EUROPHYSICS LETTERS 1 July 1996

Europhys. Lett., **35** (1), pp. 7-12 (1996)

Induced superconductivity distinguishes chaotic from integrable billiards

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(received 6 May 1996; accepted 17 May 1996)

PACS. 05.45+b – Theory and models of chaotic systems.
PACS. 74.50+r – Proximity effects, weak links, tunneling phenomena, and Josephson effects.
PACS. 74.80Fp – Point contacts; SN and SNS junctions.

Abstract. – Random matrix theory is used to show that the proximity to a superconductor opens a gap in the excitation spectrum of an electron gas confined to a billiard with a chaotic classical dynamics. In contrast, a gapless spectrum is obtained for a non-chaotic rectangular billiard, and it is argued that this is generic for integrable systems.

The quantization of a system with a chaotic classical dynamics is the fundamental problem of the field of "quantum chaos" [1], [2]. It is known that the statistics of the energy levels of a two-dimensional confined region (a "billiard") is different if the dynamics is chaotic or integrable [3]-[5]: A chaotic billiard has Wigner-Dyson statistics, while an integrable billiard has Poisson statistics. The two types of statistics are entirely different as far as the level correlations are concerned [6]. However, the mean level spacing is essentially the same: Particles of mass m in a billiard of area A have density of states $mA/2\pi\hbar^2$, regardless of whether their dynamics is chaotic or not.

In the solid state, chaotic billiards have been realized in semiconductor microstructures known as "quantum dots" [7]. These are confined regions in a two-dimensional electron gas, of sufficiently small size that the electron motion remains ballistic and phase-coherent on long time scales. (Long compared to the mean dwell time $t_{\rm dwell}$ of an electron in the confined region, which itself is much longer than the ergodic time $t_{\rm erg}$ in which an electron explores the available phase space.) A tunnelling experiment measures the density of states in the quantum dot, if its capacitance is large enough that the Coulomb blockade can be ignored. As mentioned above, this measurement does not distinguish chaotic from integrable dynamics.

In this paper we show that the density of states becomes a probe for quantum chaos if the electron gas is brought into contact with a superconductor. We first consider a chaotic billiard. Using random matrix theory, we compute the density of states $\rho(E)$ near the Fermi level ($E = 0$), and find that the coupling to a superconductor via a tunnel barrier induces an

energy gap $E_{\rm gap}$ of the order of the Thouless energy $E_{\rm T} \simeq \hbar/t_{\rm dwell}$ More precisely,

$$E_{\rm gap} = cN\Gamma\delta/2\pi, \tag{1}$$

where N is the number of transverse modes in the barrier, Γ is the tunnel probability per mode, 2δ is the mean level spacing of the isolated billiard, and c is a number which is weakly dependent on Γ (c decreases from 1 to 0 6 as Γ increases from 0 to 1) Equation (1) requires $1 \ll N\Gamma \ll \Delta/\delta$, where Δ is the energy gap in the bulk of the superconductor In this limit $\rho(E)$ vanishes identically for $E \leq E_{\rm gap}$ In contrast, for a rectangular billiard we do not find an energy gap in which $\rho = 0$, but instead find that the density of states vanishes linearly with energy near the Fermi level We present a general argument that in an integrable billiard ρ has a power law dependence on E for small E

The system considered is shown schematically in the inset of fig 1 A confined region in a normal metal (N) is connected to a superconductor (S) by a narrow lead containing a tunnel barrier The lead supports N propagating modes at the Fermi energy Each mode may have a different tunnel probability Γ_n, but later on we will take all Γ_n's equal to Γ for simplicity The proximity effects considered here require time-reversal symmetry, so we assume zero magnetic field (The case of broken time-reversal symmetry has been studied previously [8]-[10]) The quasi-particle excitation spectrum of the system is discrete for energies below Δ We are interested in the low-lying part of the spectrum, consisting of (positive) excitation energies $E_n \ll \Delta$ We assume that the Thouless energy $E_{\rm T} \equiv N\Gamma\delta/2\pi$ is also much smaller than Δ ([1])

There are two methods to compute the spectrum in the regime $E, E_{\rm T} \ll \Delta$ The first method is a scattering approach, which leads to the determinant equation [11]

$$\mathrm{Det}[1 + S_0(E)S_0^*(-E)] = 0 \tag{2}$$

The $N \times N$ unitary matrix $S_0(E)$ is the scattering matrix of the quantum dot plus tunnel barrier at an energy E above the Fermi level Equation (2) is a convenient starting point for the case that the quantum dot is an integrable billiard For the chaotic case, we will use an alternative —but equivalent— determinant equation involving an effective Hamiltonian [10],

$$\mathrm{Det}(E - H) = 0,\ H = \begin{pmatrix} H_0 & -\pi W W^{\rm T} \\ -\pi W W^{\rm T} & -H_0^* \end{pmatrix} \tag{3}$$

The $M \times M$ Hermitian matrix H_0 is the Hamiltonian of the isolated quantum dot (The finite dimension M is taken to infinity later on) Because of time-reversal symmetry, $H_0 = H_0^*$ The $M \times N$ coupling matrix W has elements

$$W_{mn} = \delta_{mn}\left(\frac{2M\delta}{\pi^2}\right)^{1/2}\left(2\Gamma_n^{-1} - 1 + 2\Gamma_n^{-1}\sqrt{1-\Gamma_n}\right)^{1/2},\ m = 1,2,\quad ,M,\quad n = 1,2,\quad ,N \tag{4}$$

The energy δ is one-half the mean level spacing of H_0, which equals the mean level spacing of H if $W = 0$

We now proceed to compute the density of states We first consider the case of a chaotic billiard The Hamiltonian H_0 then has the distribution of the Gaussian orthogonal ensemble [6],

$$P(H_0) \propto \exp\Big[-\frac{1}{4}M\lambda^{-2}\,\mathrm{Tr}\,H_0^2\Big],\ \lambda = 2M\delta/\pi \tag{5}$$

([1]) The opposite regime $E_{\rm T} \gg \Delta$ has a trivial discrete spectrum, consisting of N states with energies E_n just below Δ

We seek the density of states $\rho(E) = -\pi^{-1}\mathrm{Im}\,\mathrm{Tr}\,\langle(E+i0^+-H)^{-1}\rangle$, where $\langle\cdots\rangle$ denotes an average over H for fixed W and H_0 distributed according to eq. (5). The method we use to evaluate this average is a perturbation expansion in $1/M$, adapted from ref. [12], [13]. Because of the block structure of H (see eq. (3)), the Green function $\mathcal{G}(z) = \langle(z-H)^{-1}\rangle$ consists of four $M\times M$ blocks $\mathcal{G}_{11}, \mathcal{G}_{12}, \mathcal{G}_{21}, \mathcal{G}_{22}$. By taking the trace of each block separately, one arrives at a 2×2 matrix Green function

$$G = \frac{\lambda}{M}\begin{pmatrix} \mathrm{Tr}\,\mathcal{G}_{11} & \mathrm{Tr}\,\mathcal{G}_{12} \\ \mathrm{Tr}\,\mathcal{G}_{21} & \mathrm{Tr}\,\mathcal{G}_{22}\end{pmatrix}. \tag{6}$$

(We have multiplied by $\lambda/M = 2\delta/\pi$ for later convenience.) One more trace yields the density of states,

$$\rho(E) = -\frac{1}{2}\delta^{-1}\mathrm{Im}\mathrm{Tr}\,G(E+i0^+). \tag{7}$$

To leading order in $1/M$, the matrix G satisfies

$$G = \frac{\lambda}{M}\sum_{n=1}^{M}\begin{pmatrix} z-\lambda G_{11} & \pi w_n^2+\lambda G_{12} \\ \pi w_n^2+\lambda G_{21} & z-\lambda G_{22}\end{pmatrix}^{-1}, \tag{8}$$

where we have abbreviated $w_n^2 = (WW^{\mathrm{T}})_{nn}$. Equation (8) is a matrix generalization of Pastur's equation [14]. A unique solution is obtained by demanding that G goes to λ/z times the unit matrix as $|z|\to\infty$.

We now restrict ourselves to identical tunnel probabilities, $\Gamma_n \equiv \Gamma$. For $M \gg N \gg 1/\Gamma$ eq. (8) simplifies to

$$NG_{11}\delta = \pi z G_{12}(-G_{12}+1-2/\Gamma),\quad G_{22}=G_{11},\ G_{21}=G_{12},\ G_{12}^2 = 1+G_{11}^2. \tag{9}$$

This set of equations can be solved analytically ([2]). The result is that $\rho(E)=0$ for $E\le E_{\mathrm{gap}}$, where E_{gap} is determined by

$$\frac{k^6-k^4}{(1-k)^6}x^6 - \frac{3k^4-20k^2+16}{(1-k)^4}x^4 + \frac{3k^2+8}{(1-k)^2}x^2 = 1,$$

$$x = E_{\mathrm{gap}}/E_{\mathrm{T}},\ k = 1-2/\Gamma. \tag{10}$$

The solution of this gap equation is the result (1) announced in the introduction. The complete analytical solution of eq. (9) is omitted here for lack of space. In fig. 1 we plot the resulting density of states. In the limit $\Gamma = 1$ of ideal coupling it is given by

$$\rho(E) = \frac{E_{\mathrm{T}}\sqrt{3}}{6E\delta}[Q_+(E/E_{\mathrm{T}}) - Q_-(E/E_{\mathrm{T}})], \tag{11a}$$

$$Q_\pm(x) = \left[8-36x^2 \pm 3x\sqrt{3x^4+132x^2-48}\right]^{1/3}, \tag{11b}$$

$$E > E_{\mathrm{gap}} = 2E_{\mathrm{T}}\,\gamma^{5/2} \approx 0.6\,E_{\mathrm{T}}, \tag{11c}$$

([2]) It is worth noting that eqs. (9)-(12) also apply to the case that the chaotic billiard is coupled via two identical leads to two superconductors, having a phase difference ϕ. The density of states of such a quantum-dot Josephson junction is obtained by the replacement $\Gamma \to \Gamma_{\mathrm{eff}} = 2\cos(\frac{1}{2}\phi)[\cos(\frac{1}{2}\phi) - 1+2/\Gamma]^{-1}$. For a phase shift of π the excitation gap closes (since $\phi\to\pi$ corresponds to $\Gamma_{\mathrm{eff}}\to 0$), in agreement with Altland and Zirnbauer [9].

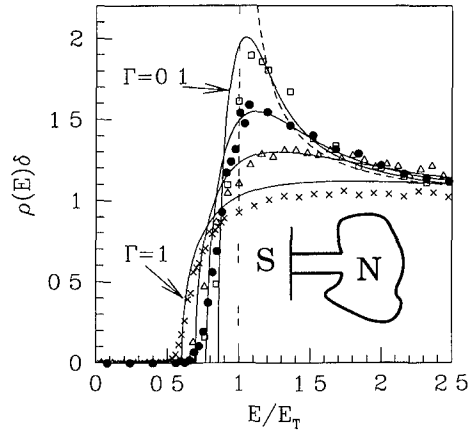

Fig 1

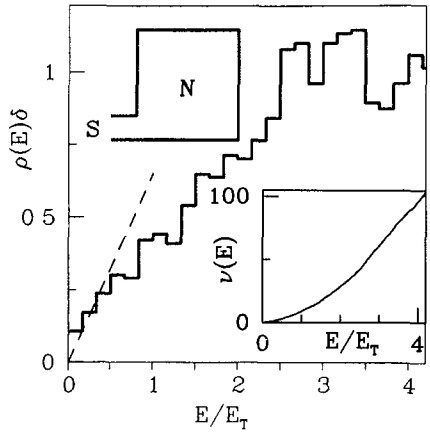

Fig 2

Fig 1 – Density of states of a chaotic billiard coupled to a superconductor (inset), for various coupling strengths The energy is in units of the Thouless energy $E_T = N\Gamma\delta/2\pi$ The solid curves are computed from eqs (7) and (9), for $\Gamma = 1$, 0 5, 0 25, 0 1 The dashed curve is the asymptotic result (12) for $\Gamma \ll 1$ The data points are a numerical solution of eq (3), averaged over 10^5 matrices H_0 in the Gaussian orthogonal ensemble ($M = 400$, $N = 80$) The deviation from the analytical curves is mainly due to the finite dimensionality M of H_0 in the numerics

Fig 2 – Histogram density of states for a rectangular billiard (shown to scale in the upper left inset), calculated numerically from eq (2) Dashed curve Bohr-Sommerfeld approximation (14) The lower right inset shows the integrated density of states, which is the quantity following directly from the numerical computation The energy $E_T = N\delta/2\pi$, with $N = 200$ modes in the lead to the superconductor

where $\gamma = \frac{1}{2}(\sqrt{5} - 1)$ is the golden number In the opposite limit $\Gamma \ll 1$ of weak coupling we find

$$\rho(E) = E\delta^{-1}(E^2 - E_T^2)^{-1/2},\ E > E_{gap} = E_T \qquad (12)$$

To check the validity of the perturbation theory, we have computed $\rho(E)$ numerically from eq (3) by generating a large number of random matrices H_0 in the Gaussian orthogonal ensemble The numerical results (data points in fig 1) are consistent with eq (9), given the finite dimensionality of H_0 in the numerics

We now turn to a non-chaotic, rectangular billiard A lead perpendicular to one of the sides of the rectangle connects it to a superconductor (The billiard is drawn to scale in the upper left inset of fig 2) There is no tunnel barrier in the lead The scattering matrix $S_0(E)$ is computed by matching wave functions in the rectangle to transverse modes in the lead The density of states then follows from eq (2) To improve the statistics, we averaged over 16 rectangles with small differences in shape but the same area A (and hence the same $\delta = \pi\hbar^2/mA$) The number of modes in the lead (width W) was fixed at $N = mv_F W/\pi\hbar = 200$ (where v_F is the Fermi velocity) In the lower right inset of fig 2 we show the integrated density of states $\nu(E) = \int_0^E dE'\rho(E')$, which is the quantity following directly from the numerical computation The density of states $\rho(E)$ itself is shown in the main plot

We have also computed the Bohr-Sommerfeld approximation to the density of states,

$$\rho_{\rm BS}(E) = N \int_0^\infty {\rm d}s\, P(s) \sum_{n=0}^{\infty} \delta\left(E - \left(n+\frac{1}{2}\right)\pi\hbar v_{\rm F}/s\right) \tag{13}$$

Here $P(s)$ is the classical probability that an electron entering the billiard will exit after a path length s Equation (13) is the Bohr-Sommerfeld quantization rule for the classical periodic motion with path length $2s$ and phase increment per period of $2Es/\hbar v_{\rm F} - \pi$ The periodic motion is the result of Andreev reflection at the interface with the superconductor, which causes the electron to retrace its path as a hole The phase increment consists of a part $2Es/\hbar v_{\rm F}$ because of the energy difference $2E$ between electron and hole, plus a phase shift of $-\pi$ from two Andreev reflections For $s \to \infty$ we find $P(s) \to 8(A/W)^2 s^{-3}$, which implies a linear E-dependence of the density of states near the Fermi-level,

$$\rho_{\rm BS}(E) \to \frac{4E}{N\delta^2} = \frac{2E}{\pi E_{\rm T}\delta}, \qquad E \to 0 \tag{14}$$

In fig 2 we see that the exact quantum-mechanical density of states also has (approximately) a linear E-dependence near $E = 0$, but with a smaller slope than the semi-classical Bohr-Sommerfeld approximation

We argue that the absence of an excitation gap found in the rectangular billiard is generic for the whole class of integrable billiards Our argument is based on the Bohr-Sommerfeld approximation It is known [15], [16] that an integrable billiard has a power law distribution of path lengths, $P(s) \to s^{-p}$ for $s \to \infty$ Equation (13) then implies a power law density of states, $\rho(E) \propto E^{p-2}$ for $E \to 0$

To conclude, we have shown that the presence of an excitation gap in a billiard connected to a superconductor is a signature of quantum chaos, which is special in two respects It appears in the spectral density rather than in a spectral correlator, and it manifests itself on the macroscopic energy scale of the Thouless energy rather than on the microscopic scale of the level spacing Both these characteristics are favourable for experimental observation Our theoretical results are rigorous for a chaotic billiard and for an integrable rectangular billiard We have presented an argument that the results for the rectangle are generic for the whole class of integrable billiards, based on the semi-classical Bohr-Sommerfeld approximation There remains the challenge to develop a rigorous general theory for the integrable case

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This work was supported by the Dutch Science Foundation NWO/FOM and by the Human Capital and Mobility program of the European Community (grant number 674664)

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