion (2) of the potential  $V$  in equation (17).

The first-order averaged Hamiltonian (76) for the rotating case in the absence of a first-order resonance contains coefficients  $\mu_{ij}$  that are given by

$$\mu_{11} = \frac{\sigma^2}{4\beta^2} [3a_{11} + 2\beta^2 a_{12} + 3\beta^4 a_{22}],$$

$$\mu_{12} = \frac{\sigma^2}{2\alpha\beta} [3\alpha^2 a_{11} + (1 + \alpha^2 \beta^2) a_{12} + 3\beta^2 a_{22}],$$

$$\mu_{22} = \frac{\sigma^2}{4\alpha^2} [3\alpha^4 a_{11} + 2\alpha^2 a_{12} + 3a_{22}],$$

$$\mu_{13} = \frac{\sigma}{2\beta\omega_3} [a_{13} + \beta^2 a_{23}],$$

$$\mu_{23} = \frac{\sigma}{2\alpha\omega_3} [\alpha^2 a_{13} + a_{23}],$$

$$\mu_{33} = \frac{3}{4\omega_3^2} a_{33},$$

where  $\alpha$ ,  $\beta$  and  $\sigma$  were defined in Section 5.1.

The additional coefficients that occur in the first-order averaged Hamiltonian (77) for the rotating 1:1:1 resonance are:

$$\mu_4 = \frac{1}{2} \left( \frac{\sigma}{\beta} \right)^{3/2} \left( \frac{\sigma}{\alpha} \right)^{1/2} [-3\alpha a_{11} + \beta(1 - \alpha\beta) a_{12} + 3\beta^3 a_{22}],$$

$$\mu_5 = \frac{\sigma^2}{4\alpha\beta} [-3\alpha^2 a_{11} + (1 + 4\alpha\beta + \alpha^2 \beta^2) a_{12} - 3\beta^2 a_{22}],$$

$$\mu_8 = \frac{1}{2} \left( \frac{\sigma}{\beta} \right)^{1/2} \left( \frac{\sigma}{\alpha} \right)^{3/2} [-3\alpha^3 a_{11} - \alpha(1 - \alpha\beta) a_{12} + 3\beta a_{22}],$$

$$\nu_1 = \frac{\sigma}{4\beta\omega_3} [a_{13} - \beta^2 a_{23}],$$

$$\nu_2 = \frac{1}{2\omega_3} \left( \frac{\sigma^2}{\alpha\beta} \right)^{1/2} [-\alpha a_{13} + \beta a_{23}],$$

$$\nu_3 = \frac{1}{2\omega_3} \left( \frac{\sigma^2}{\alpha\beta} \right)^{1/2} [\alpha a_{13} + \beta a_{23}],$$

$$\nu_4 = \frac{\sigma}{4\alpha\omega_3} [-\alpha^2 a_{13} + a_{23}].$$