BULLETIN OF THE ASTRONOMICAL INSTITUTES
OF THE NETHERLANDS.

1932 August 17 Volume VI. No. 238.

COMMUNICATION FROM THE OBSERVATORY AT LEIDEN.

The force exerted by the stellar system in the direction perpendicular to the galactic plane and some related problems, by *J. H. Oort.*

Notations.

z	distance from the galactic plane,
Z	velocity component perpendicular to the galactic plane,
Z_0	the value of Z for $z = 0$,
l	modulus of a Gaussian component of the distribution of Z (formula (5), p. 253),
$K(z)$	the acceleration in the direction of z,
Δ	the star-density,
ρ	the distance of a star from the sun,
$\Phi(M)$	the number of stars per cubic parsec between $M - \frac{1}{2}$ and $M + \frac{1}{2}$,
$A(m)$	the number of stars per square degree between $m - \frac{1}{2}$ and $m + \frac{1}{2}$,
b	galactic latitude,
ϖ	distance to the axis of rotation of the galactic system,
δ	$\partial \log \Delta / \partial \varpi$.

Summary of the different sections.

1 and **2.** In these sections a short discussion is given of KAPTEYN's previous investigation on the subject and of the reasons why the problem has been treated anew. In the second section the formulae are given which show the connection between $K(z)$, $\Delta(z)$ and the velocity distribution (formulae (5) and (6)).

3. The distribution of Z and its dependence upon spectral type and visual and photographic absolute magnitude is studied in some detail. The adopted results are in Tables 7 (spectral types), 9 (visual absolute magnitudes) and 11 (photographic absolute magnitudes). The average velocities of giants and dwarfs of the same spectrum appear to be practically identical in the z direction. On account of their irregular distribution the B0 — B9 stars have been excluded in forming the velocity laws for the different groups of absolute magnitude.

It is shown that stars at various distances north and south of the galactic plane indicate no signs of systematic motions in the z-direction (Table 12).

4. From VAN RHIJN's tables in *Groningen Publication* No. 38 the density distribution $\Delta(z)$ has been computed for four intervals of visual absolute magnitude (Table 13 and Figure 1). Figures 2 and 3 show $\log \Delta(z)$ for A stars and yellow giants, as derived by LINDBLAD and PETERSSON.

5. With the aid of the data contained in the two preceding sections I have computed the acceleration $K(z)$ between $z = 0$ and $z = 600$. The computations were made by successive approximations; the B stars were eliminated first. The results are in Table 14 and Figure 4, $K'(z)$ giving the values finally adopted. The good agreement between the practically independent values of $K(z)$ derived from the separate absolute magnitude groups is a strong argument in favour of the approximate correctness of the data up to $z = 400$. The result may be summarized by stating that the absolute value of $K(z)$ increases proportionally with z from $z = 0$ to $z = 200$; between $z = 200$ and $z = 500$ it remains practically constant and equal to $3 \cdot 8 . 10^{-9}$ cm/sec^2.

6. In this section the different spectral classes are investigated separately. A comparison of numbers computed with the aid of $K(z)$, with direct counts in high galactic latitude revealed a great discrepancy for the K stars, probably due to an error in the adopted luminosity law (compare *B. A. N.* No. 239). A slight correction to the average velocity of the A stars was also indicated. Both corrections have been applied throughout the greater part of the present investigation.

For comparison with future observations of fainter stars the computed numbers of each spectral type and visual apparent magnitude are given in Table 17, for 20°, 40° and 80° galactic latitude. The table also shows the relative numbers of giants and dwarfs to be expected for each magnitude. Finally, Table 18 shows the corresponding average colour indices and the mean square deviations from the average. No great accuracy can be claimed for these values.

7. From the best sources available mean values of $\log A(m)$ were computed for visual as well as photographic magnitudes and for latitudes $\pm 50°$ and $\pm 80°$. The results, referring to stars later than B9, are in Figures 5 and 6. Table 21 gives the corrections necessary to reduce the visual counts to SEARES' photovisual scale.

8. As a check on the computations of the 5th section numbers of stars of each visual magnitude were computed with the aid of the acceleration $K(z)$ derived in that section. The necessary formulae are shown on page 263 (formulae (10) and (11)). As was to be expected the agreement with the counts according to the Harvard scale is perfect (Table 23). For SEARES' photovisual scale the accordance is not as good. This would indicate a somewhat steeper rise of $K(z)$ between $z = 0$ and $z = 150$, but even then the agreement cannot be made very satisfactory. The difference may be due to inaccuracies of $\Phi(M)$.

With the aid of the corrected K star luminosity law new photographic and visual luminosity laws for all spectra together (except B stars) were computed and tabulated in the 2nd and 4th columns of Table 22.

For the computation of $A\ (m_{pg})$ for fainter magnitudes we need an extrapolation of $K\ (z)$ to heights of about 5000 ps. Two different extrapolations were tried, based on different assumptions as to the main attracting mass of the galactic system. These have been denoted by K_a and K_b (Figure 7). K_a is very probably nearest to the truth and has been adopted for the computations in the last two sections (the first part being taken in accordance with $K'\ (z)$ in Table 14). The comparison with the observed numbers of faint stars (shown in Table 25) is difficult for two reasons. Firstly there are indications of a considerable error in the photographic luminosity law and secondly our knowledge of the frequency and distribution of high velocities appears to be altogether insufficient. A more extensive knowledge of the latter is indispensable if we want to obtain a somewhat trustworthy picture of the density distribution and of the luminosity law at heights above 1000 or 1500 ps. In order to illustrate this the computations of $A\ (m)$ in Table 25 were made with four different assumptions as to the distribution of high velocities.

A general impression of the distribution of stars at 80° latitude and of photographic magnitudes 11, 14, 17 and 18 over different absolute magnitudes and distances may be obtained from Figures 8 and 9. The 18th magnitude stars appear to be mostly high velocity dwarfs of K and M types. It may be noted that formulae (10) and (11) (p. 263) may be used to compute the approximate distribution in distance for stars of any magnitude and in any region of the sky above 15° latitude if the density gradients derived in the 9th section are taken into account.

9. In Table 26 and Figure 10 a comparison is given between mean counts of stars at different latitudes with those computed on the assumption that the layers of equal density are parallel to the galactic plane. The close parallelism of the observed and computed values indicates that the influence of absorption or of an eventual local system only becomes sensible below 15° latitude. The average absorption at 20° latitude can hardly be larger than $0^{m}\!\cdot\!3$. At latitudes above 20° this unknown which renders density determinations in the galactic plane almost impossible appears to vanish. The present result seems in remarkable contrast with the average absorption of more than a magnitude indicated by the extragalactic nebulae in this zone.

The star counts between 20° and the poles have now been analysed according to galactic longitude. There is distinct evidence of a density increase in the direction of the centre of the large system (Figure 11) and it is shown that very satisfactory determinations of the density gradient can be obtained (Table 28). The density gradient found is of the same order as that found theoretically from the distribution of high velocities (*B. A. N.* No. 159). Some data are also given for the change of this gradient with z and ϖ. Star counts between 30° and 60° latitude appear to be best suited for determining the *regular* features of the galactic system (compare Figure 11).

The equidensity surfaces computed with these empirical numbers make angles of about 9° with the galactic plane. For $\Delta = \cdot 04$ and $\Delta = \cdot 01$ they are shown as dots and crosses in Figure 12. They can be closely represented by ellipsoids symmetrical about the axis of the great galactic system and with axial ratios of 9·3 and 6·4 respectively. It should be noted that only part of the *outer* envelopes of the system (beyond $z = 500$) have been derived in this way.

10. In this section the density distribution in the direction perpendicular to the galactic plane has been computed for different types of stars as well as for the integrated light and the mass (Table 29). Half of the photographic light is situated between $z = -166$ and $z = +166$ ps. From a comparison of this number with the distance from the sun to the centre (which is of the order of 10000 ps) we may infer that the denser part of the galactic system must have an exceedingly flat shape. Because of this extreme flatness it seems impossible to investigate the density distribution in the galactic plane, even from areas at very low latitude, without taking into account the decrease of star density with z.

I have also computed the change of the luminosity law with z, which change is very considerable (Table 30; B stars are included).

Table 31 shows average distances from the galactic plane corresponding to different average velocities, assuming Gaussian distributions of the latter. Some examples are added to show how these can be utilized in computing average parallaxes of distant stars or average velocities perpendicular to the galactic plane. A remarkable difference of about 8 times between the average radial velocity and the average value of $|Z|$ is noted for the O stars.

11. It is found that the total density of matter near the sun is equal to $6{\cdot}3 . 10^{-24}$ g/cm³ or ·092 solar masses per cubic parsec. The observed total mass of the stars down to + 13·5 visual absolute magnitude is found to be ·038 solar masses per ps³ (Table 34). It is probable that this value would still be greatly increased if we could have taken the next 5 absolute magnitudes into account, so that the total mass of meteors and nebular material is probably small in comparison with that of the stars. There is an indication that the invisible mass is more strongly concentrated to the galactic plane than that of the visible stars (Table 33).

Integrating over a column perpendicular to the galactic plane I find that an average unit of photographic light corresponds to a mass of 1·8 (if both are expressed in the sun as unit), approximately agreeing with the proportion found in the central region of the Andromeda nebula, the only available case where a comparison is possible.

In a note on VAN MAANEN's star the probability of a large mass for this star has been discussed.

1. *Introductory; previous investigations.*

It cannot be doubted that in many respects the galactic system is very irregular. It is important to find out whatever regularities there are and to utilize these to obtain a deeper knowledge of the general characteristics of the system.

To some extent the distribution of the later type stars in a direction perpendicular to the galactic plane may be classed among the regular features, as is evident, for example, from the great resemblance between the distribution on both sides of the galactic plane. It is evident, also, from the results of the present article in which it will be shown that the assumption of a well-mixed condition, together with a knowledge of the force, $K(z)$, exerted in this direction by the stellar system, enables us to represent all known features about stellar distribution perpendicular to the galactic plane.

There is good reason to suppose that the gravitational force is approximately of such a size as to keep the stars together during times which are as short as 10^8 or 10^9 years. It would seem unreasonable to assert that the extreme flatness of the galactic system is a peculiarity of this particular moment and that, after the course of 10^8 years, it would have undergone such a radical change in appearance as, say, a ten-fold increase of its thickness. The steady state hypothesis is more directly supported by the fact that the stars at different distances on both sides of the galactic plane do not show an observable systematic motion towards or away from the galactic plane (compare Table 12).

The supposition that $K(z)$ is such that it will approximately maintain the present density distribution is the working hypothesis of the following investigation, in which I have tried to derive the magnitude of this acceleration at different distances from the galactic plane from a comparison of velocity and density distributions in this direction without making any supposition about the density distribution *in* this plane. Earlier investigations, mainly by JEANS [1]), have proved that a rigorous application of the steady state conditions lead to the result that the distribution of the velocity components perpendicular to the galactic plane would ultimately be the same as that of the components directed to the axis of rotation. However, the steady state implied by this condition seems to be of a quite different class from the simple state of being well-mixed which is presupposed for the present investigation. It seems indeed probable that in a system like the one we are investigating the average peculiar motions of the stars in directions perpendicular and parallel to the galactic plane will, at least for small motions, remain practically independent of each other during times comparable with the time of development of the galactic system. The empirical fact that for most types of stars the average velocity components perpendicular to the galactic plane are considerably different from those directed towards the centre of the system points in the same direction.

A second aim has been to derive new values for the density distribution perpendicular to the galactic plane for various absolute magnitudes, as a good knowledge of this function is a prerequisite for all investigations of star-density, also at low latitudes. With the aid of these densities the density arrangement of the great galactic system in the entire region between the poles and 20° latitude has been investigated.

A third purpose was the derivation of an accurate value for the total amount of mass, including dark matter, corresponding to a unit of luminosity in the surroundings of the sun.

So far as I know the first significant numerical investigation along these lines is contained in KAPTEYN's well-known "First attempt at a theory of the arrangement and motion of the sidereal system" [2]). His formula

[1]) *Problems of Cosmogony*, Chapter X; *Astronomy and Cosmogony*, Chapter XIV.

[2]) *Astrophysical Journal*, **55**, 302, 1922; *Mt. Wilson Contr.* No. 230.

(8) is equivalent with formula (4) of the present paper. A comparison between the acceleration computed from KAPTEYN's results with that derived in the present paper may be found in Table 33.

Though KAPTEYN's general dynamical model may not agree with our present conceptions his conclusions about the acceleration $K(z)$ are not greatly affected by this change in the model of the galactic system. That I have thought it worth while to redetermine this acceleration is thus not due in the first place to the changed conception but rather to the expectation that it would be possible to get a more accurate result with present-day data. KAPTEYN's investigation of $K(z)$ was preliminary in several respects. In the first place it rests on the assumption that the density distribution in the direction perpendicular to the galactic plane (called z in the present article) resembles that of a gas in isothermal equilibrium. This is plausible in the case of a Gaussian distribution of velocities, but it will be different if the distribution deviates from the Gaussian law (see the second section). JEANS [1]) has repeated KAPTEYN's calculations without making the assumption of isothermal equilibrium. He replaced it by the assumption that the ratio between the mass density and the density of luminous stars is constant throughout the system. JEANS did not make use of any data about the distribution of velocities in the z-direction; he determined the factor needed to reduce KAPTEYN's accelerations G to absolute units from the relative velocity of the star-streams, so that his result depends directly upon the special dynamical model considered.

In the second place KAPTEYN assumes an average z-velocity of 10·3 km/sec for all stars. In the present article we shall take account of the fact that the intrinsically faint stars move much more rapidly. In the third place the density distribution needed revision. It had been derived on the assumption that it is the same for all absolute magnitudes, whereas it will be shown below that it must depend upon the absolute magnitude of the stars considered. This fact will again cause a change of the luminosity curve with z so that it will not be permissible to derive densities on the assumption that the luminosity curve is the same throughout space. The densities used by KAPTEYN rest, at least partly, on this assumption.

Other investigations touching upon the problem have been made at Upsala by LINDBLAD and his collaborators who have determined the density distribution of A and K type stars in a direction perpendicular to the galactic plane. Detailed references to the results of these and some other studies will be given later on.

[1]) *Monthly Notices, R. A. S.*, **82**, 122, 1922.

For the investigation projected we shall want a good knowledge of the velocity distribution and of the density distribution in the direction of the galactic poles. The first will be mainly determined from radial velocities of stars at high galactic latitudes, the latter will be derived from VAN RHIJN's data given in *Groningen Publications* No. 38. Considerable use will also be made of VAN RHIJN's luminosity laws and of his discussions of counts of stars of different magnitudes.

Unless one would be inclined to believe that obscuring matter is more abundant near the galactic poles than in the plane of the galaxy there is no reason to fear that the densities and counts of stars used have been seriously influenced by absorption or obstruction of light in space. KAPTEYN and VAN RHIJN's preliminary investigation on the distribution of stars in space [1]) shows that in the galaxy the apparent density does not show a sensible decrease for distances below 400 parsecs; so it is very probable that the observed density distribution in the z-direction gives a true picture at least up to this same distance and probably much further.

2. *Theory.*

As a first approximation let us suppose that at any point of the orbit of a star the acceleration $K(z)$ is equal to the acceleration in a point at a corresponding height z vertically above the sun. Let us call this latter value briefly $K(z)$. There is every reason to believe that this supposition will lead to a good approximation: because of the approximately symmetrical distribution of motions in the galactic plane there will, roughly speaking, for every star moving to a region where the acceleration is greater than this normal $K(z)$, be another star moving to a region where the acceleration is correspondingly lower. The assumption permits us to get a simpler insight into the formulae connecting velocity- and density distribution and $K(z)$, but it is not a necessary assumption, for JEANS has derived the same formula for the general case of a dynamically steady system [2]).

Let Z represent the z-component of the peculiar velocity of a star at a certain height z and let us call Z_0 the value of Z at the time when the star passes though the plane $z = 0$, the plane of the galaxy. If $K(z)$ is the corresponding component of the force, counted positive in the direction of increasing z, we have

$$Z = \sqrt{Z_0^2 + 2\int_0^z K(z)\,dz}. \qquad (1)$$

[1]) *Astrophysical Journal*, **52**, 23, 1920; *Mt. Wilson Contr.* No. 188.

[2]) *Monthly Notices, R. A. S.*, **82**, 124, 1922.

If, further, $T(Z_0)$ is the period of the star's oscillation in the z-direction, the fraction of the star's time spent in the layer between z and $z + dz$ is

$$\frac{2\,dz}{Z\,T(Z_0)}.$$

Now, if the total number of stars above or below a surface element of the galactic plane and having a Z_0 between Z_0 and $Z_0 + dZ_0$ is called $f(Z_0)\,dZ_0$, and if $\varphi(z, Z)\,dz\,dZ$ represents the average [1]) number of stars in an element of volume between z and $z + dz$ having velocities between Z and $Z + dZ$, we have evidently:

$$\varphi(z, Z)\,dz\,dZ = f(Z_0)\,dZ_0 \cdot \frac{2\,dz}{Z\,T(Z_0)}\,. \tag{2}$$

As it follows from (1) that $dZ_0/Z = dZ/Z_0$ we find

$$\varphi(z, Z) = \frac{2f(Z_0)}{Z_0\,T(Z_0)}\,, \tag{3}$$

so φ is a pure function of Z_0, i. e. of

$$\sqrt{Z^2 - 2\int_0^z K(z)\,dz}\,.$$

In the special case of a Gaussian distribution of velocities at $z = 0$ it follows from this property that the velocities Z will show the same Gaussian distribution at any value of z; for, if

$$\varphi(0, Z) = \Delta(0)\,\frac{l}{\sqrt{\pi}}\,e^{-l^2 Z_0^2}\,,$$

where $\Delta(0)$ and l are constants, it follows from the fact that φ is a pure function of Z_0 that

$$\varphi(z, Z) = \Delta(0)\,e^{2l^2\int_0^z K(z)\,dz} \cdot \frac{l}{\sqrt{\pi}}\,e^{-l^2 Z^2}\,.$$

Integrating φ over all values of Z we obtain the density. At $z = 0$ this is evidently equal to $\Delta(0)$; at an arbitrary value of z the density is found to be

$$\Delta(z) = \Delta(0)\,e^{2l^2\int_0^z K(z)\,dz}\,. \tag{4}$$

It is well known that in this particular case the density follows the same law as in a gas in isothermal equilibrium.

Formula (4) solves our present problem as it enables us to derive $K(z)$ from a knowledge of $\Delta(z)$ and l (the modulus of the velocity distribution). The practical problem is very slightly more complicated, as the velocity distribution is usually found to deviate somewhat from a Gaussian distribution. However it can always be represented by a sum of two or three Gaussian components with different moduli. In the case of two Gaussian components we have

$$\varphi(0, Z_0) = \Delta(0)\left(\theta_1\frac{l_1}{\sqrt{\pi}}\,e^{-l_1^2 Z_0^2} + \theta_2\frac{l_2}{\sqrt{\pi}}\,e^{-l_2^2 Z_0^2}\right), \tag{5}$$

from which we obtain

$$\Delta(z) = \Delta(0)\left\{\theta_1 e^{2l_1^2\int_0^z K(z)\,dz} + \theta_2 e^{2l_2^2\int_0^z K(z)\,dz}\right\}, \tag{6}$$

from which $K(z)$ can easily be determined by successive approximations.

For the case of an arbitrary velocity distribution we can derive the following formula connecting $K(z)$, Δ and the average square velocity, $\overline{Z^2}$:

$$K(z) = \frac{1}{\Delta}\,\frac{\partial(\Delta\overline{Z^2})}{\partial z}\,. \tag{7}$$

The equation is identical with that derived by JEANS for a star system in dynamical equilibrium (l. c.).

3. *The velocity distribution.*

The velocity distribution has been determined mainly from radial velocities of stars near the galactic poles. As it seemed advisable to investigate a possible dependence of the average velocity on absolute magnitude the stars have been arranged according to spectral type as well as to spectroscopic absolute magnitude. For the dwarfs a determination has been made from the space velocities of the nearest stars. The strong galactic concentration of the B stars made it necessary to determine their Z-distribution from proper motions.

Radial velocities.

A complete list was made of radial velocities of stars lying between $\pm 40°$ and $\pm 90°$ galactic latitude and of which the spectroscopic absolute magnitudes were known (except for the B type, where all available radial velocities were used irrespective of the existence of a spectroscopic absolute magnitude). Only Mt. Wilson and Victoria absolute magnitudes [1]) were used and it was tried to reduce these to a somewhat homogeneous system by applying the following corrections, representing the results of *Groningen Publications* Nos. 34 and 37 after slight modifications:

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[1]) The statement contained in formula (2) is rigorously true if we suppose this average to be taken over a time interval as long as a few oscillations. As we cannot wait for so long a time we shall in the practical problem indentify $\varphi(z, Z)\,dz\,dZ$ with the number of stars which at the present moment lies between the limits stated. The hypothesis underlying this identification might be expressed by stating that in the direction of z the stars must be "well mixed up".

[1]) *Mt Wilson Contributions* Nos. 199, 244, 262 and 319; *Astrophysical Journal*, **53**, 13; **56**, 242; **57**, 294 and **64**, 225; *Victoria Publications*, **3**, 1, 1924.

TABLE 1.

Adopted corrections to spectroscopic absolute magnitudes.

$M_{\text{spect.}}$	Mt. Wilson		Victoria		
	F stars	K stars	F stars	G stars	K stars
— 1·0				— 0·4	— 0·7
0·0	— 0·9	— 1·5	— 1·2	— 0·4	— 0·2
+ 0·5	— 0·8	— 0·9	— 1·1	— 0·3	+ 0·1
+ 1·0	— 0·7	— 0·3	— 1·0	— 0·3	+ 0·3
+ 2·0	— 0·4	+ 0·5	— 0·8	— 0·1	+ 0·9
+ 3·0	— 0·1	0·0	— 0·6	0·0	0·0
+ 5·0	+ 0·5	0·0	— 0·2	0·0	0·0
≥ 6·0		0·0		0·0	0·0

The Mt. Wilson absolute magnitudes of G stars and the Victoria ones of M stars needed no correction. The Mt. Wilson A and M star absolute magnitudes were also adopted as they stood. The spectral classification used for the application of these corrections, as well as for all the other divisions into spectral classes made in this section, is that of Mt. Wilson ("estimated") or Victoria. For all stars occurring in *Contribution* No. 387 (M stars) the Mt. Wilson absolute magnitudes were taken from this source.

Especially among the fainter stars in our list there is an undue proportion of large proper motions. As the distribution curve of radial velocities is likely to depend somewhat upon the proper motion I have counted each large proper motion star as a fraction of one star, the fraction being determined so as to make the material representative as regards the distribution of proper motions. The fractions or weights required depend upon apparent magnitude, spectrum and proper motion, and were determined by comparison of counted numbers from the present list with the complete numbers derived in *Groningen Publication* No. 30.

The radial velocities averaged from all available sources (without systematic corrections) were corrected for the motion of the sun. The following values were assumed: $V = 20$ km/sec, apex: $\alpha = 17^h 59^m$, $\delta = +31°$. All probable members of the Ursa Major group were excluded.

In determining the velocity distribution, or the average velocities in the direction perpendicular to the galactic plane, from these radial velocities we have to take into account that the stars cover an area down to 40° latitude and that, accordingly, the radial velocities will contain a not inconsiderable component of the velocities parallel to the galactic plane. In order to derive the factor F by which we must multiply the observed average velocity for obtaining the true average velocity perpendicular to the galactic plane, we must have some approximate knowledge of the other axes of the velocity distribution. Having derived this from various sources I find the following values of F, which have been used in the present paper. [1])

TABLE 2.

Type [2])	± 56° to ± 90°	± 40° to ± 55°
A	·90	·77
F	·91	·79
G giants	·91	·80
K giants	·93	·84
M giants	·93	·84
dwarfs	·83	·67

Firstly the dependence of average peculiar velocity upon absolute magnitude and spectral type was investigated and a comparison was made between the average velocities in the region of the north galactic pole with those in the region of the south galactic pole. As there appeared to be no distinct systematic difference between the two regions I shall only quote the combined results. Table 3 shows the averages without sign of the peculiar radial velocities after reduction to the Z co-ordinate. No large velocities have been excluded.

TABLE 3.

Average peculiar velocities perpendicular to the galactic plane.

Spectrum	Abs. magn.	$\|Z\|$	n
B0 — B5		7·1 km/sec	15
B6 — B9		9·2	14
A	$+0·5^M$ to $+1·4^M$	9·3	48
	+ 1·5 » + 2·4	7·6	35
F	+ 0·5 » + 2·4	12·7	24
	+ 2·5 » + 3·4	8·0	21
	+ 3·5 » + 4·4	15·7	20
G	— 1·5 » + 0·4	9·5	40
	+ 0·5 » + 1·4	13·2	42
	+ 1·5 » + 3·4	16·4	15
	+ 3·5 » + 6·4	13·9	22
K	— 1·5 » — 0·6	15·9	14
	— 0·5 » + 0·4	15·3	45
	+ 0·5 » + 2·4	15·0	53
	+ 5.0 » + 8·0	10 7	8
M	— 1·5 » — 0·6	15·9	22
	— 0·5 » + 0·4	15·3	70

A few remarks may be made.

The results for the B stars were obtained exclusively from velocities with small probable errors. It is very improbable that this result, necessarily obtained from B stars at considerable distance from the galactic plane applies to the B stars in general. Proper motions show a two or three times lower average peculiar velocity

[1]) For their derivation see the first Note concluding this article.

[2]) Throughout this article, for convenience in comparison with results obtained at Groningen, the sub-division into spectral types has been taken the same as in the *Groningen Publications*. Thus A stands for A0 to A9, F for F0 to F9, etc.

(cf. page 257), and the same low value is indicated by counts of B stars in high galactic latitudes (section 6).

The average velocity for the A stars is also doubtful; the total average velocity for the ± 56° to ± 90° zone is much lower than that for the ± 40° to ± 55° zone (6·8 and 11·2 km/sec respectively). Counts of stars in high latitude seem to indicate that a value in the neighbourhood of 7 km/sec is better than the general average of 8·6 km/sec derived from Table 3. Accordingly this lower value has been used for the final computations of $A(m)$.

In general there is no outspoken run of average velocity with absolute magnitude so that I have taken the same velocity distribution for all A stars, similarly for all F stars, all K and M giants. For the G stars there is an indication that the brightest stars move more slowly and I have divided them into three groups, G I from $-1^{M}\!\cdot\!5$ to $+0^{M}\!\cdot\!4$, G II from $+0^{M}\!\cdot\!5$ to $+3^{M}\!\cdot\!4$, G III fainter than $+3^{M}\!\cdot\!4$.

In all cases the distribution curve of the peculiar velocities is very nearly a simple Gaussian curve $n\frac{l}{\sqrt{\pi}}e^{-l^2Z^2}$, as shown in Table 4. In the direct counts (columns headed O) the zones ± 56° to ± 90° and ± 40° to ± 55° have been combined; the intervals as indicated in the first column have been used for the velocities in the ± 56° to ± 90° zone, the corresponding intervals counted for the other zone were taken somewhat larger in order to correct for the greater influence of the star-streams in this zone.

TABLE 4.

Distribution of radial velocities near the galactic poles. Comparison with Gaussian distributions.

v	A stars O	A stars ·060	F stars O	F stars ·044	G, $M \leq +0·4$ O	G, $M \leq +0·4$ ·054	G, $+0·5 \leq M \leq +3·4$ O	G, $+0·5 \leq M \leq +3·4$ ·036	G, $M \geq +3·5$ O	G, $M \geq +3·5$ ·034	K giants O	K giants ·034	M giants O	M giants ·034
0— 4	23·7	24·9	14·6	16·1	12·5	10·8	11·0	10·2	1·2	3·2	19·2	19·7	19·0	16·4
5— 9	27·3	23·8	18·7	16·4	13·5	10·5	11·6	10·7	4·8	3·4	20.9	20·8	15·0	17·4
10—14	15·5	16·9	13·1	13·7	6·5	8·0	10.0	9·5	5·1	3·1	22·6	18·6	12·0	15·6
15—19	11·8	10·2	9·0	10·4	0·5	5·3	5·0	7·8	1·3	2·6	13·7	15·9	11·0	13·2
20—29	2·9	7·2	10·8	11·6	2·0	4·5	11·0	10·6	4·2	3·6	21·6	22·1	24·0	18·4
30—39	1·8	1·0	4·8	3·8	4·0	0·8	3·0	5·0	0·1	1·9	7·7	11·3	10·0	9·4
40—49	0·0	0·0	1·0	0·9	1·0	0·1	3·1	1·8	1·7	0·8	5·0	4·7	2·0	3·9
50—59	0·0	0·0	0·6	0·1	0·0	0·0	1·6	0·6	0·0	0·2	3·1	1·5	1·0	1·2
60—79	1·0	0·0	0·0	0·0	0·0	0·0	0·0	0·1	0·6	0·1	1·5	0·5	2·0	0·4
≥ 80	0·0	0·0	0·2	0·0	0·0	0·0	0·0	0·0	0·0	0·0	0·0	0·0	0·0	0·0
w	84·0		72·8		40·0		56·3		19·0		115·3		96·0	

In each division of the table the second column of numbers shows the values computed with a Gaussion function, the modulus l being given at the top of the column. The weighted total numbers of stars are given in the last line of the table. In general the representation is quite good. There is a slight indication of an excess of observed velocities above 50 km/sec, on the average perhaps 1·5 °/₀ of the total number of stars. Because of the relatively large influence of high velocities on the determination of $K(z)$ this excess was tentatively taken into account by adding a small number of velocities distributed over a frequency curve with modulus $l = 0·020$. The constants of the Z-distributions finally adopted are shown in Table 7.

Space velocities of nearest stars.

For the velocity laws of the fainter dwarfs, especially for those of K and M types, we can only hope to obtain some information by studying the nearest stars. I have used a card catalogue extracted from SCHLESINGER's parallax catalogue and containing all stars with parallaxes greater than 0″·100 with probable errors below ± 0″·020 and all those with parallaxes between 0″·050 and 0″·100 with probable errors below ± 0″·010. Practically all these stars have been selected on account of large proper motion and consequently to some extent also on account of large velocity. It is not difficult to correct for this selection by giving to each card a certain weight depending on magnitude and proper motion and chosen in such a way as to get a truly representative sample of stars[1]). As relatively few stars with annual proper motions below 0″·200 have been observed and the completeness factors would become probihitive, I have omitted these stars. For the group of stars with parallaxes larger than 0″·100 this means that I have omitted all stars whose space velocities with respect to the sun are lower than 9·5 km/sec and that the Z-distribution obtained will be incomplete in the interval from − 9·5 to + 9·5

[1]) For the weights, or completeness factors required see Table 9, *Groningen Publications*, No. 40.

km/sec, or, as $Z_{\odot} = + 7{\cdot}5$ km/sec, from $Z = -2$ to $Z = +17$ km sec[1]) (Z representing the *peculiar* velocity). For the groups of lower parallaxes the interval will be still wider. In practice however the incompleteness does not appear to be serious, as in nearly all cases the incompletely observed intervals of positive Z contain larger numbers of stars than the corresponding intervals of negative Z which are supposed to be complete. The intervals where incompleteness is most to be feared have been indicated by a colon following the observed number in Table 5. They have had little or no weight in determining the constants of the theoretical distributions best fitting the counted numbers.

The columns O_{I}, O_{II} and O_{III} in Table 5 indicate respectively the counted numbers of stars with parallaxes exceeding ″·100, between ″·070 and ″·100 and between ″·050 and ″·070. The columns next to these show the computed numbers, the modulus l of the velocity distribution being shown at the top. The last line of the table shows the total weight attributed to each determination[1]).

TABLE 5.

Distribution of velocity components perpendicular to the galactic plane for stars nearer than 20 parsecs.

Z	F stars						G dwarfs						K and M dwarfs			
	O_{I}	·065	O_{II}	·048	O_{III}	·040	O_{I}	·045	O_{II}	·035	O_{III}	·035	O_{I}	$l_1 =$ ·050, $l_2 =$ ·020, $\theta_1 =$ ·75	$O_{\text{II+III}}$	$l_1 =$ ·050, $l_2 =$ ·020, $\theta_1 =$ ·78
0·0— 5·0	5·9 :	4·3	3·8 :	4·4	13·8 :	9·3	8·1 :	5·6	2·2 :	3·5	8·2 :	9·4	14·2 :	14·0	26·8 :	20·0
5·0—10·0	2·2	3·5	3·2 :	4·0	4·8 :	8·5	0·0	5·1	4·9 :	3·3	3·5 :	8·9	11·3	12·4	18·2 :	17·8
10·0—15·0	3·1	2·3	5·5	3·1	6·6 :	7·3	4·5	4·2	0·0	3·0	13·3 :	7·9	12·1	10·1	7·4 :	14·3
15·0—20·0	0·0	1·2	1·6	2·2	4·0	5·7	4·5	3·1	4·2	2·5	6·5	6·6	7·6	7·4	7·4	10·4
20·0—25·0	0·0	0·5	1·0	1·4	7·8	4·2	3·5	2·1	4·4	1·9	5·6	5·1	2·0	5·0	8·0	6·9
25·0—30·0	1·0	0·2	0·0	0·8	2·6	2·8	1·0	1·3	1·0	1·4	5·0	3·8	3·0	3·1	3·2	4·3
30·0—40·0	0·0	0·1	1·6	0·6	0·0	2·7	1·0	1·0	0·0	1·6	1·0	4·4	4·0	3·3	10·2	4·4
40·0—60·0	0·0	0·0	0·0	0·1	0·0	1·0	0·0	0·2	1·5	0·8	5·3	2·2	4·0	2·7	0·0	3·4
>60·0	0·0	0·0	0·0	0·0	2·0	0·0	0·0	0·0	0·0	0·1	0·0	0·1	1·0	1·3	2·0	1·7
w	8		7		8		16		6		10		36		24	

The velocity distribution for the K and M dwarfs has been represented by a combination of two Gaussian distributions (compare formula (5)). The constants finally adopted for all K and M dwarfs are

$$l_1 = 0{\cdot}050,\ l_2 = 0{\cdot}020,\ \theta_1 = 0{\cdot}775,\ \theta_2 = 0{\cdot}225. \quad (8)$$

The mean absolute magnitude of the stars from which this velocity law was determined is $+7{\cdot}8$.

Peculiar velocities of B stars.

Though the B stars are very irregularly distributed and evidently not well suited for an investigation which supposes a "thoroughly mixed" system of stars, it is nevertheless of interest to see to what extent their general distribution in the direction of the galactic poles conforms with what we should expect from their velocity distribution and the known value of $K(z)$.

In connection with the data given in Table 3 it has been noted that the radial velocities of B stars are unsuitable for a determination of $\overline{|Z|}$: it can hardly be expected that the velocity distribution of the rather exceptional B stars in high galactic latitudes will be identical with that of the stars situated in the large agglomerations near the Milky Way. There can be no doubt that the average peculiar velocity of the brighter B stars in the galactic regions is very much smaller than the value of ± 8 km/sec indicated in Table 3[2]). For a direct determination of this average velocity we must turn to proper motions. The results in Table 6 are from an unpublished investigation made several years ago.

The B0 — B5 stars in BOSS' Preliminary General Catalogue were separated into four groups according to their apparent magnitudes (first column). The numbers of stars are in the second column. In the following columns $\overline{|\tau_5|}$ denotes the average without regard to sign of the τ components reduced to the 5th magnitude, the average reduction factor f being in the 5th column; $\overline{r_\tau}$ is the average probable error of τ_5. In the 6th column $\overline{|\tau_5|}$ has been corrected for probable

[1]) Strictly speaking there will remain some incompleteness outside these limits, as some of the stars with space velocities above 9.5 km/sec will have transverse velocities below 9·5 km/sec and may, therefore, have been excluded. In order to correct for this additional incompleteness we must multiply the weights on the cards by factors representing the reciprocal of the chance that a star with a space velocity V has a transverse velocity below 9·5 km/sec. This factor is equal to $V/\sqrt{V^2 - 9{\cdot}5^2}$ and has been applied in the above investigation.

[1]) In the group with parallaxes exceeding 0″·100 this is simply the number n, of stars used; in the group with parallaxes between ″·070 and ″·100 $w = n/1{\cdot}5$ and in the third group $w = n/2{\cdot}25$.

[2]) Compare, for instance, KAPTEYN'S results in *Mt. Wilson Contributions* Nos. 82 and 147.

error. The next column contains the simple algebraic average of $\upsilon_5/\sin\lambda$. Regions nearer than 30° to apex or ant-apex were excluded so that the accuracy of the average secular parallax has only slightly suffered by the fact that no weights proportional to $\sin^2\lambda$ were used. Assuming a solar motion of 20 km/sec we find the average linear peculiar velocities in km/sec shown under $\overline{|V_\tau|}$ in the eighth column. In order to see to what extent a few large proper motions may have influenced the results I have repeated the computations using median values of $|\tau_5|$ instead of averages; the median peculiar velocities so obtained are shown in the last column.

TABLE 6.

Peculiar velocities of Bo — B5 stars.

1	2	3	4	5	6	7	8	9
$m_{vis.}$	n	$\overline{\vert\tau_5\vert}$	$\bar{r}_\tau$	f	$\overline{\vert\tau_5\vert}_{corr.}$	$\overline{\upsilon_5/\sin\lambda}$	$\overline{\vert V_\tau\vert}$	Median
<4·50	103	±″·0038	±″·0037	0·53	±″·0029	+″·0205	±2·8	± 1·8
4·50—5·29	97	·0087	·0054	1·00	·0059	·0261	4·5	3·4
5·30—5·80	86	·0136	·0065	1·35	·0088	·0288	6·1	4·5
>5·80	64	·0189	·0063	1·74	·0137	·0258	10·6	10·9

The stars used in the table lie only in such areas of the sky where the components make large angles with the galactic plane. On the average this angle is 63° and the τ components will therefore reflect about half of the galactic components of the peculiar motions. The values given will thus be higher than the true averages of Z. They will also be higher on account of the influence of systematic errors in the proper motions. The error caused by the latter may be responsible for most of the increase of $\overline{|V_\tau|}$ with apparent magnitude. The average peculiar velocity in the galactic plane can be estimated from radial velocities and from the τ components in other areas. From 180 bright Bo — B5 stars CAMPBELL obtained an average peculiar velocity of ± 6·5 km/sec [1]). From the τ components in two other areas I find corresponding averages of respectively ± 6·6 (55 stars) and ± 8·2 km/sec (53 stars).

It is difficult to determine the exact corrections which should be applied to the values $\overline{|V_\tau|}$ found above, but it seems probable that the true average velocity perpendicular to the galactic plane is smaller than ± 3·0 km/sec, which at least does not conflict with the average velocity derived from the numbers of B stars of different apparent magnitudes in high galactic latitude, which yield ± 2·8 km/sec (see section 6).

[1]) *Lick Bulletins*, **6**, 127, 1911.

Velocity distributions adopted.

The constants of the velocity distributions in the z-direction as finally adopted for the different types of stars are shown in Table 7. They have been so chosen that the average velocities are equal to the weighted averages found from Tables 3 and 5. Except perhaps for the K and M dwarfs the second Gaussian component is hardly more than a guess [1]). I have added it mainly for the purpose of getting an idea of the influence which the existence of such a component would have on the computations.

The moduli for the B and A type stars, the direct determinations of which remained somewhat uncertain, have been so adopted that with a preliminary value of $K(z)$ they yielded the right gradient of the numbers of stars of different magnitudes in a region near the galactic poles (section 6; compare also the remarks following Table 3 . The computations in this paper have been made in such a way that the B stars were practically excluded and could not influence the results.

TABLE 7.

Adopted constants for velocity distributions.

Spectrum	M	l_1	l_2	θ_1
Bo — B9		·200		
Ao — A9		·084	·020	·957
Fo — F9		·049	·020	·957
Go — G9	$\leq +\ \cdot 4$	·059		
» »	$\geq +\ \cdot 5$	·042	·020	·910
Ko — Mc	$\leq + 3\cdot 4$	·039	·020	·915
» »	$\geq + 3\cdot 5$	·050	·020	·775

What we need for the computations in this article is a knowledge of the velocity distribution of stars of a given absolute magnitude. In order to find this distribution we must know for each absolute magnitude the relative numbers of stars of different spectral

[1]) Among the 479 radial velocities used for the determination of the velocity distribution there are 5 peculiar velocities larger than 60 km/sec. 4 Out of these 5 lie in the zone ± 40° to ± 55° latitude, so it is not impossible that not all 5 stars represent high Z-velocities. I assume that about one third of these stars is already represented by the Gaussian component with a modulus between ·040 and ·050, so that we might suppose that there remains an excess of about 2 Z-velocities higher than 60 km/sec which would have to be represented by a lower modulus. From the available evidence on high velocities it appears rather improbable that this modulus would be smaller than ·015 (the value found in *Groningen Publications* No. 40, Table 12, from all high velocities available). I have assumed ·020. With this modulus we would have an average of about 5 % in this Gaussian component (except for the later type dwarfs where the percentage is probably much higher). I estimate that the uncertainty of this percentage is about 5 % in each direction. Accordingly I have made in the eighth section some different solutions in which a corresponding variation of the velocity law has been applied.

types. These have been derived from the luminosity distributions contained in *Groningen Publications* No. 38, Tables 68 and 71. The luminosity curve of the K stars as given in these tables is probably considerably in error (see section 6). For the Ko—K9 stars I have accordingly adopted the luminosity distributions derived in *B. A. N.* No. 239 and shown under log Φ_3 in the last column of Table 16. The computed relative proportions of different spectral types are shown in Table 8.

TABLE 8.

Percentages of different spectral types (visual absolute magnitudes).

$M_{\text{vis.}}$	Bo—B9	Ao—A9	Fo—F9	Go—G9	Ko—Mc
— 1·5 to — ·5	39%	21%	5%	6%	28%
— ·5 » + 1·5	13	29	8	17	31
+ 1·5 » + 3·5	21	17	43	8	10
+ 3·5 » + 5·5	1	1	38	48	12
+ 5·5 » + 7·5	0	0	10	41	49
≥ + 7·5	0	0	0	0	100

Excluding the Bo—B9 stars the velocity distributions may be represented by the formula $\Sigma\, \theta_i \dfrac{l_i}{\sqrt{\pi}}\, e^{-\, l_i^2\, Z^2}$ with the following constants.

TABLE 9.

Constants of velocity distributions (after exclusion of B stars). Visual absolute magnitudes.

$M_{\text{vis.}}$	l_1	l_2	l_3	θ_1	θ_2	θ_3
< + 1·5	·084	·043	·020	·33	·62	·05
+ 1·5 to + 3·5	·084	·046	·020	·22	·73	·05
+ 3·5 » + 5·5		·046	·020		·92	·08
+ 5·5 » + 7·5		·046	·020		·85	·15
≥ + 7·5		·050	·020		·78	·22

We shall also make use of the velocity distributions for stars of a given photographic absolute magnitude. The necessary data are shown in Tables 10 and 11. The B stars have been excluded. As the new data about the K star luminosity curve were not at hand when the computations for photographic magnitudes were made Tables 10 and 11 have been computed with the aid of a preliminary and somewhat less radical correction to VAN RHIJN's luminosity curve of the K stars [1]). The luminosity curve used is shown under log Φ_2 in Table 16.

[1]) Besides, another correction was applied to VAN RHIJN's photographic luminosity curves for the bright G and K stars in order to take account of the fact that for these stars, according to Table 70 of *Groningen Publication* 38, the interval of one photographic magnitude is smaller than that of one visual absolute magnitude. So far as I can find out VAN RHIJN has omitted this correction.

TABLE 10.

Percentages of different spectral types (photographic absolute magnitudes).

M_{pg}	Ao—A9	Fo—F9	Go—G9	Ko—Mc
— 2·5 to — ·5	75%	11%	4%	10%
— ·5 » + ·5	75	7	4	14
+ ·5 » + 1·5	62	9	11	18
+ 1·5 » + 3·5	29	33	9	30
+ 3·5 » + 5·5	2	59	34	5
+ 5·5 » + 7·5	0	15	58	27
≥ + 7·5	0	0	0	100

TABLE 11.

Constants of velocity distributions (after exclusion of B stars). Photographic absolute magnitudes. [1])

M_{pg}	l_1	l_2	l_3	θ_1	θ_2	θ_3
≤ + ·5	·084	·044 [1])	·020	·71 [1])	·24 [1])	·05
+ ·5 to + 1·5	·084	0·44	·020	·59	·35	·06
+ 1·5 » + 3·5	·084	0·44	·020	·28 [1])	·66 [1])	·06
+ 3·5 » + 7·5		·046	·020		·90	·10
≥ + 7·5		·050	·020		·78	·22

The velocity distribution at different distances from the galactic plane.

It is of importance to verify that there are no systematic motions in the direction perpendicular to the galactic plane. The problem has been investigated by arranging the stars on each side of the galactic plane according to their distance. In each shell of distance the average radial velocity was computed. The radial velocities were corrected for solar motion, probable members of the Ursa Major system, stars with very uncertain velocities and stars with velocities larger than 50 km/sec were excluded. The average velocities obtained are tabulated in Table 12; the numbers of stars are added between parentheses. In the first part of this table the stars were arranged according to spectroscopic parallax (corrected as indicated on page 254). The limits indicated refer to stars in the zone ± 56° to ± 90° latitude. The corresponding limits for stars in the zones between ± 40° and ± 55° were taken 20% smaller in order to get roughly equal heights above and below the galactic plane for all stars contributing to a given average. The last line of Table 12 shows the results obtained from a grouping according to VAN RHIJN's mean parallaxes derived from spectrum, magnitude and

[1]) For eventual new computations it will be preferable to use the definitive corrections to the K star luminosity curve as shown in the last column of Table 16. The constants in Table 11 should then be changed as follows: In the first line we should take $l_2 =$ ·046, $\theta_1 =$ ·76 and $\theta_2 =$ ·20, in the third line $\theta_1 =$ ·34 and $\theta_2 =$ ·60; the other constants remain practically the same.

proper motions [1]). It contains all stars between ± 56° and ± 90° latitude of which the radial velocity has been measured. In this case no high velocities were excluded; the parallax limit between the first and second and between the next to the last and last group was taken ″·0070, slightly different from the limit ″·0055 indicated at the top of the table.

TABLE 12.

Algebraic average velocities at different distances from the galactic plane.

Stars \ π	north galactic cap				south galactic cap			
	″·002 to ″·005	″·006 to ″·010	″·011 to ″·020	> ″·020	> ″·020	″·011 to ″·020	″·006 to ″·010	″·002 to ″·005
Ao to A9		+ 0·3 (13)	+ 5·3 (29)	+ 1·5 (26)	— 4·5 (6)	+ 6·6 (8)	+ 11· (1)	
Fo » F9	+ 14· (1)	+ 1·2 (11)	— 3·4 (14)	+ 1·0 (46)	+ 3·0 (24)	— 0·5 (2)	+ 2·5 (2)	
Go » G9	+ 5·7 (6)	+ 2·0 (15)	+ 2·9 (26)	— 1·8 (51)	—16·8 (13)	+ 5·6 (14)	+ 4·6 (5)	— 3·5 (2)
Ko » K9	— 0·3 (15)	+ 2·3 (32)	— 2·8 (24)	+ 7·4 (29)	—11·4 (11)	+ 1·7 (10)	+ 4·6 (11)	+ 3·0 (2)
Mo » M9	— 8·1 (13)	— 2·6 (36)	+ 0·7 (14)	+ 10·0 (2)	+ 4·0 (2)	+ 2·4 (9)	— 9·0 (12)	— 14·8 (5)
Sum	— 1·8 (35)	+ 0·3 (107)	+ 1·2 (107)	+ 1·5 (154)	— 5·2 (56)	+ 4·0 (43)	— 0·6 (31)	— 8·3 (9)
G.P. No. 34	+ 4·4 (37)	+ 1·6 (67)	— 1·3 (121)	+ 2·2 (155)	— 7·0 (66)	+ 2·4 (81)	+ 1·2 (34)	+ 3·0 (7)

On the whole the results are entirely satisfactory in so far as they show no systematic motions in a direction perpendicular to the galactic plane. The fact that for stars further than 100 parsecs to the north and to the south of the galactic plane there appears to be no trace of systematic relative motion lends some support to the assumption stated in the first section, that in the z-direction the stars are thoroughly mixed.

When we consider Table 12 in detail we observe a few averages which deviate rather strongly: the stars with large parallaxes in the region surrounding the southern galactic pole appear to give a negative average velocity; also the most distant M type stars on both sides of the galactic plane show values which are systematically negative. It is doubtful whether these apparent deviations should be considered as real. The first mentioned phenomenon comes out a little more strongly if we limit ourselves to still nearer and absolutely fainter stars; on account of selection according to large proper motion among these stars the average peculiar velocity is rather large in this group and I think that the deviation is probably accidental. The effect shown by the M stars may possibly be real, but the larger deviations rest on very few stars.

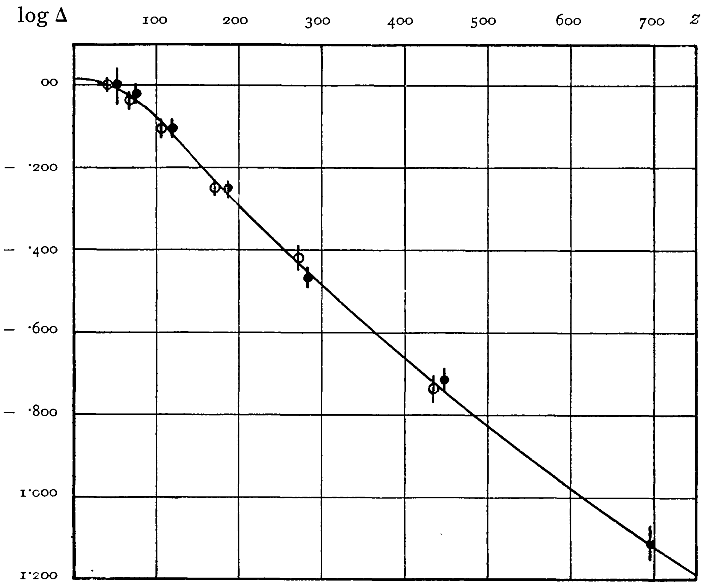
I have also investigated whether the average velocity without regard to sign is the same at different heights z. Only K and M giants have been studied. The differences between the averages are reasonably small. REDMAN has observed some faint K stars near the north galactic pole and has been kind enough to communicate his unpublished results to me. His 15 velocities if corrected for the usual solar motion average + 5·0 km/sec, and without regard to sign ± 14·2 km/sec; the latter average agrees well enough with the average z-velocity determined from brighter K stars (c.f. Table 3). The average visual magnitude of REDMAN's stars is $7^{\cdot m}3$.

[1]) *Groningen Publications*, No. 34.

4. *The density distribution.*

The variation of the density with height above and below the galactic plane has been taken from VAN RHIJN'S results in *Groningen Publications*, No. 38. With the aid of the distribution of the parallaxes of stars of given proper motion and apparent magnitude as found in an earlier publication VAN RHIJN derived the numbers of stars between given limits of parallax and absolute magnitude. A small correction has been applied to the mean parallaxes of faint stars of very small proper motions, as indicated in Table 11, *Groningen Publications*, 38. This correction does away with an a priori improbable systematic difference in the density distributions of absolutely bright and faint stars in low galactic latitudes which was observed when the uncorrected mean parallaxes were used. VAN RHIJN's corrected numbers are in Table 12 of his publication. The numbers in the third part of this table were used to compute the density distribution in a direction perpendicular to the galactic plane for four intervals of absolute magnitude, as indicated in Table 13 below. The computation was made as follows: for each column in VAN RHIJN's Table 12 we formed the differences in log Δ between the successive parallax groups and also the weight of this difference. For the absolute magnitude group — 1·5 to — ·5 the differences found from the column headed — 6·46 were combined with those from the column headed — 5·48, using the weights just computed except that those for the — 5·48 column were halved. For the other groups the differences in log Δ were computed in a similar way, the group — ·5 to + 1·5 being obtained from the columns — 5·48, — 4·50 and — 3·50, giving half weights to the first and the last mentioned column, etc. The logarithms of the densities obtained were tentatively corrected for the asymmetrical position of the sun with

FIGURE 1.



Logarithms of densities at different distances from the galactic plane.

Dots and circles represent stars of absolute magnitudes — ·5 to + 1·5 and + 1·5 to + 3·5 respectively. The accuracy of the points is indicated by vertical lines, the half lengths of which are equal to the mean errors.

respect to the galactic plane. From a discussion of the positions of δ Cephei variables HERTZSPRUNG found that the sun is situated 38 parsecs to the north of the average plane of these variables [1]). From several different types of stars GERASIMOVIČ and LUYTEN have recently derived a similar value (33 parsecs) [2]). Let us assume, for a moment, that the sun is at a distance of + 30 parsecs from the true plane of symmetry and that the densities in *Groningen Publications* No. 38 rest equally on stars north and south of the galactic plane; the corrections to be applied to the tabulated values in order to reduce them to the corresponding distances from the plane of symmetry can then be computed. The correction to log Δ is not very important, it averages + ·009 for $z < 80$ parsecs and is negligible for larger values of z. Though it is doubtful whether the actual conditions are such as mentioned above the correction has tentatively been applied.

The final values of log Δ are shown in columns 3 to 6 of Table 13. In each column I have added a constant of such size that the first number is reduced to zero. The first column shows the parallax limits, the second column shows the average distance from the plane of symmetry to which each density in the first part of the table refers. The estimated mean errors of log Δ have been added; they were computed on the assumption dat the unit of weight used by VAN RHIJN corresponds to a mean error of ± ·082. A graphical representation of the change of log Δ with z for the groups with absolute magnitudes between — ·5 and + 3·5 will be found in Figure 1.

TABLE 13.

Logarithms of densities at different distances.

π	z \ M	± 40° to ± 90° galactic latitude				0° to ± 20° galactic latitude			
		—1·5 to —·5	—·5 to +1·5	+1·5 to +3·5	+3·5 to +5·5	—1·5 to —·5	—·5 to +1·5	+1·5 to +3·5	+3·5 to +5·5
> ″·0158	44		·000 ±·043	·000 ±·017	·000 ±·020		·000	·000	·000
″·0100 to ″·0158	69	·000 ±·092	—·023 ±·025	—·036 ±·022	—·014 ±·033	·000	+ ·018	+ ·048	+ ·119
·0063 » ·0100	110	—·061 ±·050	—·104 ±·020	—·109 ±·020	—·081 ±·039	— ·062	+ ·011	+ ·073	+ ·123
·0040 » ·0063	174	—·232 ±·032	—·246 ±·018	—·251 ±·019	—·170 ±·036	— ·153	— ·044	+ ·035	+ ·070
·0025 » ·0040	275	—·504 ±·039	—·465 ±·018	—·422 ±·027	—·365 ±·028	— ·345	— ·131	— ·019	
·0016 » ·0025	438	—·777 ±·035	—·706 ±·026	—·738 ±·031		— ·566	— ·266	— ·117	
·0010 » ·0016	694		-1·113 ±·039						

In the last part of the table the densities found in corresponding shells of distance in the area from 0° to ± 20° galactic latitude have been added. Except in the first group there is no evidence of any decrease in density for distances smaller than 250 parsecs. This fact gives some additional confidence in the general method according to which the densities have been computed. The decrease shown by the first group is probably due to the strong galactic concentration of the B and A type stars; part of it may be due to systematic errors in the mean parallaxes of the absolutely brightest stars. On account of this uncertainty a relatively small weight has been given to this group in the computation of the mean force $K(z)$.

In his article "On the decrease of star-density with distance from the galactic plane" [3]) LINDBLAD

[1]) *A. N.* No. 4692, 1913.

[2]) *Proceedings National Academy*, *Washington*, **13**, 387, 1927.

[3]) *Arkiv f. Matematik*, *Astronomi o. Fysik*, Bd. **19 B**, No. 15, 1926; *Meddelanden Upsala*, No. 14.

discusses the density distribution of Go — K2 giants and B8 — A3 stars in a region centered at roughly 103° galactic longitude and + 46° latitude. Assuming the stars to be distributed in plane parallel layers we may use these data for a determination of the density distribution in the direction perpendicular to the galactic plane. The relevant data will be found in Table II of LINDBLAD's article (for the Go — K2 giants the average of the columns DI an DII was assumed). A similar investigation has been made by J. H. PETERSSON [1]) who investigated a region near 190° galactic longitude and + 27° latitude. On account of the lower latitude of this region a greater extrapolation is necessary to arrive at the density distribution in the z-direction. It will be seen from Figures 2 and 3, where LINDBLAD's and PETERSSON's densities have been plotted, that the two determinations are in good agreement.

FIGURE 2.

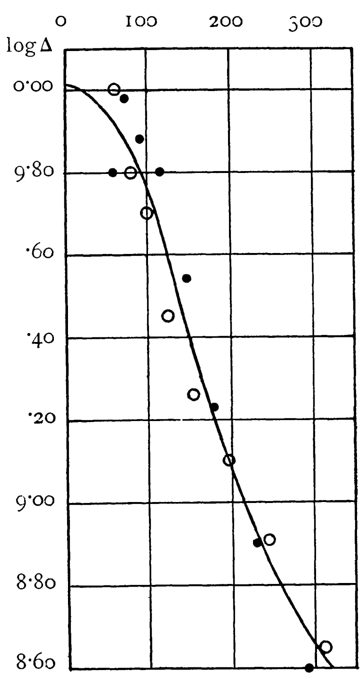
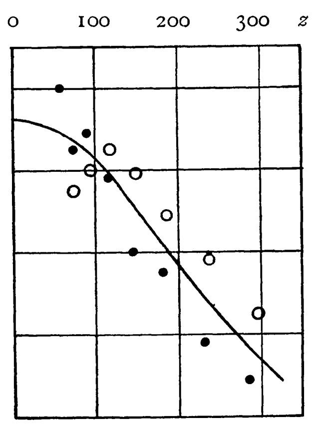


FIGURE 3.



Density distributions of A stars (Figure 2) and Go—K2 giants (Figure 3) in a direction perpendicular to the galactic plane.
Dots refer to LINDBLAD's results, circles to those of PETERSSON.

The density distributions were derived from accurate observations of the spectra of stars down to about 11·0 photographic magnitude, the classification being made according to Draper classes as well as to new criteria which were especially studied by LINDBLAD and his collaborators. The main difficulty is in the calibration of the absolute magnitude criteria and especially in the determination of the distribution of the true absolute magnitudes of stars of given spectral characteristics. In this respect the results mentioned may still be considered as more or less uncertain. In the next section the force $K(z)$ derived from these data will be compared with that derived from the densities of *Groningen Publication*, No. 38.

A special region surrounding the north galactic pole has been investigated by MALMQUIST [1]) who has determined colour indices of all stars in this region down to about the 13th magnitude. In the paper quoted the density distribution of the white stars (colour index between ·00 and ·24) has been discussed. I have not used these densities in the present investigation because the results appeared to be uncertain for two reasons. In the first place it is to be feared that the accidental errors of the colour indices have caused an undue proportion of faint stars which in reality possess somewhat larger colour indices to be included in the group of white stars, thereby spoiling the density results. In the second place 15 out of the total number of 39 white stars discussed were excluded because of alleged membership of the Coma cluster. The reality of this cluster is, however, still doubtful and the exclusion is more or less arbitrary.

5. *Derivation of the acceleration $K(z)$ up to $z = 500$ parsecs.*

If $K(z)$ is expressed in cm/sec², z in parsecs, and if $\overline{Z^2}$ denotes the average value of the square of the velocities in km/sec, we find from formula (7)

$$K(z) = 7{\cdot}48.10^{-9}\ \overline{Z^2}\ \partial \log (\overline{Z^2}.\Delta)/\partial z. \qquad (9)$$

For stars with a Gaussian distribution of the peculiar velocities $\overline{Z^2}$ is independent of z and the only function we need is $\partial \log \Delta/\partial z$. In order to obtain a first approximation of $K(z)$ I started from this assumption. For each of the four intervals of absolute magnitude shown in Table 13 $\partial \log \Delta/\partial z$ was derived graphically from plots showing the relation of log Δ to z. This relation is illustrated in Figure 1 in which I have plotted the results for the groups with absolute magnitudes between —·5 and + 1·5 (dots) and between + 1·5 and + 3·5 (open circles) [2]); the beautiful agreement between the densities in the two intervals speaks well for VAN RHIJN's densities.

It is probable that over the first 100 parsecs or so $K(z)$ will increase about proportionally with z, so that log Δ will be proportional with z^2. Accordingly each log Δ curve was drawn in such a way that the first part fitted a parabola, $\log \Delta = a z^2$. In Figure 1 a has been taken equal to —0·9.10⁻⁵; the parabola fits the observed points up to $z = 100$.

[1]) *Meddelanden Upsala*, No. 29, 1927. Compare especially Table XXVIII, p. 50.

[1]) *Lund Meddelanden*, Serie II, No. 37, 1927.
[2]) In order to prevent overlapping the dots and open circles were shifted a little apart in directions parallel to the curve.

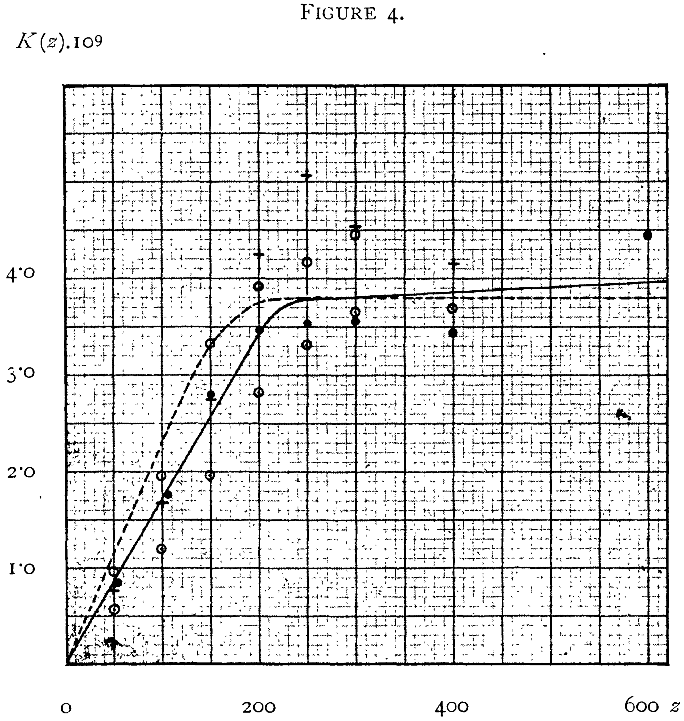
Having derived a first approximation of $K(z)$ from these graphs we can proceed to compute $\overline{Z^2}$ for different distances and use formula (9) to derive a second approximation of $K(z)$. By this method of successive approximations I arrived at the value of $K(z)$ shown in the second column of Table 14 and by the dotted curve in Figure 4.

In the course of the later investigations it became evident that the velocity distributions preliminarily assumed in the computations of this $K(z)$ (especially that of the B stars) needed considerable corrections. Accordingly it was decided to make a new computation of the force (to be denoted by $K'(z)$) with the corrected velocity distributions. In these new computations the first step was to eliminate the B stars. Assuming $l = \cdot 200$ (Table 7) and a percentage of B stars as indicated in Table 8 the preliminary force $K(z)$ enables us to compute in each interval of absolute magnitude the densities of B stars at different heights. Let us denote by Δ' the densities remaining after subtraction of the B star densities and let us compute $\overline{Z^2}$ for different values of z with the aid of $K(z)$ and the definitive velocity distributions in Table 9. A graphical determination of $\partial \log (\overline{Z^2} . \Delta')/\partial z$ yields the corrected values of the force as shown, separately for the four groups of absolute magnitude, in columns 3 to 6 of Table 14; the relative weights of the four determinations are added between parentheses. The weighted average $K'(z)$ is in the 7th column. A smooth graphical representation of $K'(z)$ and of the points derived from the separate groups is shown in Figure 4 (full drawn curve). The agreement of the results of the various groups as shown in this figure and in the table is satisfactory and gives confidence in the general correctness of the derived force.

For the convenience of eventual later computations of density distributions the values of $\int_0^z K'(z)dz$, integrated with the aid of the smooth curve in Figure 4, are added in column 8. ($K'(z)$ is always expressed in cm/sec², z in parsecs).

I have also derived $K(z)$ from LINDBLAD's and PETERSSON's data. In this case the assumption of a constant average velocity will not be far wrong. For the yellow giants the average velocity is ± 15·3 km/sec according to Table 7. Table 7 yields an average velocity for A stars amounting to ± 7·5 km/sec. It was found from radial velocities that for A0 — A3 stars the average velocity was about 15 % lower than that

FIGURE 4.



The acceleration $K(z)$.

z Is expressed in parsecs, $K(z)$ in cm/sec². The values finally adopted are represented by the full-drawn curve; the dotted curve is the preliminary result in the 2nd column of Table 14. The points plotted refer to different groups of absolute magnitude: — 1·5 to — ·5 crosses, — ·5 to + 1·5 dots, + 1·5 to + 3·5 dots surrounded by circles and + 3·5 to 5·5 circles.

TABLE 14.

z	$K(z) . 10^9$	Corrected values of $K(z) . 10^9$				Mean $K'(z) . 10^9$	$10^9 . \int_0^z K'(z)dz$	LINDBLAD and PETERSSON $K(z) . 10^9$	
		—1·5 to —·5	—·5 to +1·5	+1·5 to +3·5	+3·5 to +5·5			B8 —A3	gG0 — gK2
50	— 1·15	— ·78 (0)	— ·78 (1/2)	— ·57 (1)	— ·96 (1)	— ·77	21·	— 1·3	— 2·8
100	2·30	1·68 (1/4)	1·68 (1)	1·20 (1)	1·96 (1/2)	1·55	86·	2·7	5·5
150	3·15	2·74 (1/4)	2·80 (1)	1·96 (1)	3·32 (1/2)	2·59	193·	3·7	7·9
200	3·80	4·25 (1/2)	3·48 (3/2)	2·81 (1)	3·91 (1)	3·52	342·	2·8	6·9
250	3·80	5·06 (1/2)	3·53 (3/2)	3·31 (1)	4·17 (1/2)	3·78	526·	2·2	6·2
300	3·80	4·53 (1/2)	3·55 (3/2)	3·67 (1)	— 4·46 (1/2)	3·86	716·	— 1·8	— 5·8
400	3·75	— 4·14 (1)	3·44 (2)	— 3·70 (1)		3·68	1100·		
600	— 3·80		— 4·44 (1)			— 4·44	1884·		

for A0 — A9 stars, accordingly a value of $\pm$ 6·5 km/sec was adopted. However, this result is still rather uncertain. For $\overline{Z^2}$ we find 66 and 367 for the two kinds of stars respectively; combining these results with the values of $\partial \log \Delta / \partial z$ determined graphically from the curves in Figures 2 and 3 we arrive at the numbers shown in the last two columns of Table 14. The accordance is not particularly good; in both cases the force decreases more rapidly than in the former determinations; for $z > 150$ the two deviate in opposite directions from $K'(z)$. It may be noted that the regions treated are at fairly low latitudes and only about 45° from the direction opposite the centre so that the results may not be quite comparable to those derived above.

6. *Numbers of stars of different magnitudes for each spectral class separately.*

If the distribution of visual absolute magnitudes is denoted by $\Phi(M)$, the number of stars per square degree and of apparent magnitude $m - 0{\cdot}5$ to $m + 0{\cdot}5$ by $A(m)$, the relative density by Δ (chosen in such a way that $\Delta = 1$ for $z = 0$) and the distance by ρ, we have

$$A(m) = -\,3{\cdot}046.10^{-4}\int_0^\infty \rho^2\, \Delta\, \Phi\,(m + 5 - 5\log\rho)\, d\rho =$$

$$= 7{\cdot}01.10^{-4}\int_{-\infty}^{+\infty} \rho^3\, \Delta\, \Phi\,(m + 5 - 5\log\rho)\, d\,(\log\rho). \qquad (10)$$

If the frequency of the velocities Z is represented by a sum of Gaussian distributions, $\Sigma\,\theta_i\, l_i \frac{l_i}{\sqrt{\pi}} e^{-l_i^2 Z^2}$, the density Δ may be expressed as follows

$$\Delta = \Sigma\,\theta_i\; 10^{2{\cdot}67.10^8\, l_i^2 \int_0^z K(z)\,dz}. \qquad (11)$$

$z = \rho \sin b$ Is expressed in parsecs, $K(z)$ in cm/sec² and Z in km/sec.

Numbers of stars between given limits of apparent magnitude have been computed for each spectral class with the aid of formulae (10) and (11) and are compared with counts of stars in the Henry Draper Catalogue in Table 15. The numerical integration of formula (10) was made by steps of 0·20 in log ρ, which proved to be sufficiently small for all practical purposes.

The counts were made in the region from 11^h0^m to 14^h30^m right ascension and from + 10° to + 50° declination, extended to + 5° declination between 11^h30^m and 14^h0^m, and an equivalent region at the opposite point of the sky. The counts for the first region, surrounding the north galactic pole, are shown under O_n in the table, those of the region around the south galactic pole are shown under O_s. The columns headed O contain the average of the two, except in the last division where it is identical with O_s. The average latitude of the stars counted is estimated to be about 75°. Each region contains 1970 square degrees. According to SHAPLEY and others the Draper Catalogue is supposed to be practically complete up to 8·2 visual magnitude. In the southern hemisphere the limit is probably considerably fainter; for this reason the interval $8^m{\cdot}26$ to $8^m{\cdot}75$ has tentatively been added. It should be noted, however, that very considerable errors may arise from systematic and accidental uncertainty in the magnitudes.

In order to carry out the computation of Δ according to formula (11) we want a knowledge of $K(z)$. Up to $z = 600$ ps the force used has been practically that shown in the second column of Table 14 (dotted curve in Figure 4). The extrapolation beyond 600 ps has been made by means of the curve K_b in Figure 7, (for the explanation compare page 271). The extrapolation is of little consequence for the comparisons with the counts of stars in the Draper Catalogue, but it has been of influence on the computed numbers of fainter stars in Table 17.

The luminosity distributions, Φ, required for the calculation of the theoretical numbers were taken from *Groningen Publications*, No. 38, Table 68. The velocity distributions used will be found in Table 7. In the original computations for the B and A types the average velocity was supposed to be higher, viz. $\pm$ 4·03 km/sec ($l = {\cdot}140$) for the B stars and $\pm$ 9·1 km/sec ($l_1 = {\cdot}068$, $l_2 = {\cdot}020$, $\theta_1 = {\cdot}957$) for the A stars. Both velocity laws were known to be rather uncertain. A comparison of the numbers calculated with these velocity laws (shown under C in the table below) with the numbers observed in the Draper Catalogue shows very large discrepancies; the factors by which the observed and computed numbers differ show a pronounced increase with the apparent magnitude, indicating that the error is not in the adopted luminosity distribution but in the velocity law. The computations were, therefore, repeated with different velocity laws, the average velocity being so much decreased that the agreement between observed and computed values became satisfactory. The numbers finally computed are shown under C', the final velocity laws being those given in Table 7. (The average velocities found in this way are $\pm$ 2·82 km/sec and $\pm$ 7·6 km/sec for the B and A stars respectively).

A comparison of the columns O and C for the spectral types later than A shows that the agreement is very good for the F stars and reasonably so for the G stars. The K type stars show a very serious disaccordance, the computed numbers being about 2·5 times too large. As there is no distinct dependence of this factor upon the magnitude it is very probable

TABLE 15.

Comparison of counts in the Draper Catalogue with numbers of stars computed with the aid of $K(z)$. The numbers refer to a surface of 1970 square degrees and to about 75° galactic latitude. Preliminary or tentative numbers have been placed between parentheses.

$m_{vis.}$ / Spectr.	4·00 to 5·99					6·00 to 7·97					7·98 to 8·25					8·26 to 8·75				
	O_n	O_s	O	C	C'	O_n	O_s	O	C	C'	O_n	O_s	O	C	C'	O_n	O_s	O	C	C'
Bo to B9	2	9	6	(10)	7	9	15	12	(32)	11	1	3	2	(6)	1	0	2	2	(8)	1
Ao » A5	42	30	36	(33)	29	164	133	148	(219)	154	48	39	44	(62)	41	71	72	72	(140)	97
Fo » F8	22	21	22	22		288	224	256	274		153	138	146	116		283	301	301	326	
Go » G5	13	16	14	20		128	252	190	201		129	125	127	81		299	332	332	216	
Ko » K5	38	50	44	(106)	50	396	377	386	(1009)	473	213	167	190	(384)	175	362	318	318	(941)	429
Ma » Md	10	7	8	9	(8)	49	47	48	69	(57)	16	12	14	24	(20)	16	18	18	55	(45)

that neither the adopted force nor the velocity distribution (which, moreover, is well determined for these stars) can have caused the discrepancy. It is probable that the fault lies with the assumed luminosity distribution. In *B. A. N.* No. 239 a luminosity curve has been derived from data given by STRÖMBERG (see the last column of Table 16, log Φ_3). It deviates widely from VAN RHIJN's curve and the deviation is such that it explains all of the above discrepancy, as will be evident by comparing the observed numbers of K stars in Table 15 with those in the column C', which have been computed with the new luminosity curve.

The M stars in Table 15 also show some discordance, the computed numbers being about 1·5 times too large. In this case it is unlikely that the failure is due to the brighter part of the luminosity curve, as the average results derived from STRÖMBERG's data do not deviate appreciably from VAN RHIJN's curve. The very uncertain part of the luminosity curve between $+2^M$ and $+8^M$ may have influenced the computations, but even under the rather extreme assumption that these intermediate stars are entirely absent the computed numbers (shown under C') remain somewhat too high. It should be remarked that the difference between O and C in the last division of the table is probably due to incompleteness of the counts for red stars fainter than 8·2 visual magnitude.

Table 16 shows the luminosity distribution of K stars. The distribution adopted by VAN RHIJN is given under log Φ_1; log Φ_2, in the third column, is a combination of the luminosity curve as determined by VAN RHIJN from proper motion parallaxes and the results derived by the same author from trigonometric parallaxes [1]); log Φ_2 has been used for a preliminary correction to the luminosity curve in the computations of $A(m_{pg})$ in the eighth section. The last column of the table gives the new luminosity curve as derived in *B. A. N.*, No. 239, and extended towards the faint absolute magnitudes with the aid of VAN RHIJN's trigonometric parallax results.

TABLE 16.

Luminosity curves of K stars.

$M_{vis.}$	log Φ_1 + 10	log Φ_2 + 10	log Φ_3 + 10
− 4·0	3·02		2·41
3·0	3·49	3·00	3·11
2·0	4·06	3·95	3·68
− 1·0	4·90	4·78	4·49
0·0	5·80	5·43	5·65
+ 1·0	6·25	5·95	5·64
2·0	6·20	6·30	5·65
3·0	5·82	6·13	5·99
4·0	5·94	6·18	6·24
5·0	6·64	6·65	6·70
6·0	7·18	7·04	7·07
7·0	7·32	7·23	7·26
8·0	7·36	7·36	7·37
+ 9·0	7·40:	7·40:	7·40:

In Table 17 the computation of the numbers of stars of various types has been extended to the 12th visual magnitude. The corrected luminosity law, log Φ_3, was used for the K stars. Giants and dwarfs, and also in some cases the intermediate stars, have been tabulated separately. The table gives the number of stars between $m - 0·5$ and $m + 0·5$ (Harvard visual magnitude), m being given at the top of each column. The computations were made for three different latitudes, ± 80°, ± 40° and ± 20°, as indicated in the table. It was assumed that up to the distances of 12th magnitude stars in latitudes of 20° or greater the average density is constant in layers parallel to the galactic plane. It will be shown in section 9 that this assumption is very nearly fulfilled if densities averaged over all galactic longitudes are considered.

[1]) *Groningen Publications*, No. 38, Table 46, 2nd column (Method I, A) and 5th column respectively. The transition from the "Method I" curve to the "Trig." curve has been made between the absolute magnitudes + 2·5 and + 3·5 (international scale).

TABLE 17.

Computed numbers of stars per square degree for different visual magnitudes.

Draper Spectrum	Visual abs. mag.	SEARES' colour index	galactic latitude 80°					galactic latitude 40°					galactic latitude 20°				
			$4^m·0$	$6^m·0$	$8^m·0$	$10^m·0$	$12^m·0$	$4^m·0$	$6^m·0$	$8^m·0$	$10^m·0$	$12^m·0$	$4^m·0$	$6^m·0$	$8^m·0$	$10^m·0$	$12^m·0$
B0—B9		— ·07	·00069	·0028	·002	·00	·00	·00115	·0057	·013	·00	·00	·00250	·0127	·058	·07	·00
A0—A5		+ ·08	200	170	71	·24	1·20	234	250	139	·43	2·10	250	340	310	1·37	4·50
F0—F8		+ ·57	129	170	191	1·35	5·25	132	185	220	2·04	10·80	137	207	273	3·08	24·50
G0—G5	$\leq +$ ·4	+ 1·03	42	27	5	·00	·00	55	44	17	·01	·00	71	72	50	·12	·03
»	+ ·5 to + 3·4	+ 1·03	52	76	81	·46	1·38	52	80	104	·79	3·00	52	83	122	1·39	8·70
»	$\geq +$ 3·5	+ ·79	18	28	44	·58	5·25	17	27	43	·64	7·30	17	27	43	·66	9·50
K0—K5	$\leq +$ 1·4	+ 1·60	282	324	240	·81	1·58	294	378	360	1·79	4·17	320	454	540	4·27	16·70
»	+ 1·5 to + 3·4	+ 1·34	15	23	32	·28	1·12	15	24	35	·40	2·31	15	24	38	·52	4·90
»	$\geq +$ 3·5	+ 1·08	6	9	14	·19	1·82	6	9	14	·21	2·44	6	9	15	·22	3·16
Ma—Md	$\leq +$ 3·4	+ 1·88	56	60	38	·13	·27	60	75	61	·28	·73	64	92	103	·71	2·70
»	$\geq +$ 3·5	+ 1·76	·00000	·0009	·000	·00	·05	·00000	·0000	·000	·00	·06	·00000	·0000	·000	·00	·07
Sum			·00800	·0887	·716	4·04	17·92	·00980	·1129	1·006	6·59	32·91	·01182	·1435	1·552	12·41	74·76
Direct counts			·0085	·089	·68	4·4	17 4	·0115	·111	·96	6·0	30·8	·0153	·151	1·40	10·2	61·1

Readers should be warned that the numbers of A stars and giants in the columns referring to the 12th magnitude are quite uncertain, the uncertainty being due to insufficient knowledge of the exact percentage and distribution of high velocities.

Among other things the table informs us about the relative number of giants and dwarfs among stars of different apparent magnitudes. We see, for example, that, whereas among 6th magnitude K stars near the galactic pole about 1 out of 40 will be a dwarf fainter than $+ 3^{M}·4$, the chance that a high latitude K star of the 12th magnitude is such a dwarf is about 40%; at 20° latitude the same chance for a 12th magnitude star is only about 13%.

Sums of the numbers of stars of all types have been formed at the bottom of the table. The last line shows the results derived directly from counts of stars (Harvard visual scale). In the first division of the table (80° latitude) the B stars were omitted in the formation of these sums and the counted numbers were read from the curve in Figure 5. For the other latitudes the B stars were included and the counted numbers were derived from Table V, *Groningen Publications*, No. 27. It will be noted that the agreement between computed and counted total numbers is remarkably close down to the 12th magnitude.

TABLE 18.

$m_{\text{vis.}}$	$b = \pm 80°$ c. i.	$b = \pm 80°$ d	$b = \pm 40°$ c. i.	$b = \pm 40°$ d	$b = \pm 20°$ c. i.	$b = \pm 20°$ d	SEARES
4·0	+ ·90	± ·69	+ ·84	± ·70	+ ·74	± ·72	+ ·62
6·0	·99	·64	·94	·66	·88	·69	·67
8·0	1·05	·55	1·03	·59	·97	·66	·73
10·0	·95	·47	1·01	·50	1·05	·57	·79
12·0	+ ·85	± ·39	+ ·88	± ·42	+ ·97	± ·48	+ ·85

With the aid of the numbers given in Table 17 it is easy to derive the distribution of colours for stars of a given magnitude and from these the average colours of stars of given visual or photographic magnitude. The results are shown under c.i. in Table 18. They are arranged according to Harvard visual magnitude. The colour indices used for the different spectra and absolute magnitudes are those given by SEARES [1]) and are shown in the third column of Table 17. The mean square deviation of the colour indices from their average is given under d, in the third column of Table 18.

The average colour indices c_v, computed with SEARES' formula [2]) $c_v = + ·50 + ·029\, m_v$, are added in the last column. According to SEARES' curve in Figure 2 (l. c.) a correction of roughly + ·20 would reduce these to 80° latitude; for 40° latitude the correction would be + ·10 while for 20° it is approximately zero. The agreement between the present values and those computed from SEARES' data is not particularly good, the difference being probably due to a difference in division into spectral classes and to the fact that SEARES assumes a considerably greater proportion of dwarfs among the K stars.

7. *Counts of stars in high galactic latitudes.*

The only source from which we may get information about the force $K(z)$ at heights greater than 500 parsecs

[1]) *Astrophysical Journal*, **55**, 198, 1922; *Mt. Wilson Contributions*, No. 226, Table XII. I have assigned the colour indices given by SEARES for G giants to all G stars brighter than + 3·5 absolute magnitude; those given for K giants have been used for the K stars brighter than + 1·5 absolute magnitude, whereas to the K stars between + 1·5 and + 3·4 absolute magnitude a colour index intermediate between that given for dwarfs and for giants has been assigned.

[2]) *Astrophysical Journal*, **61**, 114, 1925; *Mt. Wilson Contributions*, No. 287.

is the number of faint stars observed in the direction of the galactic poles. At the same time counts of stars in these regions will afford a very welcome check on the general correctness of the force at smaller distances from the galactic plane. A discussion of the direct counts will be given in the present section. In the next section the counts will be compared with the computed numbers.

Visual magnitudes.

The principal data have been taken from the well-known study by VAN RHIJN [1]). Between the magnitudes 7 and 9 these data were supplemented by the results of PANNEKOEK's careful reduction of the Bonn and Cordoba Durchmusterung to the Harvard scale [2]). For the visual magnitudes VAN RHIJN's numbers rest on an earlier investigation by KAPTEYN [3]), the numbers down to $m = 10$ being identical with those given by KAPTEYN. For the fainter magnitudes VAN RHIJN has applied a considerable scale correction to KAPTEYN's results so as to bring them into accordance with the counts of the Selected Areas (the later Harvard-Groningen Durchmusterung) which, for the purpose of this comparison, were reduced to the Harvard visual scale.

On account of the uncertainty of the velocity distribution of these stars it has been thought preferable to exclude the B type stars in the following comparisons. The numbers of B stars of different apparent magnitudes were found from counts in KAPTEYN's M.S. copy of BOSS' Catalogue and from the Draper Catalogue [4]). The numbers adopted are shown in Table 19. It was assumed that the average colour index, Mt. Wilson photographic minus Harvard visual, was -0.20. In the zones $\pm 50°$ the numbers of stars fainter than 6·0 were not counted but computed on the roughly verified assumption that the average numbers are about 3 times higher than those in the 80° zone. It is easy to extrapolate the table to fainter magnitudes and to prove that beyond $m = 9$ the exclusion or inclusion of the B stars has practically no effect on the total numbers of stars.

The logarithms of the numbers of stars between $m - 0.5$ and $m + 0.5$ computed from VAN RHIJN's table and diminished by the numbers of B stars according to Table 19 are plotted in Figure 5 ($\pm 80°$ latitude) and in Figure 6 ($\pm 50°$ latitude); the curves represent the average of the counts in the northern and southern galactic hemisphere. As PANNEKOEK's counts must

TABLE 19.
Numbers of B0—B9 stars per square degree.

$m_{vis.}$ \ b	$+80°$	$+50°$	$-50°$	$-80°$
< 2·2	·0000	·0001	·0005	·0000
2·2 to 3·2	0	1	4	0
3·2 » 4·2	0	3	13	0
4·2 » 5·2	6	10	40	26
5·2 » 6·2	15	26	58	20
6·2 » 7·2	25	75	123	41
7·2 » 8·2	20	60	123	41
8·2 » 9·2	·0024 :	·0072 :	·0120 :	·0040 :

be considered more trustworthy than the older B.D. counts the $\log A(m)$ curves finally adopted were drawn through the points computed from PANNEKOEK's data (after exclusion of the B stars).

The computation of these latter points from the data in *Amsterdam Publ.*, No. 1, requires some explanation. They were made with the aid of PANNEKOEK's Table 58 which summarizes the results of the B. D. counts in regions of about 400 square degrees arranged according to galactic latitude and longitude. In this table Δ signifies $\log \bar{A} - \log \bar{A}_0$, where $\bar{A}$ represents the average number of stars per square degree found for each region from the detailed data in Table 57 and A_0 the number of stars between $m - 0.5$ and $m + 0.5$ in the schematical system from which PANNEKOEK has started. Log A_0 is tabulated in the second section of PANNEKOEK's Table 4. From this table I have recomputed the mean numbers, $\bar{A}_0$, for the zones 40° to 60°, 60° to 80° and 80° to 90° galactic latitude and for the magnitudes 5·6, 5·8, 6·9, 7·7, 8·3 and 8·9 to which Δ_1 to Δ_6 of Table 58 refer (compare page 71). For each of the Δ's tabulated we can compute the corresponding value of $\log \bar{A}$, from which the schematical A_0 has now been eliminated. The A's obtained were averaged over each of the 6 latitude zones considered and are tabulated in the first 7 columns of Table 20. In the last 4 columns the B stars have been excluded and the results have been smoothed and reduced to the latitudes $\pm 80°$ and $\pm 50°$.

In recent years considerable doubt has been thrown on the correctness of the Harvard visual scale by the investigations of SEARES.[1]) As my original computations, on the Harvard scale, showed a great divergence from the counts, it was feared that errors in the magnitude scale might have had an important influence on the results. Though it was found afterwards that the original discrepancy was mainly due to an error in the luminosity curve, yet the influence of possible scale errors appeared to be important enough to justify

[1]) *Groningen Publications*, No. 27, Table V, 1917.
[2]) *Publications of the Amsterdam Astronomical Institute*, No. 1, 1924.
[3]) *Groningen Publications*, No. 18, 1908.
[4]) For the region counted see page 263.

[1]) *Astrophysical Journal*, **61**, 284, 1924; *Mt. Wilson Contributions*, No. 288.

TABLE 20.

Values of log $A(m)$ derived from PANNEKOEK's reduction of the Bonn and Cordoba D. M.

b / $m_{vis.}$	B stars included						B stars excluded			
	+ 83°	+ 68°	+ 49°	— 49°	— 68°	— 83°	+ 80°	+ 50°	— 50°	— 80°
5·6	8·874	8·745	8·796	8·803	8·823	8·664	8·80	8·76	8·77	8·73
5·8	8·878	8·818	8·914	8·904	8·903	8·836	8·84	8·88	8·88	8·86
6·9	9·361	9·344	9·423	9·424	9·393	9·368	9·35	9·40	9·41	9·37
7·7	9·671	9·699	9·772	9·789	9·720	9·706	9·68	9·76	9·78	9·71
8·3	9·902	9·961	0·054	0·066	0·001	9·980	9·93	0·04	0·06	9·99
8·9	0·159	0·232	0·328	0·334	0·272	0·224	0·19	0·32	0·33	0·25

a recomputation of part of the results on the basis of the Mt. Wilson photovisual scale of magnitudes, this being the scale which deviates most from the Harvard magnitudes. The Potsdam visual scale shows deviations of the same character but about two times smaller.

In the following I have tentatively applied SEARES' corrections to the Harvard magnitudes, viz.:

$+ ·07\ (m - 6)$ between $6^{m}·0$ and $9^{m}·5$ and $- ·08\ (m - 12·5)$ for $m > 9·5$; they are in the second column of the following table. The third column, $\Delta_1 \log A(m)$, has been determined from the visual curves in Figures 5 and 6 (which are on the Harvard scale) and shows the consequent correction to be applied to log $A(m)$. If we assume that the photovisual scale is correct the intervals of one Harvard magnitude over which the counts were made also need correction. The correction to log $A(m)$ caused by the errors of the intervals is shown under $\Delta_2 \log A(m)$ in the fourth column, the total corrections, $\Delta_1 + \Delta_2$, being in the last column of the table.

TABLE 21.

Corrections for reducing log $A(m)$ to the Mt. Wilson photovisual scale.

m	Δm	$\Delta_1 \log A(m)$	$\Delta_2 \log A(m)$	$\Delta \log A(m)$
6·0	+ ·01	— ·005	— ·017	— ·022
7·0	·07	32	29	61
8·0	·14	62	29	91
9·0	·21	86	— 29	115
10·0	·20	74	+ 36	38
11·0	·12	38	36	— 2
12·0	+ ·04	— ·011	+ ·036	+ ·025

Photographic magnitudes.

The data for photographic magnitudes were taken from Table 5 of *Groningen Publications*, No. 43. The magnitudes in this table are on the international scale. To a certain extent the log $A(m)$ values have been smoothed by VAN RHIJN and for the fainter magnitudes, which have not been observed south of —15° declination, they have been tentatively corrected so as to represent the averages of complete latitude zones. The B0 to B9 stars were eliminated with the aid of Table 19 of the present article and the log $A(m)$ obtained are shown by the lower curves in Figures 5 and 6.[1])

The remarkable smoothness of these curves (when drawn on a larger scale) speaks strongly for the correctness of the Mt. Wilson magnitude scale over a long range of magnitudes. It should be noted that up to $m = 10·5$ or $11·0$ the curves have been derived entirely from visual magnitudes (from 5·0 to 10·5 they were checked by the present author who recomputed the numbers of stars of each photographic magnitude from PANNEKOEK's counts in Table 20 with the aid of the colour distribution found in Table 18). It is certainly remarkable that the numbers computed in this way fit so beautifully to the part of the curve beyond $m = 12$, which rests on counts of stars on photographic plates.

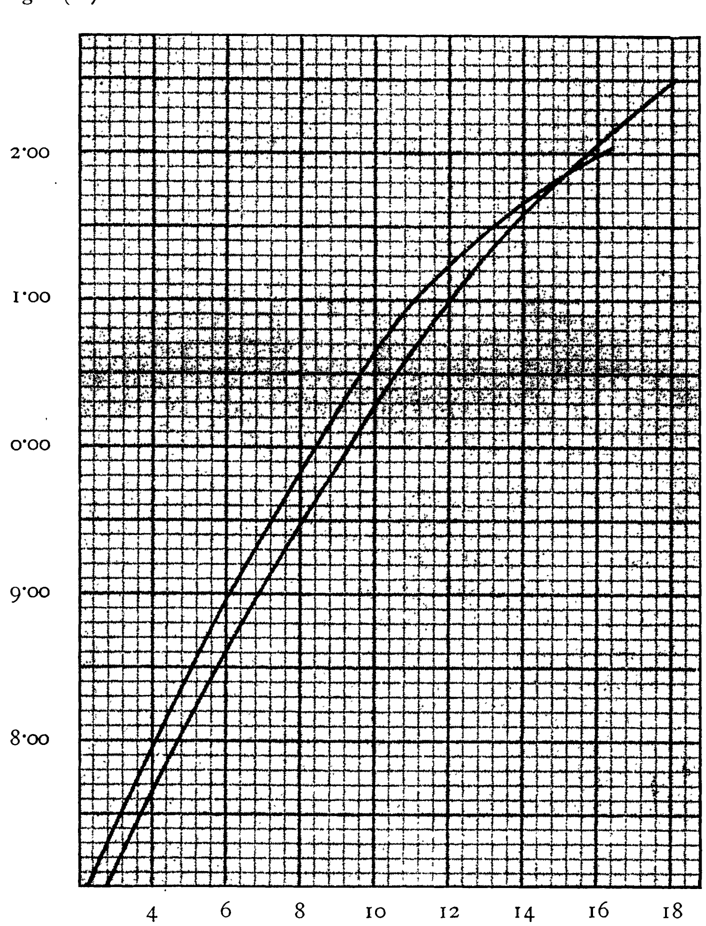
The regularity of the curves is also illustrated by the closeness with which they can be represented by a quadratic formula. Nowhere between $m = 3$ and $m = 18$ does the curve for $b = \pm 80°$ deviate more than ·025 from the formula

$$\log A(m) = - ·0122\, m^2 + ·612\, m - 4·60.$$

The average residual without regard to sign is only about ± ·011.

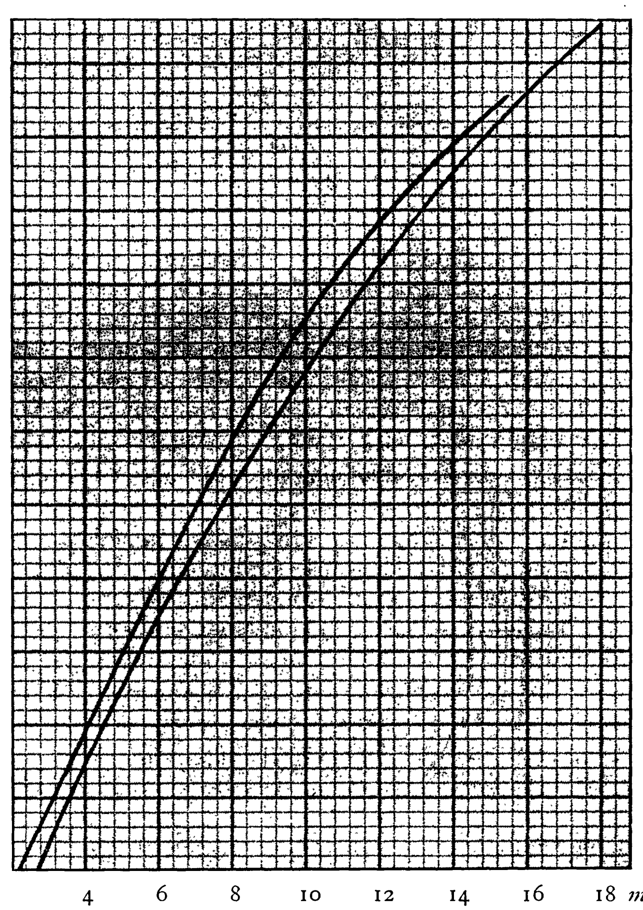
[1]) In *Mt. Wilson Contributions*, No. 301 (*Astrophysical Journal*, **62**, 343, 1925) by SEARES, VAN RHIJN, JOYNER and RICHMOND, numbers of stars have been derived from a material which is very similar to that used in VAN RHIJN's publication cited above. The mean distribution of stars as derived by the authors is shown in Table XVII of the *Contribution*. It is a little surprising to find that, for the fainter stars at least, the $A(m)$ derived from this table for 90° latitude come out nearly 25 % smaller than the corresponding $A(m)$ derived from VAN RHIJN's table, or from Figure 5 of the present article. Upon closer inspection it was found that most, if not all, of the discrepancy is due to an erroneous smoothing and extrapolation in the high latitude region (compare Figure 1, *Mt. Wilson Contributions*, No. 301). The curves connecting log $N(m)$ and the galactic latitude should evidently become horizontal near 90° latitude. The curves drawn deviate markedly from this apparently necessary condition and moreover they have been traced systematically below the observed points at 73° and 79° latitude.

FIGURE 5.



Star counts at $\pm$ 80° latitude.

FIGURE 6.



Star counts at $\pm$ 50° latitude.

Abscissae are apparent magnitudes, ordinates are logarithms of numbers of stars between $m - {}^1/_2$ and $m + {}^1/_2$. The upper curves refer to visual counts, the lower ones to photographic counts.

In comparing the curves for visual and photographic magnitude it is evident that beyond $m = 12$ there must be a systematic error in one of them, which causes the rapid approach of the two curves. There is every reason to believe that the error is in the visual rather than in the photographic curve.

8. *The acceleration $K(z)$ for z greater than 500 parsecs. Check of the acceleration for smaller distances.*

In the first part of this section the numbers of stars of given visual magnitudes will be computed and compared to the direct counts. The main purpose of these computations is to furnish a check on the accelerations derived in the fifth section.

The computations for visual magnitudes were made with the final values of the acceleration $K'(z)$ as given in the 7th column of Table 14 (Figure 4, full curve). Except for the faintest magnitudes the extrapolation of $K'(z)$ beyond 500 ps has relatively little influence on these computations. The extrapolation was made in agreement with K_b in Figure 7.

The computations of $A(m)$ have been made in the same way as those for the separate spectra (compare p. 263); the velocity distributions are given in Tables 9 and 11. The luminosity law has been found by addition of the luminosity laws for the separate spectra as derived by VAN RHIJN[1]). B0 — B9 stars have been omitted and a corrected luminosity curve was used for the K stars. For the visual magnitudes the definitive K star luminosity curve (given as $\log \Phi_3$ in Table 16 of the present paper) was used; the computations for photographic magnitudes were made before this luminosity curve had been derived and only a preliminary correction was applied; this preliminary luminosity curve is obtained from $\log \Phi_2$ in

[1]) *Groningen Publications*, No. 38, Tables 68 and 71.

Table 16 by applying the reductions from visual to photographic absolute magnitudes [1]). With the considerable uncertainty still attaching to the reduction of the visual luminosity curves to photographic ones it was not thought worth while to repeat the rather laborious computations of $A(m)$ with the definitive luminosity curve, the less so because it is easy to form a rough estimate of the corrections which this change would bring about. The visual and photographic luminosity distributions used are shown in the second and third columns of Table 22. For the convenience of eventual later computations I have added in the last column the photographic luminosity curve computed with the definitive luminosity curve of the K stars, but this has not been used in the computations for the present article.

TABLE 22.

Visual and photographic luminosity curves. (B stars excluded).

M	Visual	Photographic	
	$\log \Phi + 10$	$\log \Phi + 10$	$\log \Phi' + 10$
$-4·0$	2·74		
3·0	3·45	2·82	2·70
2·0	4·17	3·72	3·69
$-1·0$	5·00	4·63	4·60
0·0	5·91	5·43	5·40
$+1·0$	6·29	6·05	6·06
2·0	6·51	6·47	6·46
3·0	6·92	6·83	6·66
4·0	7·28	7·15	7·12
5·0	7·56	7·42	7·37
6·0	7·60	7·57	7·62
7·0	7·51	7·46	7·49
8·0	7·60	7·33	7·36
9·0	7·64	7·37	7·38
10·0	7·74	7·48	7·50
11·0	7·88	7·61	7·61
12·0	8·02	7·76	7·76
13·0	8·08	7·88	7·88
$+14·0$		7·97	7·97

From $8^{M}·0$ on the visual luminosity curve has been mainly determined from VAN RHIJN's curve for all spectral types together (his Figure 1). From recent trigonometric parallaxes I have lately computed some new points on the faintest part of the luminosity curve; these have been used for a slight correction of VAN RHIJN's curve.

In Table 23 the results of the computations for visual magnitudes are compared with the direct counts of stars in the zones $\pm 80°$ and $\pm 50°$ latitude respectively. The counted numbers (under O) were read from Figures 5 and 6. The computed numbers are shown under C. On the Harvard scale the agreement may be called perfect for both zones. As is evident from Table 20 the results of the comparison would not have been less favourable if the southern and northern polar regions had been compared separately, which speaks in favour of our assumption of regularity in the distribution of the stars. The agreement appears especially satisfactory if we consider that the luminosity curve has been almost entirely derived from stars of the 6th magnitude and brighter and that the derivation of the force, $K'(z)$, is mainly dependent upon proper motions and only to a slight degree upon the total numbers of stars of each magnitude.

The slight deviations between observed and computed numbers for $m = 4$ and $m = 7$ are probably due to errors in the assumed luminosity curve; the negative values of $O - C$ for $m = 13$ are probably due to a systematic error in the Harvard visual magnitudes.

TABLE 23.

Log $A(m_{\text{vis.}})$.

Comparison of observed and computed numbers of stars.

$m_{\text{vis.}}$	Harvard scale				Mt. Wilson photovis.		
	$b = \pm 80°$		$b = \pm 50°$		$b = \pm 80°$		
	O	C	O	C	O	C	C'
4·0	7·94	7·90	7·97	7·91	7·94	7·88	7·86
7·0	9·40	9·43	9·45	9·51	9·34	9·41	9·38
9·0	0·26	0·25	0·36	0·35	0·15	0·23	0·20
10·0	0·65	0·61	0·76	0·74	0·60	0·59	0·56
11·0	0·98	0·95	1·10	1·09	0·98	0·93	0·89
13·0	1·47	1·53	1·68	1·74	1·51	1·51	1·46

The observed log $A(m)$ on the Mt. Wilson photovisual scale was obtained from the second column by adding the corrections in the last column of Table 21. The corresponding correction to be applied to the luminosity curve and consequently to the computed log $A(m)$ is rather uncertain. VAN RHIJN's mean parallaxes for the separate spectra have been derived for a mean magnitude about 6·0. At this magnitude a correction of $-·02$ should be applied to the logarithms of the numbers of stars; this correction has been applied to the values of log $A(m)$ computed above.

It will be observed that between $m = 7$ and $m = 9$ the computed numbers are somewhat too high. If the computations are made with the force $K(z)$ given in the second column of Table 14 and shown as dotted curve in Figure 4, somewhat smaller values are obtained, as will be seen from the last column, C', of the above table. Though these values are in better agreement with the observed values for the interval 7·0 to 9·0 they are considerably too low for the fainter and brighter magnitudes so that, on the whole, K' seems to give the better solution. For the present

[1]) *Groningen Publications*, No. 38, Table 70. Compare also the footnote in the first column of page 258 of the present article.

this function should, therefore, be considered as the final result for the acceleration up to $z = 500$ ps.

It is necessary to go to stars fainter than the 13th magnitude in order to get information about the acceleration for distances greater than 500 ps. If the velocity laws were accurately known our knowledge of the numbers of stars down to the 18th photographic magnitude would probably be sufficient to get a fair estimate of $K(z)$ up to 3000 or 4000 ps; our present knowledge about the velocity laws is, however, not sufficiently accurate for this purpose.

If Z_0 is expressed in km/sec, z in parsecs, $K(z)$ in cm/sec², we find that the velocity needed by stars near $z = 0$ in order to reach a height z must be greater than or equal to $Z_0 = \sqrt{6{\cdot}17.10^8 \int_0^z K'(z)\,dz}$.

Assuming that beyond 300 ps the acceleration is given by K_b in Figure 7 it is found that, whereas velocities Z_0 equal to 30 km/sec and 46 km/sec are sufficient to reach distances of respectively 500 and 1000 parsecs from the galactic plane, distances greater than 2000 parsecs are only reached by stars with $Z_0 > 67$ km/sec. The computation of the densities at such large distances depends entirely on these high velocities, of which only a few have been observed in our neighbourhood.

The amount of uncertainty of $K(z)$ involved is best illustrated by repeating the computations with different velocity laws deviating within plausible limits from the velocity laws adopted in Table 11. The uncertainty of the percentage of stars with a velocity modulus $l = {\cdot}020$ was estimated to be about 5 °/₀ (compare the footnote on page 257). It should be noted that for the K and M dwarfs, to which nearly all 17th and 18th magnitude stars near the galactic poles belong, the uncertainty of the percentage of stars in this component is probably larger. The computations of $A(m)$ have been carried out with four different sets of velocity laws which I will denote by S_1, S_2, S_3 and S_4. S_1 represents the velocity distributions given in Table 11; in S_2 θ_3 has been taken zero for the first three absolute magnitude groups, equal to ·05 in the fourth group and ·17 in the last group, θ_1 and θ_2 being increased proportionally so as to keep the sums $\theta_1 + \theta_2 + \theta_3$ equal to 1·00; in S_3 θ_3 has been taken ·10 for the first three groups, ·15 for the fourth group and ·27 for the last group, θ_1 and θ_2 being decreased in proportion. In S_4 θ_1, θ_2 and θ_3 are the same as in S_1, but 2 °/₀ stars with $l = {\cdot}010$ have been added. This is a rather extreme supposition as the high velocities in general show a higher modulus, even in directions parallel to the galactic plane.

The luminosity law used in the computations was log Φ in the third column of Table 22. The acceleration used practically coincides with $K(z)$ (dotted curve in Figure 4) up to $z = 400$. Beyond this distance two different extrapolations of $K(z)$ were tried, as shown in Figure 7. These were chosen in accordance with the following considerations:

If we assume that the great galactic system is in rotation around a distant centre, say at a distance of 10000 ps, and if we adopt the following values for the constants of the rotation effects: $A = +{\cdot}019$ and $B = -{\cdot}011$ km/sec. ps, the corresponding velocity of rotation being 300 km/sec, we find that the attracting force, K, of this system on a unit of mass near the sun is 29.10^{-9} g.cm/sec². Let us assume, for a moment, that this force is composed of two parts, K_1 being exerted by a spherical mass around the centre and K_2 by a homogeneous ellipsoid symmetrical around the centre but extending beyond the sun [1]). In this case we have $K_1/K = 4A/3(A-B) = {\cdot}84$, so that $K_1 = 24.10^{-9}$ cm/sec², corresponding with a total spherical mass at the centre of $1{\cdot}71.10^{11}$ solar masses. In a region surrounding the sun and extended in a perpendicular direction above and below the galactic plane this mass would cause accelerations as shown in the second column of Table 24 (dots in Figure 7). We may note that the value assumed for the distance to the centre (or for the velocity of rotation) has hardly any influence on the computed accelerations. If the main attracting mass is supposed to have a very flattened shape and to extend almost to the sun, the accelerations $K(z)$ caused by this mass may come out rather different. For example, if it is supposed to be an ellipsoid of revolution around the axis of the great system with an axial ratio 1 : 10 and extending in the galactic plane to a distance of 9/10 of that to the sun, the total mass required is less than half as large as in the case of a spherical mass, viz. $8{\cdot}0.10^{10}$ solar masses (the corresponding density in this ellipsoid is ·34 solar masses per cubic parsec). In this case we find that the accelerations $K(z)$ due to the main attracting mass are as indicated in the last column of Table 24 and represented by circles in Figure 7.

TABLE 24.

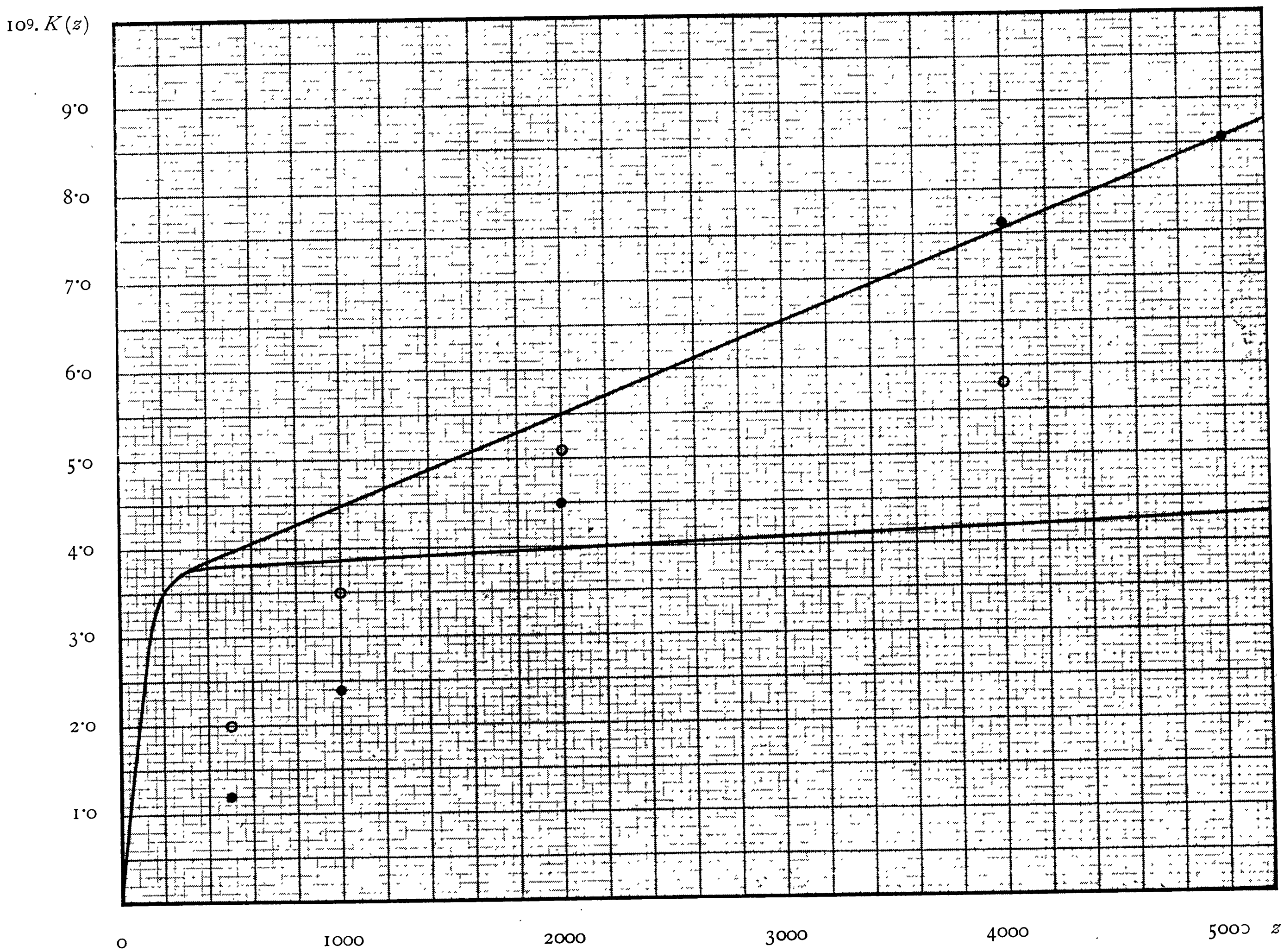
Hypothetical accelerations, $K(z)$, caused by the central attracting mass of the great galactic system.

z	Spherical	Ellipsoidal
250	$\cdot 6.10^{-9}$	$1{\cdot}0.10^{-9}$
500	1·2 »	2·0
1000	2·4 »	3·5
2000	4·5 »	5·1
4000	7·6 »	5·8
8000	9·1 »	4·7
16000	5·7 »	2·7

[1]) Compare *B.A.N.*, No. 120, p. 281 and No. 132, p. 88, 1927.

For the first extrapolation of $K(z)$, which I shall briefly denote by K_a, it was assumed that for $z = 5000$ the acceleration was equal to that given by the spherical central mass, viz. $8 \cdot 5 . 10^{-9}$ cm/sec², and the acceleration was assumed to increase linearly with z between 400 and 5000 ps. The second extrapolation, K_b, has also been taken as a straight line but the abscissa at $z = 5000$ is twice as small as in the case of K_a (compare Figure 7). It will be shown that even the 18th magnitude stars must practically all be situated within 4000 ps of the galactic plane so that a further extrapolation is hardly necessary. It is evident from

FIGURE 7.



Extrapolation of $K(z)$ up to $z = 5000$; K_a : upper line, K_b : lower line.
Dots and circles represent accelerations due to spherical and ellipsoidal central masses respectively.

the numbers in Table 24 that, from a theoretical point of view, K_b is almost certain to be too low; even the rather extreme ellipsoidal case gives considerably higher accelerations due to the central mass only. If we take account of the matter in the larger ellipsoids surrounding the sun these numbers should still be increased by some 20 °/₀ or more. The reason for its use is that with the velocity distributions adopted it gives a better representation of the numbers of very faint stars (compare Table 25, S_1) and that the comparison of the results obtained with K_a and K_b will give a good impression of the effect of a definite change in $K(z)$ on the computed numbers. I believe, however, that K_a will probably be rather nearer to the truth.

The logarithms of the computed numbers of stars are shown in Table 25. The various columns will not need much explanation. The first column gives the Mt. Wilson, or international, photographic magnitude, the following columns give the logarithms of the counted numbers of stars near the north and south galactic pole and the average as read from Figure 5; the further columns contain numbers computed with different assumptions about the velocity distributions.

TABLE 25.

Log $A(m_{pg})$.

Comparison of observed and computed numbers of stars for ± 80° galactic latitude.

m_{pg}	Counts			S_1		S_2		S_3		S_4	
	+80°	−80°	Mean	K_a	K_b	K_a	K_b	K_a	K_b	K_a	K_b
8·0	9·46	9·52	9·48	9·56	9·56	9·52	9·52	9·58	9·59	9·58	9·58
11·0	0·61	0·70	0·65	0·79	0·80	0·70	0·70	0·85	0·86	0·85	0·86
14·0	1·55	1·61	1·58	1·62	1·68	1·45	1·46	1·74	1·79	1·83	1·90
17·0	2·33	2·20	2·27	2·11	2·28	1·89	2·03	2·27	2·44	2·53	2·75
18·0	2·57	2·38	2·49	2·19	2·40	1·99	2·16	2·33	2·55	2·68	2·96

In comparing the numbers in Table 25 the principal attention should be given to the magnitudes 14, 17 and 18; there is no doubt that with the present state of our knowledge the magnitudes 11·0 and brighter can be better discussed from counts and computations according to visual magnitudes. These showed an almost perfect agreement and I believe we may safely conclude that the large negative residual $O - C$ shown for $m = 11·0$ in the above table is due the fact that the preliminary luminosity curve of the K stars was used and possibly also to errors in the assumed colour indices. If we assume S_1 to be correct the faintest magnitudes are well represented by K_b. As has been stated above I believe, however, that K_a is more probable than K_b; we should then have to conclude that the dwarfs fainter than + 7 absolute magnitude contain a greater percentage of considerable velocities than was assumed in S_1 (and still somewhat greater than in S_3), but that for the absolutely brighter stars S_1 is more nearly correct.

An unsatisfactory feature of the table is the fact that for all computations (especially if we leave the somewhat improbable S_4 out of consideration) the difference between log $A(18)$ and log $A(17)$ comes out much smaller than the corresponding difference in the observed numbers. This may be due to the special type of velocity distribution used, or, more likely perhaps, to the influence of star like extragalactic nebulae which may have sensibly increased the counts for $m = 18$.

It is interesting, though not encouraging for the determination of $K(z)$ at large distances, to compare the divisions headed S_1, S_2, S_3 and to notice how strongly the computed numbers are influenced by relatively slight changes in the velocity distributions. It will be necessary to extend considerably our knowledge of the velocity laws before we can hope to obtain a satisfactory determination of $K(z)$ beyond $z = 1000$. This may be done through a new discussion of the space velocities of the nearest stars [1]); another, and in some respects more satisfactory, way would be to determine the radial velocities of faint stars near the galactic poles; but as a considerable number of velocities would be required this seems hardly possible with the present telescopes. As an example to show what might be expected the following average velocities $\overline{|Z|}$ have been computed for stars of the 11th photographic magnitude with the four different assumptions as to the velocity distributions at $z = 0$ (the percentage of stars belonging to the Gaussian component with $l = ·020$ is added between parentheses):

S_1 ± 17·0 km/sec (29 % with $l = ·020$)

S_2 ± 13·2 „ (5 % „ „ „)

S_3 ± 18·9 „ (41 % „ „ „)

S_4 ± 22·4 „ (14 % with $l = ·010$)

For certain investigations it may be desirable to know approximately how the absolute magnitudes and distances of stars of a given apparent magnitude are distributed. These distributions have been obtained in the course of the computations of log $A(m)$; those obtained with S_1 and K_b are shown graphically in Figures 8 and 9. The irregular shape of some of these curves is due to the way in which the velocity distribution has been represented by a sum of different Gaussian components. It will be unnecessary to stress again the uncertainty of the curves for the fainter magnitudes. They are intended to get an insight into the nature of this uncertainty and as an illustration of what might be obtained with better knowledge of the velocity laws, rather than as a true representation of these distributions.

It will be noticed from an inspection of Figure 9 that the bulk of the 18th magnitude stars are dwarfs fainter than + 5·0 photographic magnitude (presumably mostly of K and M types and of high velocity) situated at distances from 500 to 4000 parsecs. As Professor HERTZSPRUNG remarked the 18th magnitude stars in high latitudes may accordingly be expected to have generally observable annual proper motions, of the order of ″·01 on the average.

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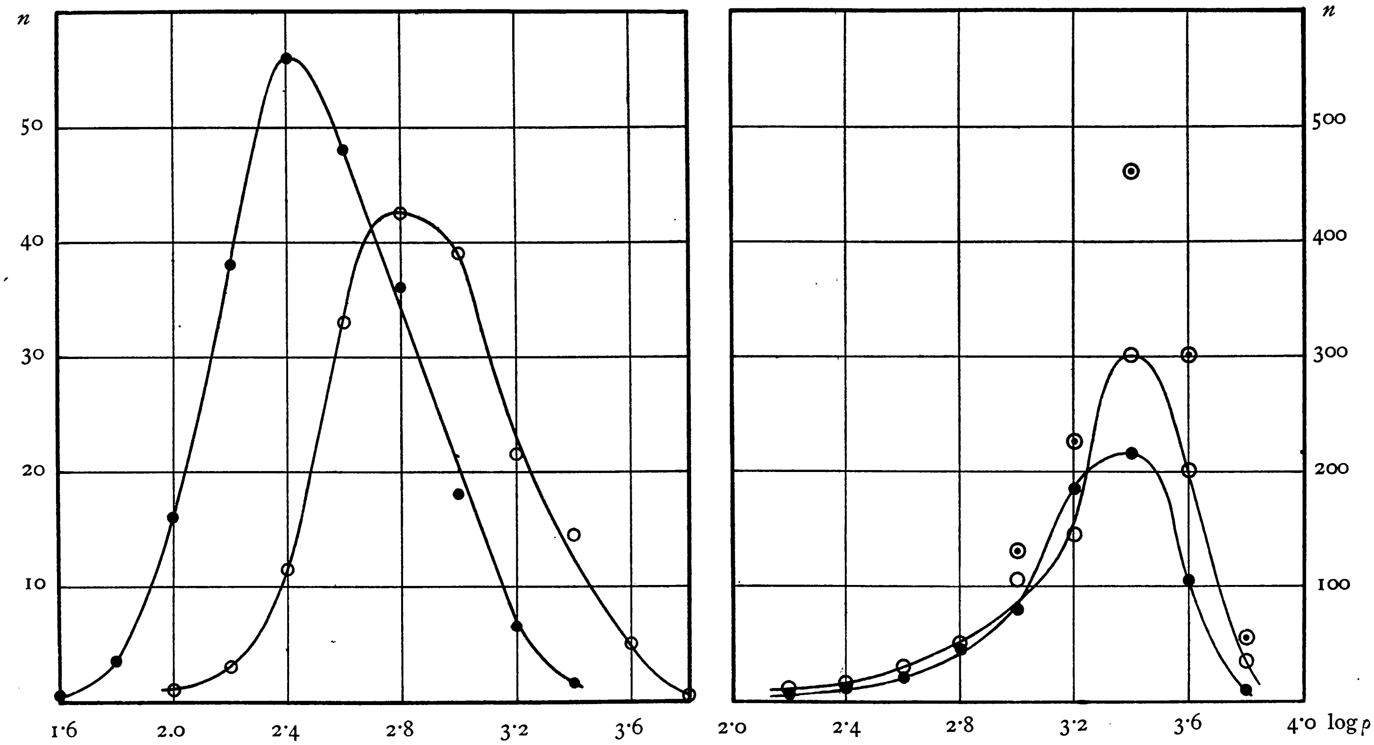
[1]) The data used rest on SCHLESINGER's parallax catalogue. In a few years, as many new southern parallaxes become available, it will certainly be worth while to redetermine the velocity distribution.

FIGURE 8. FIGURE 9.



Distribution in distance of stars at 80° galactic latitude.

Figure 8 shows the distribution for photographic magnitudes 11 (dots) and 14 (circles), Figure 9 for $m = 17$ (dots) and $m = 18$ (circles). The computations were made with K_b and S_1 (see text). For $m = 18$ the results with K_b and S_3 are also indicated (dots surrounded by circles). Abscissae are logarithms of the distance, ordinates are numbers of stars within intervals of ·2 of $\log \rho$. The numbers refer to an area of 3·56 square degrees, except for $m = 11$ where they refer to a ten times larger area.

9. *The distribution of stars between 20° and 60° latitude and the systematic change of density in the great galactic system.*

If the equidensity surfaces were planes parallel to that of the galaxy it would be possible from the data on the distribution perpendicular to the galactic plane to compute the distribution in any other direction provided it does not lie too close to the galactic plane. Table 26 and Figure 10 indicate how far the above supposition gives correct results if we consider means over all longitudes. Under O_n and O_s the table shows the logarithms of the average number of stars per square degree between $m - \frac{1}{2}$ and $m + \frac{1}{2}$ as computed from Table 6, *Groningen Publications*, No. 43. O_n indicates the number for northern latitude, O_s for southern. In Figure 10 these have been entered as dots and small circles respectively. The curves and the columns C in Table 26 indicate the results obtained by direct computation by formulae (10) and (11) (page 263); the different functions involved were taken the same as in the preceding section. The extrapolation used for $K(z)$ was that which has been denoted by K_b. The velocity distributions were taken in accordance with Table 11 (S_1).

In comparing the curves to the observed numbers we should abstract from the constant differences between the observed and computed values which persist through all latitudes down to 20°. These differences have been discussed in the preceding section; they may be due to relatively slight errors in the adopted velocity and luminosity laws.

Probably the main effect of these errors is to displace the computed curves in a vertical direction without perceptibly changing their shapes. To some extent this has been verified in the case of $m = 18$, where the computations were repeated for $b = 80°$ and $b = 20°$ with K_a instead of K_b and with a slightly different set of velocity laws (the same as S_3 in the preceding section except that 1% stars with $l = ·010$ have been added to the groups fainter than $+4^M$). The results are shown under C' in the last column of Table 26.

It is interesting to see how closely the average observed and computed values agree if we correct for the constant difference. Only near 10° the points begin to fall systematically too low by about 0·14.

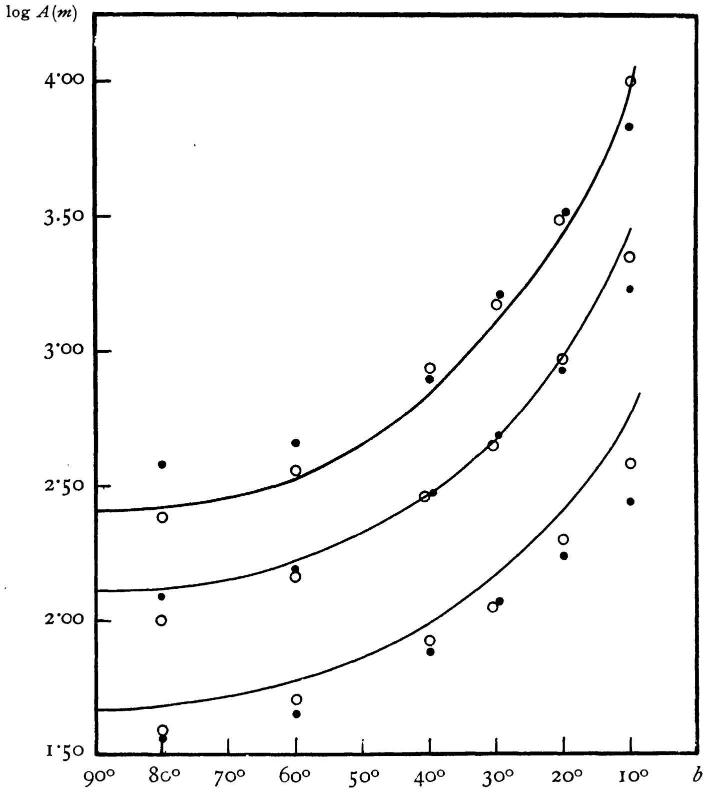
TABLE 26.

Log $A(m_{pg})$.

Comparison of observed and computed numbers of stars at different galactic latitudes.

gal. lat.	$m = 14$			$m = 16$			$m = 18$			
	O_n	O_s	C	O_n	O_s	C	O_n	O_s	C	C'
80°	1·56	1·59	1·68	2·09	2·00	2·12	2·58	2·38	2·40	2·57
60	1·65	1·70	1·77	2·19	2·16	2·22	2·66	2·55	2·53	
40	1·88	1·92	1·99	2·46	2·47	2·47	2·90	2·94	2·84	
30	2·05	2·05	2·16	2·68	2·66	2·68	3·20	3·18	3·11	
20	2·24	2·30	2·41	2·93	2·97	2·97	3·50	3·50	3·43	3·55
10	2·43	2·57	2·76	3·23	3·35	3·45	3·83	4·00	3·98	

FIGURE 10.



Relation between observed and computed numbers of stars at different galactic latitudes.

The curves represent computed values of $\log A(m)$ for the 18th, 16th and 14th magnitude. Dots and circles are observed values for the northern and southern galactic hemispheres respectively.

The inference which can be made from this result is that even for stars at 20° latitude and as faint as the 18th magnitude the supposition of plane parallel density layers is practically correct if we consider means over all longitudes. It would also appear that down to this latitude there is no appreciable absorption of light in the galactic system; if there was, the observed numbers of stars would have been smaller than the values computed with the assumption that the absorption could be neglected. The inference is rendered somewhat uncertain by the rather large deviations from plane parallel density distribution which are found below, but I think it is not likely to be very much in error. At $m = 16$ or $m = 18$ an absorption of half a magnitude would have lowered $\log A(m)$ by roughly 0·15. It seems very difficult to reconcile the results in the present section with an absorption of such an amount (cf. the graphs in Figure 11, where it would be necessary, in case of this absorption, to lower the zero lines in the 20° zones by 0·15). Now the distribution of extragalactic nebulae gives rather convincing evidence that in the galactic system the light of these nebulae must have undergone an absorption of about three times this amount, a very conservative estimate giving $1^{m}\!\cdot\!5$ (or an equivalent total obscuration of 88 °/₀ of the sky) for 20° latitude. The contrast between the results for stars and nebulae is all the more noteworthy as the faintest stars considered are, on the average, about as distant as the density surface where $\Delta = 1/100$th of the density near the sun.

It should be noted that the present findings are in general agreement with those of KAPTEYN and VAN RHIJN who have shown that, averaging over all longitudes, the layers of equal density appear sensibly parallel to the galaxy over a long range of distance [1]). This result is now seen to hold even up to the distance of 18th magnitude stars.

The foregoing remarks apply only to the average number of stars over all longitudes. It is now well known that the stars are far from being uniformly distributed in galactic longitude. Our knowledge of the systematic variation of the number of stars with galactic longitude is due entirely to KAPTEYN's Plan of Selected Areas and to the careful discussions and measures by SEARES [2]) and VAN RHIJN [3]). SEARES has drawn attention to the fact that great part of the variation with longitude can be represented by the following simple formula:

$$\Delta \log N(m) = a + b \cos (l - L')$$

The longitude of the centre, L', appeared to be the same for all latitudes (in the following some indications will be given that this law may be seriously disturbed below 20° latitude), but varied somewhat with the apparent magnitude [4]).

The fact that, for the fainter magnitudes where L' could be accurately determined, L' appeared to coincide

[1]) Compare Figure 2 of *Mt. Wilson Contributions*, No. 188; *Astrophysical Journal*, **52**, 37, 1920.

[2]) *Astrophysical Journal*, **67**, 24 and 123, 1927; *Mt. Wilson Contributions*, Nos. 346 and 347.

[3]) *Groningen Publications*, No. 43, 1929.

[4]) A similarity of the same kind was found by VAN RHIJN for the various coefficients of the Fourier series by means of which he has analysed the star counts.

with the direction to SHAPLEY's globular cluster system suggested that these numbers might teach us something about the density distribution in the greater galactic system. In his discussion of this problem SEARES starts from the simple assumption of a homogeneous and sharply bounded stellar system. In trying to get numerical results for the density distribution in the large system he has limited himself to a discussion of the low galactic latitudes, at least for fainter stars. From this he concludes that the true features of the density distribution in the general system are probably masked to a large extent by absorption effects and by the influence of a local cluster. The probable importance of absorption effects near the galactic plane is now still more generally admitted than at the time SEARES wrote his articles. Therefore, I propose to follow a more or less opposite way, viz. to see what information we can get about the large scale features of the density distribution from the regions far from the Milky Way, for which we know from the above discussion that the influence of absorption as well as of an eventual local system is unimportant.

For this purpose I have made extensive use of VAN RHIJN's discussion of the star counts. The dots in Figure 11 were computed directly from Table 10 of *Groningen Publication* 43. They refer to the numbers of stars between $13^{m}\cdot5$ and $14^{m}\cdot5$, $15^{m}\cdot5$ and $16^{m}\cdot5$, $17^{m}\cdot5$ and $18^{m}\cdot5$, respectively. The numbers in VAN RHIJN's table were obtained by means of Fourier analyses in various latitude zones. As, according to VAN RHIJN's statements, the values computed from the formulae fit the observed numbers very closely there appears to be no objection to and considerable advantage in using these smoothed values instead of the numbers actually counted. The graphs in Figure 11 show the variation of $\log A(m)$ with galactic longitude for the latitude zones indicated to the left of the figure. Positive and negative latitudes have been combined in order to eliminate errors in the adopted position of the galactic pole (which may be quite important). It should be remembered that for the region south of $-20°$ declination the magnitude scales have been rather inaccurately determined. The errors in the scales may have influenced the parts between 180° and 360° longitude, especially those for $m = 18$ which, in this part of the sky, rest largely on an extrapolation. An extension of the Mt. Wilson Durchmusterung to the southern hemisphere is indeed a great desideratum. It seems improbable, however, that the errors in the southern magnitudes are of sufficient size and sufficiently systematic to have seriously spoilt the general character of the $\log A(m)$ curves. The quite satisfactory regularity shown by the $\log A(m)$ curves if drawn separately for $-60°$ and $-40°$ latitude is an argument in favour of this supposition. It is true that the curves for the $+60°$ and $+40°$ zones seem still more regular and I have considered the possibility of using the amplitudes derived from these curves [1]) instead of those from the combined northern and southern zones, but in view of the influence of errors in the galactic pole the latter were preferred.

Except for the curve for 0° and $-2°$ galactic latitude where it simply represents the average value of $\log A(m)$ the zero lines represent the numbers of stars to be expected if the equidensity surfaces were parallel to the galactic plane. The zero lines were inferred from the computed numbers in Table 26, after correction by an amount constant for each magnitude and determined in such way that the differences $O-C$ vanished near the galactic poles.

The principal features of the graphs in Figure 11 are a maximum near 320°, approximately the longitude of the centre of the large system, and a minimum in the opposite longitude. In general the maxima are much flatter than the minima; also, the deviations from the zero line seem much larger in 140° longitude than in the longitude of the centre, which indicates that at smaller distances from the centre the equidensity surfaces become less inclined to the galactic plane (compare Figure 12). It may be noted that the dark nebulae in Taurus, situated between $-5°$ and $-30°$ latitude, and between 125° and 150° longitude have undoubtedly increased the depth of the minimum in the 20° zone, but they cannot be responsible for the whole phenomenon. The features described hold for all magnitudes between 14 and 18 and for all latitudes larger than 20°, but they are most regular at 40° and 60° latitude. At lower latitudes the irregularities become absolutely as well as relatively more pronounced. Below 20° they seem to become so important as to make it impossible to trace the main effect with any confidence. The star counts for $\pm 10°$ and $0°/-2°$, $m = 16$, have been plotted in order to illustrate this. For this and other reasons, cited above, it was decided not to consider any zones below 20° latitude.

Some traces of the general effect are still visible for stars of the 12th magnitude, but not sufficiently distinct to be of any use for computations of the density gradient.

The graphs of Figure 11 are in many respects similar to those published by SEARES and miss JOYNER [2]), but there are some noteworthy deviations, mostly due

[1]) For $m = 16$ and 18 these are about 50% larger than the values given in Table 27.

[2]) *Astrophysical Journal*, **67**, Figures 11, 12 and 13; *Mt. Wilson Contributions*, No. 346, 1926.

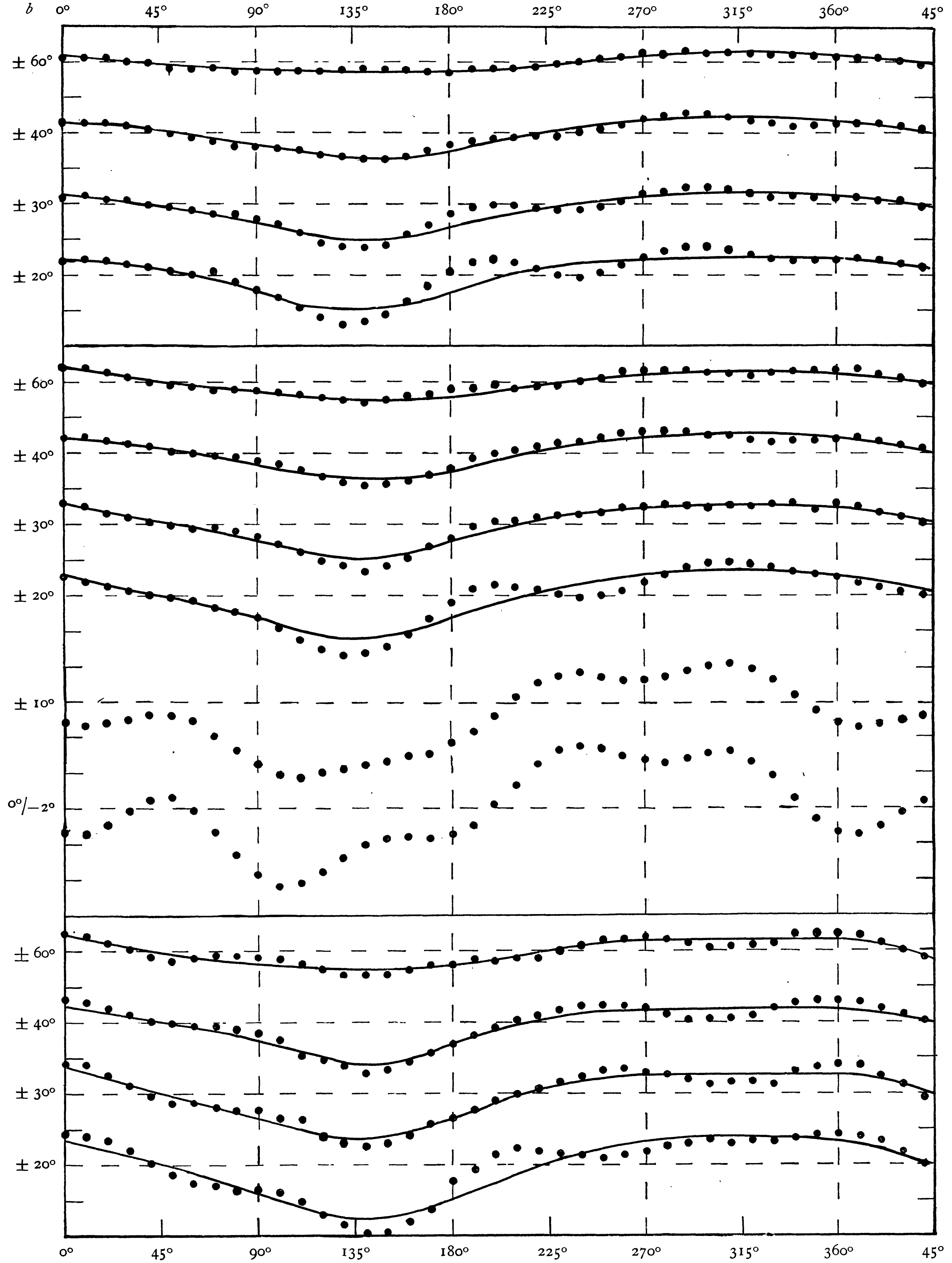


FIGURE 11. Variation of log $A(m)$ with galactic longitude.
The various zones of galactic latitude are indicated at the left side. The upper division refers to $m = 14$, the central division to $m = 16$ and the bottom part to $m = 18$. Each division of the vertical scale corresponds to a difference of ·25 in log $A(m)$.

to the fact that SEARES and miss JOYNER did not make use of the counts of the Harvard-Groningen Durchmusterung. At the time their article was written only the total numbers of stars in each area had been discussed. In order to get provisional data for the part of the sky south of $-15°$ declination they used these total numbers, assuming that the limiting magnitude of all plates had been the same.

Now that complete counts of the Harvard-Groningen Durchmusterung have been published and reduced by VAN RHIJN there is little doubt that these results are to be preferred. For though there is considerable uncertainty about the scale errors in the southern areas it is evident from the discussion of the areas common to Mt. Wilson and Harvard that the variation in these errors is much less than that of the limiting magnitudes.

For $m = 13{\cdot}5$ SEARES and miss JOYNER rely largely on counts in Astrographic zones. I believe that there are serious objections to the use of these counts for our present purpose. The magnitude scale of each zone of declination could only be determined indirectly by comparing the star counts in the zone to mean counts from other sources. As VAN RHIJN has remarked, there is a danger that the assumption of a constant scale for each zone might be to some extent invalidated by seasonal influences. Moreover, the effect of the galactic longitude on the determination of the scales has been determined exclusively from the Mt. Wilson Catalogue of Selected Areas, and for the southern zones from the total numbers of stars in the Harvard-Groningen Durchmusterung, for which a constant limiting magnitude was assumed. [1]) For the southern regions the results had to rest on this assumption as well as on the hypothesis that the variation of $\log N(m)$ with galactic longitude is the same for 17th magnitude stars as for those of $13^m.5$, which is difficult to believe. For these reasons I have given preference to the direct counts from the Selected Areas which are now also available for $m = 14$. It may be noted that the points for $m = 14$ in Figure 11 show good evidence of the density increase towards the centre of the large system. Even these bright stars do not seem to be spoilt by the influence of local phenomena.

[1]) Compare Tables VII and X of the work cited.

Through the dots in Figure 11 I have drawn smooth curves having a maximum near 320° and a minimum near 140°. Except for these conditions I have tried to draw them without prejudgement, balancing positive and negative residuals as well as possible. It will be seen that several of the deviations from these smooth curves show a systematic character; for instance, many curves show an excess of density near 200° longitude. Perhaps still more pronounced is a secondary minimum near 320°, which is especially well shown by all magnitudes in the zones at $+30°$ and $+40°$ latitude. A discussion of these deviations does not lie within the scope of the present paper. Our present purpose is to determine the regular part of the density distribution in the great galactic system.

For this purpose the maximum and minimum deviations were read from the smooth curves. They are given in Table 27. On account of the relative uncertainty of the correct position of the zero lines it was, however, decided not to use the positive and negative deviations separately but only the total variations, given under a. The probable errors of a were rather arbitrarily, but consistently, computed by taking 4/5th of the maximum positive and negative deviations between the points and the curves.

To some extent the values of a are comparable to the amplitudes of the first harmonics in VAN RHIJN's discussion [1]) and, for $m = 16$ and 18, to twice the values of b computed by SEARES and miss JOYNER [2]). In general there is a reasonable agreement. It is to be noted that SEARES' and VAN RHIJN's results refer to $\log N(m)$, the present ones to $\log A(m)$.

If $\Delta(\varpi, z)$ is the star density at a distance ϖ from the axis of rotation of the galactic system and at a distance z from the galactic plane we want to compute $\partial \log \Delta / \partial \varpi$, which quantity will be denoted by δ. It is clear that in general δ will vary with z and ϖ, and to some extent also with M. The variation with ϖ and M will be neglected in the following solution, and it will be supposed that δ is constant in two large

[1]) *Groningen Publ.* 43, Table V, p. 30.
[2]) L. c., Table XVI.

TABLE 27.

Observed and computed variations of $\log A(m)$ with galactic longitude.

b	$m=14$ max.	$m=14$ min.	$m=14$ a	$m=14$ p. e.	$m=14$ C	$m=16$ max.	$m=16$ min.	$m=16$ a	$m=16$ p. e.	$m=16$ C	$m=18$ max.	$m=18$ min.	$m=18$ a	$m=18$ p. e.	$m=18$ C
20°	+ ·13	— ·25	·38	± ·17	·40	+ ·18	— ·33	·51	± ·11	·61	+ ·20	— ·39	·59	± ·11	·83
30°	+ ·08	— ·25	·33	± ·06	·30	+ ·15	— ·27	·42	± ·04	·44	+ ·15	— ·33	·48	± ·07	·55
40°	+ ·09	— ·17	·26	± ·04	·21	+ ·14	— ·19	·33	± ·04	·28	+ ·10	— ·28	·38	± ·06	·35
60°	+ ·06	— ·07	·13	± ·02	·12	+ ·08	— ·12	·20	± ·02	·18	+ ·08	— ·14	·22	± ·04	·20

regions, viz. $z < 1500$ ps and $z > 1500$ ps. The first value will be denoted by δ_1, the latter by δ_2. The quality of the data, and especially our lack of knowledge of the variation of Δ with z for large values of z do not seem to justify a much greater refinement.

Denoting the distance of the sun from the centre of the galactic system by ϖ_0, we have the following equations of condition:

$$\frac{A(m)_{l=320^\circ}}{A(m)_{l=140^\circ}} = \frac{\int_{z=0}^{z=1500} n.\,10^{-\delta_1 x}\, d(\log z) + \int_{z=1500}^{z=\infty} n.\,10^{-\delta_2 x}\, d(\log z)}{\int_{z=0}^{z=1500} n.\,10^{\delta_1 x}\, d(\log z) + \int_{z=1500}^{z=\infty} n.\,10^{\delta_2 x}\, d(\log z)} \tag{12}$$

where $x = \rho \cos b$; $n = \rho^3 \Delta(z) \Phi(M)$, $\Delta(z)$ being defined by formula (11) on page 263; $\rho = z/\sin b$.

A solution by least squares gave the following results:

TABLE 28.

Gal. lat.	δ_1	p. e.	δ_2	p. e.
20° and 30°	—·000156	±·000024	—·000037	±·000018
40°	— 190	± 36	— 65	± 14
60°	— 161	± 31	— 72	± 19
Combined	— 148	± 16	— 63	± 10

The agreement between the independent results from the three zones of galactic latitude is very satisfactory. It should be kept in mind, however, that the uncertainty of the densities for $z > 1500$ ps enters directly into the scale of ϖ and thus into the result for δ_2 in all three zones. The exact value of this constant should, therefore, not be accepted without considerable reserve. The combined results in the last line of Table 28 have been used for the computation of the values under C in Table 27. In comparing these with the observed ranges the general agreement is quite sufficient, the only large residual being that for $m = 18$, $b = 20°$.

If separate solutions are made for the gradients in the direction of 320° and 140° longitude the latter comes out roughly twice as large absolutely as the former, the average of the two being equal to the values in Table 28.

A rough computation of the average height above the galactic plane for which the computed values of δ_1 and δ_2 are valid gives 950 and 2750 parsecs respectively. Supposing δ to vary smoothly with z and to assume the values δ_1 and δ_2 at these two points we proceed to investigate the general features of the surfaces of equal density. The angle, i, between such a surface and the plane of the galaxy is given by

$$tg\, i = \frac{\partial \log \Delta}{\partial \varpi} : \frac{\partial \log \Delta}{\partial z}.$$

Let Δ represent the total intensity of photographic light in an element of volume. The distribution in the z direction is then given by Table 29 in the next section. We find:

z	i
500	7°:
1000	11°
1500	11°
2000	9°
3000	5°:

Figure 12 shows the equidensity surfaces $\Delta = 1/100$th (crosses) and $\Delta = 1/25$th (dots) of the integrated light near the sun. The figure shows a cross section with a plane perpendicular to the galaxy, passing through the sun and the galactic centre, which is situated towards the right. For the computation of the crosses and the dots it was assumed that the gradient δ varied smoothly not only with z but also with ϖ, the absolute values of δ_1 and δ_2 increasing respectively with ·000030 and ·000004 per 1000 ps increase of ϖ. These changes have been roughly determined from the difference between the deviations of $\log A(m)$ in 140° and 320° longitude (Table 27).

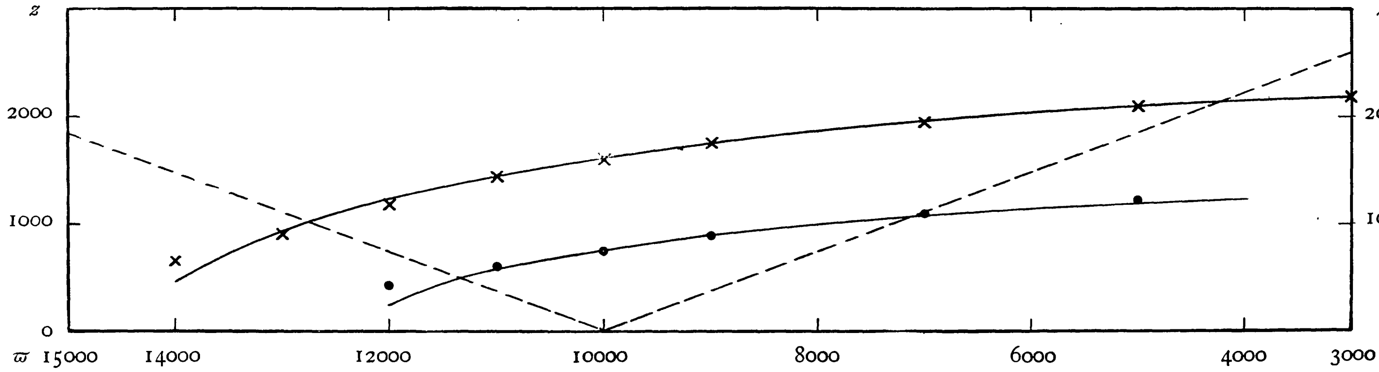
Both the dots and crosses so computed appear to fit very closely to an ellipse having the same centre as the galactic system (supposed to be at 10000 ps distance from the sun). For distances greater than, say, 700 ps from the galactic plane the equidensity surfaces thus seem to approximate ellipsoids symmetrical around the axis of the large galactic system. For $\Delta = 1/100$ the axes of the ellipsoid are 14300 and 2240 ps, for $\Delta = 1/25$ they are 12200 and 1310 ps.

The results obtained in the present section teach us nothing about the density distribution in the regions less than 500 ps from the galactic plane; they only indicate the shape of a part of the outer envelopes of the galactic system. It may be noted that the results rest almost wholly upon stars with high peculiar velocities, for the stars with low velocities Z_0 do not reach the regions in question. This fact may be mainly responsible for the comparatively regular character of the density distribution.

In the direction opposite the centre, where the absorption is likely to be fairly small, it may be possible to utilize star counts in latitudes below 20° and to find the approximate limits of the system in the direction of the galactic plane. But an eventual absorption should certainly be taken into account and the investigation can hardly be tried without an exhaustive analysis of all star counts within 60° of the anticentre.

The quantity $\partial \log \Delta/\partial \varpi$ for high velocity objects can be estimated in an entirely different way from

FIGURE 12.

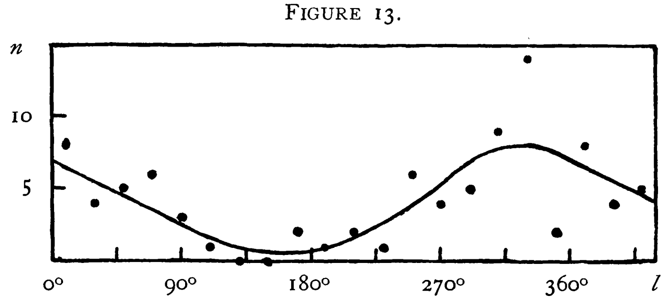


Cross section of the galactic system with a plane perpendicular to the galaxy, passing through the centre and the sun. Abscissae are distances to the centre (the sun is situated at $\varpi = 10000$), ordinates are distances from the galactic plane. The dots and crosses, which were derived from star counts (cf. Figure 11), show points where the density of photographic light is 1/25th and 1/100th of that near the sun. The lines drawn are parts of ellipses having the same centre as the galactic system. The axes are $a = 12200$, $b = 1310$ and $a = 14300$, $b = 2240$ parsecs, respectively. The dotted lines indicate the directions of 20° latitude.

the asymmetry in the velocity distribution. This problem has been discussed in a former paper [1]). The average value of $\partial \log \Delta / \partial \varpi$ derived was $-0{\cdot}0003$. If we had used the values of the rotation constants A and B as adopted in the present paper this number would have been reduced by a factor 0·70. The gradient thus reduced corresponds very satisfactorily to that found by an extrapolation of the present results to the lower galactic regions. To some extent the above result may be considered as a confirmation of the dynamical theory underlying the former article.

It may have some interest to compare the density gradients found in the present section with other results about the increase in density towards the galactic centre. Besides the globular clusters for which the gradient ∂ is of order of $-{\cdot}00005$ ($z = \pm 8000$ ps), the only objects clearly indicating the effect and which have been studied more or less systematically over the whole sky are the planetary nebulae. The remarkable distribution of these high velocity objects is well known.

FIGURE 13.



Distribution in longitude of planetary nebulae within 10° of the galactic circle. Dots show counted numbers in 20° intervals.

1) *B. A. N.*, No. 159, section 8, 1928.

It is illustrated in Figure 13 which shows the numbers of nebulae within ± 10° latitude in intervals of 20° of longitude. If we compare the ordinates of the smooth curve at the points 90° from the centre with that in the direction of the centre we find that they differ by a factor of 2·4, corresponding to $d \log \Delta = 0{\cdot}38$. The average distance of these nebulae from the sun, as found from the effects of the rotation of the galaxy upon their radial velocities is roughly 1400 ps, so that we find $\partial \log \Delta / \partial \varpi = {\cdot}38/1400 = {\cdot}00027$. A similar comparison for the direction opposite to the centre gives a much larger gradient (about ·00050). If the assumed distance is correct both gradients are nearly twice as large as the density gradients indicated by the above results extrapolated to $z = 0$.

VAN GENT has recently investigated the cluster type variables in a region near the centre ($l = 327°$, $b = -18°$) and finds a strong increase of the density for large distances [1]). More complete data will soon be available; from the data at hand it is evident that the density gradient for these objects is considerably larger numerically than the value derived for the stars in general.

10. *The density distribution perpendicular to the galactic plane. Change of the luminosity curve with z.*

The data derived in the present article have been used for computing the distribution perpendicular to the galactic plane for some special types of stars as well as for the total amount of light and mass in a unit of volume. The results are in Table 29. The accelerations used in these computations are $K'(z)$ (Table 14) and K_a (Figure 7) which seem to come

1) *B. A. N.*, No. 227, 1932.

nearest to the truth. The amounts of photographic light per cubic parsec are expressed in the light of the sun as unit, N_{pg} denoting with how many stars of $M_{pg} = +6{\cdot}0$ the total light per cubic parsec is equivalent. The velocity distributions used in the computations of N_{pg} are practically those of Table 11 (corrected according to the remark in the footnote); only in the last two groups θ_3 has been somewhat increased (compare the remarks in section 8). The numbers of stars in each absolute magnitude interval near $z = 0$ were found from the corrected luminosity law, $\log \Phi'$, of Table 16. The light of the B stars (which are not included in these luminosity laws) was added separately. The B star luminosity curve used is that given in Table 71, *Groningen Publication* No. 38, except that the numbers of stars for photographic absolute magnitudes $-4{\cdot}0$ and $-3{\cdot}0$ were increased by 50°/₀ in order to make the total numbers of these bright stars equal to those found from the general luminosity law for stars of all spectra. The velocity distribution of the B stars is given in Table 7.

In the third column of Table 29 the numbers N_{pg} have been reduced so as to make the density equal to 1·00 near $z = 0$ by dividing by ·0946, which represents the total photographic light per cubic parsec at $z = 0$.

The distribution of the mass, in the fourth column of the table, was computed with the aid of the mean masses of stars in different intervals of visual absolute magnitude as derived in the last section. The masses given refer to stars brighter than $+13^{M}{\cdot}5$. They would have been considerably larger if the total mass of fainter stars and invisible material had been included.

As a direct by-product of these computations we find the change of the luminosity curve with the height above the galactic plane. The results for the photographic luminosity curve (including the B stars) are summarized in Table 30. For each z a constant was added to the logarithms of the computed distributions so as to make them coincide with $\log \Phi_{z=0}$ for $M \geq +8$. These added quantities are shown in the bottom line of the table. It has been assumed that beyond $+8^{M}$ the luminosity curves are all identical in shape. The luminosity curve given under $z = 0$ practically coincides with that given by VAN RHIJN in Table 71, *Groningen Publication* No. 38. In view of the rapid change of the bright part of the luminosity curve with z and also in view of the great local variations in the density of B stars VAN RHIJN's $\Phi(M)$ really represents an average curve for a layer extending to some distance from the galactic plane, but this consideration makes no difference for the relative change in $\log \Phi(M)$ with z as shown in Table 30. As most investigations on star density are concerned with stars further than 250 parsecs from the galactic plane it is clear from Table 30 that the average luminosity curve which should be used in such investigations differs greatly from the curve for $z = 0$ which has hitherto been used by all investigators. I have not given $\log \Phi(M)$ for $z > 1500$ ps as this would have been pure guess work. A great deal more knowledge of the velocity distributions is needed before we can make a reliable forecast of the character of $\log \Phi$ at greater heights. Even that at $z = 1500$ is still quite uncertain.

As most investigations about star density deal with photographic magnitudes I have restricted myself to data about the photographic luminosity curve. The corresponding changes of the visual luminosity curve are easily computed. They are also evident from VAN RHIJN's numbers in Table 10 of *Groningen Publication* No. 38, zone ± 40° to ± 90°. The change of the luminosity curve is indicated in each column by the systematic change of $\log N - \log \varphi$ with the absolute magnitude.

For comparison with the distribution of photographic light and mass as derived in the present paper I have entered in the second column of Table 29 the density distribution derived by KAPTEYN and VAN RHIJN on the assumption of a constant luminosity law[1]). The more rapid falling off in the photographic light intensity between $z = 0$ and $z = 250$ as compared to that of KAPTEYN and VAN RHIJN's densities is largely due to the influence of the B and A type stars. If we compare the change in density from $z = 250$ parsecs upwards we find that both N_{pg} and the mass density, but in particular the latter, decrease much more slowly than the density computed by KAPTEYN and VAN RHIJN.

It should be remarked that it would perhaps have been possible to deduce somewhat more trustworthy values of the density at large distances by trying to get a better fit between the computed values of $\log A(m)$ and the actual star counts. However, I hardly think that it would pay to do this before we have more certainty about the photographic luminosity law and about its change for heights above 1000 parsecs.

Beside the distribution of photographic light I have also computed that of the visual light. According to these computations the colour index of the integrated light varies from $+0^{m}{\cdot}4$ at $z = 0$ to about $+0^{m}{\cdot}7$ at $z = 500$.

From the numbers in Table 29, roughly extrapolated to larger distances, it is found that half of the photographic light in a column perpendicular to the galactic plane is contained between $z = -166$ and $z = +166$; for the mass the corresponding limits are ± 267 ps.

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[1]) *Astrophysical Journal*, **52**, 36, 1920; *Mt. Wilson Contr.* No. 188, Table VI.

TABLE 29.

Densities at different distances from the galactic plane.

z	K. & v. Rh.	N_{pg} / ·0946	Mass / ·0378	A0—A9	F0—F9	Late giants	N	M6e to M8e	M4e to M5e	M0e to M3e	RR Lyr var.	Glob. clusters comp.	Glob. clusters obs.
0	1·000	1·000	1·000	1·000	1·000	1·000	1·000	1·000	1·000	1·000	1·000	1·000	1·00
125	·708	·495	·815	·576	·827	·890	·875	·955	·982				
250	·407	·256	·526	·136	·478	·634	·590	·836	·927				
500	·126	·092	·225	·032	·136	·286	·224	·601	·807	·948	·918	·964	·98
750	·046	·042	·124	·023	·047	·134	·081	·427	·698				
1000	·020	·024	·082		·023	·067	·027	·293	·597	·881	·817	·912	·93
1250	·009	·016	·061		·014	·037	·009	·198	·506				
1500	·005	·011	·042		·010	·024	·003	·131	·426	·811	·714	·859	·87
2000	·002	·006	·021			·010		·054	·292	·740	·617	·805	·80
3000								·007	·120	·596	·434	·689	·65
5000									·013	·350	·184	·468	·42
10000										·074	·015	·152	·25
15000										·021	·002	·062	·18
20000										·009		·032	·13
$\overline{\lvert Z \rvert}$				7·6	12·2	15·7	14	24	37	75	59	88	
$\overline{\lvert z \rvert}$		320	490	130	220	330	270	650	1200	3700	2500	4800	6800

TABLE 30.

Change of photographic luminosity curve with distance from the galactic plane.

M_{pg}	$\log \Phi + 10$ $z = 0$	$z = 125$	$z = 250$	$z = 500$	$z = 1000$	$z = 1500$
— 4·0	3·62	2·25				
— 2·0	4·59	3·72	3·27	3·04	2·89	2·85
0·0	5·56	5·28	4·98	4·75	4·60	4·56
+ 2·0	6·62	6·42	6·32	6·20	5·92	5·85
4·0	7·14	7·12	7·10	7·00	6·75	6·69
6·0	7·62	7·62	7·62	7·55	7·41	7·36
+ 8·0	7·36	7·36	7·36	7·36	7·36	7·36
Const. added	0·00	0·07	0·25	0·58	0·94	1·21

The distributions deviate widely from Gaussian curves; there is a great excess both for very small and for large values of z. Compared with the distance to the centre, which is of the order of 10000 ps, the z-dimension of the system is exceedingly small. The galactic system appears to be considerably flatter than most, or possibly all, of the extragalactic systems known.

The further columns of the table show the predicted distributions of some special types of stars. For the A and F stars and for the giants the computations were made with the velocity distributions of Table 7, for the other objects it was assumed that the distribution of the Z-components of the velocities was Gaussian; the assumed values of the average velocities in km/sec are in the second line from the bottom of Table 29. It should be noted that in several cases the true velocity distributions are certainly non Gaussian and also that it is not always certain that the average peculiar velocity given really represents the average velocity in the Z-direction. On these accounts the distributions given cannot be trusted to be more than very rough approximations. For the computation of the distribution of the high-velocity objects in the last four columns it was assumed that for $z > 4000$ the force $K(z)$ is equal to that exerted by a mass of $1{\cdot}7.10^{11}$ suns situated at the centre of the galactic system (compare Table 24, 2nd column, "spherical"). As remarked in section 8 the true total mass within the equidensity ellipsoid which passes through the sun is probably smaller than $1{\cdot}7.10^{11}$, but on the other hand we must add the force arising from the outer parts of the galaxy; it has been assumed that to some extent these two errors cancel each other.

The last column of Table 29 shows the observed distribution of the globular clusters. Only those 22 clusters were counted whose distances from the sun as projected on the galactic plane were smaller than 10000 ps. The distances used are those given by SHAPLEY [1]). The fact that the observed density between $z = 10000$ and $z = 20000$ is larger than the computed values might be taken as an indication that the actual acceleration at these distances is smaller than that assumed in the computations. But the difference may as well be due to an error (of about 20 %) in the value used for the average velocity.

If something were known of the distribution of RR Lyrae variables over different apparent magnitudes in a region near one of the galactic poles this might be used to check $K(z)$ for large values of z. But

[1]) Star Clusters, Appendix A.

beside this distribution a more accurate knowledge of the velocity law would be required.

The bottom line of the table shows the average distance from the galactic plane for the various objects considered. The relation between average velocity and distance from the galactic plane is shown in the first two columns of Table 31. The computations were made with Gaussian velocity distributions. Only objects with $z < 20000$ ps have been considered in forming the averages.

The third column shows the maximum height reached by a star whose velocity at $z = 0$ was equal to the value in the first column. In the last column the corresponding periods of the z-motions are indicated. For slow motions these are about 2/5 of the period of rotation of the galactic system (205.10^6 years, if $A = + \cdot 019$, $B = - \cdot 011$).

TABLE 31.

$\overline{\lvert Z_0 \rvert}$	$\overline{\lvert z \rvert}$	$z_{\max.}$	T
5 km/sec	70 ps	70	84.10^6 (5–15)
10	161	140	
15	301	210	
20	480	290	
30	910	500	
50	1910	1100	170.10^6
75	3710	2100	
100	5560	3200	240.10^6

The average heights above the galactic plane may in some cases be used to derive the mean parallax of a certain type of stars. For example, it is known that the N stars are sensibly concentrated towards the galaxy. The following relation will hold approximately:

$$\overline{|\sin b|} = \bar{\pi} \, . \, \overline{|z|} \tag{13}$$

where b represents the galactic latitude. For the N stars with D.M. magnitudes between 7·0 and 8·2 in *Harvard Annals*, **56**, I find $\overline{|\sin b|} = \cdot 30$; combining this with the value for $\overline{|z|}$ derived in Table 29 we obtain $\bar{\pi} = {}''\!\cdot 0011$. The average parallax derived from the proper motions of N stars of roughly the same apparent brightness is $''\!\cdot 0017 \pm {}''\!\cdot 0008$ m.e. [1]).

Inversely, in cases where the average parallax is known, the numbers in Table 31 permit an estimate of the average velocity $\overline{|Z|}$. Such cases are presented, for instance, by the O stars and the δ Cephei variables. From 43 Oe5 stars with known proper motions an average parallax of $''\!\cdot 0020 \pm {}''\!\cdot 0006$ m.e. has been derived [1]); the value has been independently confirmed by the rotation effects in the radial velocities. From *Harvard Annals*, **56**, I find an average latitude of $\pm 3°.5$ for stars of the same average magnitude (5·6); accordingly $\overline{|z|} = 30$ ps, corresponding with an average velocity $\overline{|Z|} = \pm 2 \cdot 1$ km/sec. The very small value of this velocity is rather remarkable in view of the fact that the average peculiar *radial* velocity is ± 16 km/sec. The average value of one component of the velocity parallel to the galactic plane is thus seen to be about eight times higher than that of the component perpendicular to the galactic plane. In computing the average peculiar velocity the effects of differential galactic rotation were eliminated; the elimination cannot be made accurately for each star, but it is unlikely that the derived average peculiar velocity has been considerably increased by this inaccuracy. It may be remarked that among the 27 stars considered there are 2 bright stars with residual velocities of + 52 and + 66 km/sec respectively, which appear to be well-determined and which cannot possibly be ascribed to effects of rotation [1]). The great difference between the two average velocity components seems enigmatic and gives one a feeling of hesitation in accepting the observed spectral shifts as real motions.

For the δ Cephei variables we may adopt $\overline{|z|} = 71$ ps which is the value derived by BOTTLINGER and SCHNELLER from 38 Cepheids within 1000 ps distance [2]). The corresponding value of $\overline{|Z|}$ is ± 5·1 km/sec. From radial velocities we find ± 9 km/sec for the average velocity of one component parallel to the galactic plane.

Some further data about the average distances from the galactic plane for strongly concentrated objects will be found in BOTTLINGER's memoir „Die hellen Sterne und die Rotation der Milchstrasze" [3]) which reached me while the present article was being written. In this memoir the motions and distribution of the absolutely bright stars are discussed from a somewhat different point of view, starting from a highly schematical model of the stellar system and from the assumption that $\overline{|Z|}$ is equal to the average velocity component parallel to the direction of the centre. This hypothesis has been discussed in the first section; I do not believe it is justified.

From BOTTLINGER's data about the average distance $\overline{|z|}$ for the B stars (about 38 ps) I find a confirmation of the value of $\overline{|Z|}$ derived in the sixth section from counts of B stars near the galactic poles. The latter value was ± 2·82 km/sec, whereas, according to Table 31, BOTTLINGER's result for $\overline{|z|}$ corresponds with ± 2·9 km/sec.

[1]) *B. A. N.* No. 132, Table 3, 1927. For the N stars compare WILSON, *Astronomical Journal*, **34**, 191, 1923.

[1]) It is not likely that these velocities can be explained as relativity shifts, for there are three Wolf Rayet stars with large negative residuals.

[2]) *Zeitschrift f. Astrophysik*, **1**, 340, 1930.

[3]) *Veröff. Berlin-Babelsberg*, **8**, Pt. 5, p. 27, 1931.

It is clear that for the derivation of average parallaxes from τ components and peculiar radial velocities one must take account of the fact that in cases of strong galactic concentration the average peculiar velocity perpendicular to the galactic plane may be quite different from the value inferred from radial velocities. An approximation to the true value of $\overline{|Z|}$ may always be obtained with the aid of the above Table 31.

11. *The amount of dark matter.*

From the results found for the decrease of $K(z)$ with z we may derive an approximate value of the total density of matter, Δ, in the neighbourhood of the sun. Let us suppose that we are situated inside a homogeneous ellipsoid of revolution with semi-axes a and c, and density Δ. For $z = 0$ there will then be the following relation:

$$\partial K(z)/\partial z = -4\pi\gamma x \Delta \qquad (14)$$

where $$x = \frac{a^2}{a^2-c^2} - \frac{a^2 c}{(a^2-c^2)^{3/2}} \tan^{-1}\sqrt{\frac{a^2}{c^2}-1} \qquad (15)$$

and γ is the constant of gravitation.

For the cases which will be considered x does not deviate much from unity. We find:

a/c	x
5	·750
10	·860
30	·950
∞	1·000

$\partial K(z)/\partial z$ can be determined from Table 14 or Figure 4. I find $\partial K(z)/\partial z = 5{\cdot}62.10^{-30}$ sec^{-2}.

In reality we are not situated inside a homogeneous ellipsoid. The inner part of the galactic system, nearer the centre will be denser and very probably of different shape. Part of the empirical value of $\partial K(z)/\partial z$ will be due to the attraction of these inner ellipsoids and this part should be subtracted before we can compute the local density from (14) and (15). Luckily the contribution of these inner parts of the galactic system to $\partial K(z)/\partial z$ appears to be small as long as we limit ourselves to small values of z, and a reasonably trustworthy result for the local density can be obtained notwithstanding the fact that we have very little knowledge about the more central parts of the system. Also, the exact shape of the equidensity surfaces near the sun appears to be of little consequence for the computation of the mass density. The limits of uncertainty are best shown by considering the following extreme cases:

I. The stellar system is supposed to consist of a local system symmetrical around the sun, and a large attracting mass symmetrical around the centre of the great galactic system. In the z-direction the density distribution in the local system agrees with the distribution of mass in Table 29 (the mean densities in each ellipsoidal shell being given in the 2nd column of Table 32), except for a constant factor which takes the dark matter into account. The equidensity surfaces are further supposed to be ellipsoids of revolution, the ratio of the axes being taken as 5 to 1. This local system thus resembles the KAPTEYN system. About the main attracting mass of the larger system two rather extreme assumptions will be considered:

a) The mass is spherical and equal to $2{\cdot}04.10^{11}$ solar masses.

b) The mass consists of a homogeneous ellipsoid with semi-axes of 9000 ps and 900 ps respectively and having a total mass of $0{\cdot}95.10^{11}$.

The masses have been so chosen that the forces exerted parallel to the galactic plane counterbalance the centrifugal force arising from a rotational velocity of 300 km/sec at 10000 ps from the centre. The corresponding values of $K(z)$ may be obtained from Table 24 by multiplying with a factor 1.19.

II. There is no local system; in the outer parts of the great galactic system the equidensity surfaces are supposed to be ellipsoids of revolution, symmetrical around the galactic centre. The dimensions of these ellipsoids will be determined by the conditions that near the sun they must reproduce the density distribution in the z-direction as given in the fourth column of Table 29 and that in the galactic plane, in the direction opposite to the galactic centre, they must extend to the same distance from the sun as the local ellipsoids in case I. The semi-axes of the ellipsoids are shown under a and c in Table 32. The first column gives the distance of the ellipsoids from the galactic plane at a point near the sun. The second column shows the mean density in terms of the sun's mass per cubic parsec, respectively in the first ellipsoid, in the shell between the second and the first ellipsoid, etc. The densities refer to all stars brighter than $+13{\cdot}5$ visual absolute magnitude.

TABLE 32.

z	$\bar{\Delta}$	a	c
100	·0361	10500	326
200	·0286	11000	480
400	·0168	12000	723
600	·0087	13000	938
1000	·0046	15000	1340
1500	·0023	17500	1830
2000	·0012	20000	2310
3000	·0004	25000	3280

As in the previous case two different suppositions will be made about the inner parts, the following masses being superposed upon the above ellipsoids:

a) A spherical mass equal to $1{\cdot}71.10^{11}$ solar masses.

b) An ellipsoidal mass with semi-axes 9000 and 900 ps and equal to $0{\cdot}80.10^{11}$ times the mass of the sun.

The corresponding values of $K(z)$ are in Table 24. The total masses needed are somewhat smaller than in case I as the ellipsoids in Table 32 also exert a force parallel to the galactic plane.

The densities near $z = 0$ derived with the aid of formulae (14) and (15) for each of the 4 cases enumerated are as follows:

Case	Δ
I *a*)	·108
I *b*)	·093
II *a*)	·089
II *b*)	·079

The unit is one solar mass per cubic parsec, which is equivalent to $6{\cdot}85.10^{-23}$ g/cm³. The most probable value of the total density near the sun is thus ·092 suns/ps³ or $6{\cdot}3.10^{-24}$ g/cm³. The probable error of this quantity is estimated to be about 20°/₀. The present value may be compared to the densities derived by former investigators. KAPTEYN [1]) finds 0.099; the agreement is unexpectedly good; in fact, it seems probable that part of it should be attributed to a chance coincidence, as the velocity and density data used by KAPTEYN differ rather widely from those used in the present paper. This is illustrated by the fact that the agreement between the forces $K(z)$ computed from KAPTEYN's data and those of the present paper is not nearly as good (compare the last two columns of Table 33). JEANS [2]) finds ·143 solar masses per cubic parsec. A somewhat hypothetical estimate by LINDBLAD [3]) gives ·217 for the same quantity.

If we suppose that the ratio of the total density to the density due to the stars brighter than $+13^{M}{\cdot}5$ is the same for all values of z we can compute the forces $K(z)$ exerted by the outer ellipsoids in the cases I and II. These are in the 2nd and 3rd columns of Table 33. The next two columns show the force due to the central mass (the average between the ellipsoidal and spherical case being adopted). The total forces are shown in the 6th and 7th columns. A comparison with K_a, the force derived and adopted in the present article, shows some outstanding differences. The comparison has no meaning for $z > 1000$, but even below $z = 1000$ the values just computed appear to be sensibly too high. It seems probable that the error is not wholly with K_a but partly with the above assumption as to the distribution of dark matter. It would appear from the comparison that the dark mass must be relatively more frequent near the galactic plane than far from it, but the data are too uncertain to derive numerical results. A similar conclusion was reached by KAPTEYN in the investigation quoted above.

TABLE 33.

Values of $K(z).10^9$.

z	Outer ellipsoids I	Outer ellipsoids II	Central mass I	Central mass II	Total force I	Total force II	K_a obs.	KAPTEYN
100	1·33	1·35	·38	·32	1·71	1·67	1·73	1·20
200	2·32	2·42	·76	·64	3·08	3·06	3·43	1·87
400	3·22	3·59	1·52	1·28	4·74	4·87	3·86	1·84
600	3·48	4·12	2·28	1·92	5·76	6·04	4·06	1·56
1000	3·45	4·61	3·51	2·95	6·96	7·56	4·47	1·21
1500	3·2	4·7	4·8	4·0	8·0	8·7	5·0	1·0
2000	2·9	4·7	5·7	4·8	8·6	9·5	5·5	
3000	2·4	4·4	6·9	5·8	9·3	10·2	6·5	

We have adopted a value of 0·038 solar masses for the total mass of the visible stars within a cubic parsec near the sun. Through the kindness of Mr. G. P. KUIPER I was enabled to use for the computation of this quantity his up-to-date catalogue of parallaxes, in which all data about visual or spectroscopic duplicity had been completely entered. I have also had the privilege to use a number of unpublished measures of magnitudes of near-by double stars which were made by Mr. KUIPER.

For each of the intervals of visual absolute magnitude indicated in the first column of Table 34 I determined the average mass of the stars within the distance shown in the second column. Multiple systems were counted as one star because they have been counted as single objects in the determination of the luminosity law. If direct data about the masses were available these have generally been adopted; in all other cases the mass was computed from the mass luminosity curve, somewhat corrected by Mr. KUIPER with the aid of new data. In the case of a spectroscopic double the mass of the system has been assumed to be twice the mass computed from the mass luminosity curve with the combined absolute magnitude. The factor has been taken so high in order to take rough account of the fact that some systems must have remained undiscovered.

The resulting mean masses have been entered in the third column of Table 34. The fourth column shows the number of systems on which the determination depends. In the last column the mean masses have been multiplied by the corresponding numbers of stars in a cubic parsec according to the luminosity curve in the second column of Table 22. The B stars are shown separately in the first line of the table. Their average mass is only a very rough estimate.

[1]) *Astrophysical Journal*, **55**, 302, 1922; *Mt. Wilson Contr.* No. 230.

[2]) *Monthly Notices*, *R. A. S.*, **82**, 122, 1922. Compare the remark in the first column of p. 252 of the present article.

[3]) *Upsala Meddelanden*, No. 11, p. 30, 1926.

TABLE 34.

Mean masses and total mass of the stars in a unit of volume.

$M_{\text{vis.}}$	Limiting distance	Average mass	n	Total mass
B stars		4· :		·0016 :
$< +$ 1·5	20 ps	4·09	10	12
+ 1·5 to + 3·5	10	1·87	7	22
3·5 » 5·5	10	1·34	22	74
5·5 » 7·5	10	1·09	27	79
7·5 » 9·5	6·7	·86	5	72
9·5 » 11·5	6·7	·42	14	55
+ 11·5 » + 13·5	5·0	·22 :	6	·0049 :
All $< +$ 13·5				·0378

It is hardly possible to form an estimate of the total mass of the stars of still fainter absolute magnitudes. The only thing we can say is that it may be a quite considerable quantity. Among the 5 fainter stars which we know there are probably two white dwarfs. The companion to Procyon has a mass of ·39, VAN MAANEN's star may be several times the mass of the sun [1]). Thus, the next interval of 2^{M} beyond $+ 13^{M}\cdot 5$ may easily contain a total mass of more than twice that in the preceding interval. Extrapolating the mass of the faint stars the total mass gets dangerously near the value of ·092 solar masses derived from $K(z)$. We may conclude that the total mass of nebulous or meteoric matter near the sun is less than ·05 suns/ps^3 or 3.10^{-24} g/cm^3; it is probably less than the total mass of visible stars, possibly much less.

A remark may be added concerning the ratio between the average total mass in an element of volume and the total amount of photographic light. The ratio evidently varies with z, but we may compute the average ratio in a cylinder perpendicular to the galactic plane. The total mass in such a cylinder may be estimated from the numbers given in Table 29. If it is assumed that at all heights the same relative amount of dark matter has to be added we find for the total mass contained in a cylinder perpendicular to the galactic plane and with a cross section of a square parsec 80·4 times the sun's mass. With the alternative assumption that there is no dark matter beyond $z = 200$ ps I find 51·5 solar masses.

The total amount of photographic light contained in the same column can also be found from Table 29, but I prefer to follow the more direct way of computing it from the total amount of photographic light seen per unit of surface in the direction of the galactic pole. From the numbers given by VAN RHIJN [2]) this is computed to be equivalent to the light of ·00230 stars of 0·0 photographic magnitude per square degree. As SEARES has remarked [1]) this quantity is half the surface brightness of the galactic system as seen from an outside point in a direction perpendicular to the galactic plane. From the latter quantity we can directly deduce the amount of light in the cylinder considered, which I find to be equivalent to 37·0 units (the unit being the light of a star of + 6·0 photographic absolute magnitude). From Table 29 I find 43·2 units for the same quantity; the small difference is due to the imperfect representation of $A(m)$ by the laws assumed for the computation of Table 29. Using the first value and the mean of the two estimates of the total mass in the cylinder I find that one unit of light corresponds to 1·8 solar masses. Had we considered the layer between $z = \pm 100$ only the result would have been 1·1 solar masses.

It may be of some interest to compare these numbers to some other estimates of the same quantity. From the rotational velocity of the galaxy we know approximately the total mass contained in the more central parts of the galactic system. It may be put at $1\cdot 2.10^{11}$, if we take the mass of the sun as unit. We can also form an approximate estimate of the total luminosity contained in the same part of the system by computing from VAN RHIJN's star counts the total light which we receive from the region between, say, 280° and 10° galactic longitude and ± 20° latitude. The total luminosity estimated in this way is 10^{10} units. Thus, the average mass corresponding with a unit of light would be about 12 in this case, or about 7 times larger than the value derived above. It is not necessary to conclude from this that the absolutely bright stars are relatively less frequent near the centre, or that there is a greater percentage of nebulous or dark matter in this region: we might reverse the argument and conclude that some 85 % of the light of the galactic system is obscured before it reaches us. It is now becoming generally accepted that we can see only a small part of the system, the rest being dimmed or obscured by absorbing matter. We are not yet in a position to estimate the true total amount of light of the inner part of the galactic system, but it does not seem impossible that the amount derived above should be increased so much as to make the average mass corresponding to a unit of luminosity equal to 1·8, as found above.

HUBBLE has computed the relative amount of light and mass contained in the central part of the Andromeda nebula [2]). At 200 ps from the centre the velocity of

[1]) Compare the 2nd Note to the present article.

[2]) *Groningen Publications*, No. 43, Tables 6 and 7.

[1]) *Astrophysical Journal*, **52**, 171, 1920; *Mt. Wilson Contributions*, No. 191, p. 346.

[2]) *Astrophysical Journal*, **69**, 152, 1929; *Mt. Wilson Contribution*, No. 376.

rotation would be 72 km/sec, corresponding to a spherical mass of $2 \cdot 4 . 10^8$. The true mass required will be smaller, because it has a flattened shape, the factor being 0·70 for a homogeneous ellipsoid with $a/b = 2$ and 0·61 for an ellipsoid with $a/b = 3$. Let us assume $1 \cdot 6 . 10^8$ for the true mass. HUBBLE estimates that the luminosity of the corresponding part of the nebula is 2^m fainter than that of the entire system. With HUBBLE's distance of 275000 ps this gives $1 \cdot 2 . 10^8$ times the light of the sun. For the mean mass corresponding with a unit of light we find 1·3, agreeing within the limits of uncertainty with the value found above for the surroundings of the sun. A similar result was found by OEPIK, who arrived at a good estimate of the distance of the Andromeda nebula from the assumption of the proportionality of light and mass [1]).

The rotation of N. G. C. 4594, discovered by V. M. SLIPHER, may also be briefly discussed. According to measures by PEASE the velocity of rotation at 2′ from the centre, and near the border of the nebula, which is seen edge on, is about 330 km/sec. From the observed recession of + 960 km/sec of the centre of the nebula with respect to the centre of the galaxy we may estimate its distance at $3 . 10^6$ ps. The corresponding spherical mass at the centre would be $4 \cdot 4 . 10^{10}$; the true, flattened mass may be estimated at $3 . 10^{10}$. The luminosity of the region considered will probably not be very much less than that of the whole nebula, the integrated photographic magnitude of which has been assumed as $9^m \cdot 7$ (HOLETSCHEK's magnitude increased by $1^m \cdot 0$ in order to reduce to photographic light). The corresponding absolute luminosity is $3 . 10^9$; the average mass corresponding to a unit of light is thus found to be 10, or five times larger than in the sun's neighbourhood. In this case it is certain that the observed luminosity is too small on account of absorption; there is a broad dark lane across the nebula which cuts off part of the light. The light of the rest of the nebula may also have been considerably dimmed by absorption so that, as in the case of the central region of the galaxy, it is impossible to form an estimate of the true light or, say, the brightness with which the nebula would appear to an observer viewing it from a point on its axis.

Assuming that every unit of light corresponds with a mass of 1·8 times that of the sun we get the following estimates of total masses and circular velocities:

[1]) *Astrophysical Journal*, **55**, 406, 1922.

Object	m_t	ρ	Mass	v_c
Large Magellanic Cloud	1·2 [1])	$2 \cdot 6 . 10^4$ ps	$11 . 10^8$	54 km/sec
Small Magellanic Cloud	2·8 [1])	2·9 »	$3 . 10^8$	36
ω Centauri	3·3 [2])	·68 »	$10 . 10^6$	48
Messier 3	7·2 [3])	1·22 »	$0 \cdot 9 . 10^6$	24

The magnitudes represent total photographic magnitudes. The distances were taken from SHAPLEY's monograph on star clusters. The circular velocities refer to points 1700 and 900 ps from the centres of the Magellanic Clouds, whereas the numbers given for the globular clusters represent the maximum values of the circular velocities.

In the Large Magellanic Cloud the existence of rotational velocities of the order of the systematic differences in radial velocity as found by R. E. WILSON [4]) would seem to be quite possible.

NOTES.

Derivation of the factor by which average radial velocities in a region surrounding the galactic poles must be multiplied in order to be reduced to components perpendicular to the galactic plane.

Call this factor F and call the mean square velocities in the directions of the axes of the velocity ellipsoid a, b and c. Let us assume that c is directed towards the pole of the Milky Way. If c' is the mean square velocity for a circular area around the pole and homogeneously covered with stars:

$$c'^2 = c^2 P + \frac{a^2 + b^2}{2} (1 - P)$$

(compare STRÖMBERG, *Mt. Wilson Contributions*, No. 293, p. 5), where $P = 0 \cdot 680$ if the zone extends from 40° latitude to the pole and $P = 0 \cdot 834$ if the zone extends from 56° to the pole.

I have adopted the following values for a, b and c (from proper motion results by H. RAYMOND [5]) and radial velocity results by EDDINGTON and HARTLEY [6]); for the dwarfs I have adopted STRÖMBERG's values [7]).

[1]) VAN HERK, *B. A. N.* No. 209, 1930.
[2]) SCHILT, *Astronomical Journal*, **38**, 112, 1928.
[3]) HERTZSPRUNG, *Astr. Nachrichten*, No. 4952, 1918.
[4]) *Lick Publications*, **13**, 189, 1917.
[5]) *Astronomical Journal*, **29**, 25, 1915.
[6]) *Monthly Notices, R. A. S.*, **75**, 526, 1915.
[7]) *Mt. Wilson Contributions*, No. 245, p. 19, group VI; *Astrophysical Journal*, **56**, 283, 1922.

Type	c/a	c/b
A	0·51	1
F	·54	1
G giants	·55	1
K giants	·60	1
M giants	·60	1
dwarfs	·44	0·70

It is then an easy matter to compute $F = c/c'$ from the above formula; from a combination of the factors obtained for the two areas the value of F for the annular area from 40° to 55° latitude may be obtained. The results are in Table 2 of the above article.

The mass of VAN MAANEN's *star*.

In *Nature*, May 2, 1931 (p. 661) H. N. RUSSELL and R. D'E. ATKINSON have put forward the suggestion that VAN MAANEN's star ($0^h 43^m \cdot 9$; + 4° 55′, 1900) might show a very large Einstein shift. With MILNE's theory they compute a minimum mass of 8 times that of the sun, corresponding to an Einstein shift of at least 700 km/sec.

It may be of interest to note that there is a fairly good indication that the star really shows a large red-shift, though not as large as the predicted value. A radial velocity of + 238 km/sec has been published by ADAMS and JOY[1]). The resulting velocity of the star, corrected for the classical solar motion of 20 km/sec, is 239 km/sec, directed towards 94° galactic longitude and −68° latitude. This is a most uncommon direction for a velocity of such size, as may be noticed by an inspection of the frontispiece of *Groningen Publications* No. 40. In Table 6 of this same publication there are 13 velocities larger than 200 km/sec, all directed toward longitudes between 161° and 302°, and only one being inclined more than 20° to the galaxy. It may also be remarked that with any plausible value for the rotational velocity of the galactic system the velocity of the star exceeds considerably the presumable velocity of escape from the system.

[1]) *Publications Astr. Soc. Pacific*, **38**, 122, 1926.

As the anomalous character of the space motion is entirely due to the radial component (the transverse velocity, after correction for a solar velocity of 20 km/sec, is only 42 km/sec) and as the radial velocity was marked as uncertain, I suspected at the time that the large radial motion might not be real. From Professor DE SITTER I learn that the velocity has been confirmed by more than one plate, but that the lines seem to be very diffuse (which might be expected in case of a large Einstein shift).

I believe that, in view of the abnormality of the motion if the radial velocity is taken as real, RUSSELL and ATKINSON's interpretation is the most promising one Assuming a red-shift of + 240 km/sec the star would have a mass 3 times that of the sun (radius 5000 km) and a density of 10^7 g/cm³.

As was mentioned by RUSSELL and ATKINSON there is, for the near-by white dwarfs, a direct way of distinguishing between Einstein shift and real radial velocity by determining the secular change of the proper motion. If the radial velocity of + 240 km/sec represents a real motion the annual proper motion of VAN MAANEN's star should decrease by ″·0002 annually. If a fair number of plates are taken with some large instruments and at various epochs (the first long-focus plates known to me are from 1917) it would seem quite possible to determine such an amount in an interval of 30 or 40 years.

It is a pleasant duty to express my gratitude to the members of the computing staff of the Observatory, especially to Messrs. PELS and KRIEST who are responsible for the major part of the computational work involved in the above investigation.