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PHOTON SHOT NOISE

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A recent theory is reviewed for the shot noise of coherent radiation propagating through a random medium. The Fano factor $P/\bar{I}$ (the ratio of the noise power and the mean transmitted current) is related to the scattering matrix of the medium. This is the optical analogue of Büttiker's formula for electronic shot noise. Scattering by itself has no effect on the Fano factor, which remains equal to 1 (as for a Poisson process). Absorption and amplification both increase the Fano factor above the Poisson value. For strong absorption $P/\bar{I}$ has the universal limit $1+\frac{3}{2}f$, with f the Bose–Einstein function at the frequency of the incident radiation. This is the optical analogue of the one-third reduction factor of electronic shot noise in diffusive conductors. In the amplifying case the Fano factor diverges at the laser threshold, while the signal-to-noise ratio $\bar{I}^2/P$ reaches a finite, universal limit.

1. Introduction

Analogies in the behavior of photons and electrons provide a continuing source of inspiration in mesoscopic physics.[1] Two familiar examples are the analogies between weak localization of electrons and enhanced backscattering of light and between conductance fluctuations and optical speckle.[2] The basis for these analogies is the similarity between the single-electron Schrödinger equation and the Helmholtz equation. The Helmholtz equation is a classical wave equation, and indeed the study of mesoscopic phenomena for light has been limited mostly to *classical* optics. A common theme in these studies is the interplay of interference and multiple scattering by disorder. The extension to *quantum* optics adds the interplay with vacuum fluctuations as a new ingredient.

Recently a theoretical approach to the quantum optics of disordered media was proposed,[3] that utilizes the methods of the random-matrix theory of quantum transport.[4,5] The random matrix under consideration is the scattering matrix. The basic result of Ref. 3 is a relationship between the scattering matrix and the photocount distribution. It was applied there to the statistics of blackbody radiation and amplified spontaneous emission. This work was reviewed in Ref. 6. Here we review a later development,[7] the optical analogue of electronic shot noise.

Shot noise is the time-dependent fluctuation of the current $I(t) = \bar{I} + \delta I(t)$ (measured in units of particles/s) resulting from the discreteness of the particles. The noise power

$$P = \int_{-\infty}^{\infty} dt\, \overline{\delta I(0)\delta I(t)} \tag{1}$$

quantifies the size of the fluctuations. (The bar $\overline{\cdots}$ indicates an average over many measurements on the same system.) For independent particles the current fluctuations form a Poisson process, with power $P_{\rm Poisson} = \bar{I}$ equal to the mean current. The ratio $P/P_{\rm Poisson}$ (called the Fano factor[8]) is a measure of the correlations between the particles.

For electrons, correlations resulting from the Pauli exclusion principle reduce P below $P_{\rm Poisson}$. (See Ref. 9 for a review.) The ratio $P/P_{\rm Poisson}$ is expressed in terms of traces of the transmission matrix t at the Fermi energy by[10]

$$\frac{P}{P_{\rm Poisson}} = 1 - \frac{{\rm Tr}(tt^\dagger)^2}{{\rm Tr}\, tt^\dagger}\,. \tag{2}$$

This formula holds at zero temperature (no thermal noise). In the absence of scattering all eigenvalues of the transmission-matrix product $tt^\dagger$ are equal to unity, hence $P = 0$. This absence of shot noise is realized in a ballistic point contact.[11,12] At the other extreme, in a tunnel junction all transmission eigenvalues are $\ll 1$, hence $P = P_{\rm Poisson}$.[13] A disordered metallic conductor is intermediate between these two extremes, having $P = \frac{1}{3}P_{\rm Poisson}$.[14,15]

For the optical analogue we consider a monochromatic laser beam (frequency ω_0) incident in a single mode (labelled m_0) on a waveguide containing a disordered medium (at temperature T). The radiation from a laser is in a coherent state. The photostatistics of coherent radiation is that of a Poisson process,[16] hence $P = P_{\rm Poisson}$ for the incident beam. The question addressed in this work is: How does the ratio $P/P_{\rm Poisson}$ change as the radiation propagates through the random medium? We saw that, for electrons, scattering increases this ratio. In contrast, in the optical analogue scattering by itself has no effect: P remains equal to $P_{\rm Poisson}$ if the incident beam is only partially transmitted — provided the scattering matrix remains unitary. A non-unitary scattering matrix, resulting from absorption or amplification of radiation by the medium, increases the ratio $P/P_{\rm Poisson}$. This excess noise can be understood as the beating of coherent radiation with vacuum fluctuations of the electromagnetic field.[17]

Photon shot noise has been studied extensively in systems where the scattering is one-dimensional (for example, randomly layered media).[18,19] No formula of the generality of Eq. (2) was needed for those investigations. In order to go beyond the one-dimensional case, we have derived the optical analogue of Eq. (2). The result is[7]

$$\frac{P}{P_{\rm Poisson}} = 1 + 2f(\omega_0, T)\frac{[t^\dagger(\mathbb{1} - rr^\dagger - tt^\dagger)t]_{m_0m_0}}{[t^\dagger t]_{m_0m_0}}\,, \tag{3}$$

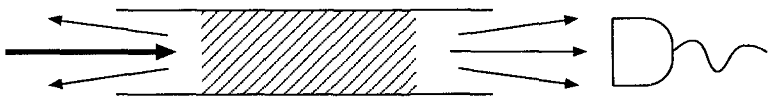

Fig. 1. Coherent light (thick arrow) is incident on an absorbing or amplifying medium (shaded), embedded in a waveguide. The transmitted radiation is measured by a photodetector.

where $f(\omega, T) = [\exp(\hbar\omega/kT) - 1]^{-1}$ is the Bose–Einstein function. Equation (3) contains both the transmission matrix t and the reflection matrix r (evaluated at frequency ω_0). For a unitary scattering matrix, $rr^\dagger + tt^\dagger$ equals the unit matrix $\mathbb{1}$, hence the term proportional to f in Eq. (3) vanishes and $P = P_{\text{Poisson}}$. Absorption and amplification both lead to an enhancement of P above P_{Poisson}. For an absorbing system the matrix $\mathbb{1} - rr^\dagger - tt^\dagger$ is positive definite and $f > 0$, so $P/P_{\text{Poisson}} > 1$. In an amplifying system $\mathbb{1} - rr^\dagger - tt^\dagger$ is negative definite but f is also negative (because $T < 0$ in an amplifying system), so P/P_{Poisson} is still > 1.

We will review the derivation of the optical shot-noise formula (3), and the application to absorbing and amplifying disordered waveguides. The amplifying case is of particular interest in view of the recent experiments on random lasers,[20,21] which are amplifying media in which the feedback required for a laser threshold is provided by scattering from disorder rather than by mirrors.

2. Optical Shot-Noise Formula

In this section we summarize the scattering formulation of the photodetection problem,[3] and derive the formula (3) for the excess noise.[7] We consider an absorbing or amplifying disordered medium embedded in a waveguide that supports $N(\omega)$ propagating modes at frequency ω (see Fig. 1). The absorbing medium is in thermal equilibrium at temperature $T > 0$. In the amplifying medium, the amplification could be due to stimulated emission by an inverted atomic population or to stimulated Raman scattering.[17] A negative temperature $T < 0$ describes the degree of population inversion in the first case or the density of the material excitation in the second case.[18] A complete population inversion or vanishing density corresponds to the limit $T \to 0$ from below. The Bose–Einstein function $f(\omega, T)$ is > 0 for $T > 0$ and < -1 for $T < 0$.[a] The absorption or amplification rate $1/\tau_{\text{a}} = \omega|\varepsilon''|$ is obtained from the imaginary part ε'' of the (relative) dielectric constant ($\varepsilon'' > 0$ for absorption, $\varepsilon'' < 0$ for amplification). Disorder causes multiple scattering with rate $1/\tau_{\text{s}}$ and (transport) mean free path $l = c\tau_{\text{s}}$ (with c the velocity of light in the medium). The diffusion constant is $D = \frac{1}{3}cl$. The absorption or amplification length is defined by $\xi_{\text{a}} = \sqrt{D\tau_{\text{a}}}$.

[a]The quantity $f(\omega, T)$ is called the "population inversion factor" in the laser literature, because if ω is close to the laser frequency Ω one can express $f = (N_{\text{lower}}/N_{\text{upper}} - 1)^{-1}$ in terms of the ratio $N_{\text{lower}}/N_{\text{upper}} = \exp(\hbar\Omega/kT)$ of the population of the lower and upper atomic levels, with $f = -1$ corresponding to a complete population inversion.

The waveguide is illuminated from one end by monochromatic radiation (frequency ω_0, mean photocurrent I_0) in a coherent state. For simplicity, we assume that the illumination is in a single propagating mode (labelled m_0). At the other end of the waveguide, a photodetector detects the outcoming radiation. We assume, again for simplicity, that all N outgoing modes are detected with unit quantum efficiency. We denote by $p(n)$ the probability to count n photons within a time t. Its first two moments determine the mean photocurrent $\bar{I}$ and the noise power P, according to[b]

$$\bar{I} = \frac{1}{t}\bar{n}, \qquad P = \lim_{t\to\infty} \frac{1}{t}\left(\overline{n^2} - \bar{n}^2\right) . \tag{4}$$

The outgoing radiation in mode n is described by an annihilation operator $a_n^{\rm out}(\omega)$, using the convention that modes $1, 2, \ldots, N$ are on the left-hand-side of the medium and modes $N+1, \ldots, 2N$ are on the right-hand-side. The vector $a^{\rm out}$ consists of the operators $a_1^{\rm out}, a_2^{\rm out}, \ldots, a_{2N}^{\rm out}$. Similarly, we define a vector $a^{\rm in}$ for incoming radiation. These two sets of operators each satisfy the bosonic commutation relations

$$[a_n(\omega), a_m^\dagger(\omega')] = \delta_{nm}\delta(\omega - \omega'), \qquad [a_n(\omega), a_m(\omega')] = 0 , \tag{5}$$

and are related by the input-output relations[18,22,23]

$$a^{\rm out} = Sa^{\rm in} + Ub + Vc^\dagger . \tag{6}$$

We have introduced the $2N \times 2N$ scattering matrix S, the $2N \times 2N$ matrices U, V, and the vectors b, c of $2N$ bosonic operators. The reflection and transmission matrices are $N \times N$ submatrices of S,

$$S = \begin{pmatrix} r' & t' \\ t & r \end{pmatrix} . \tag{7}$$

The operators b, c account for vacuum fluctuations. In order for these operators to satisfy the bosonic commutation relations (5), it is necessary that

$$UU^\dagger - VV^\dagger = \mathbb{1} - SS^\dagger . \tag{8}$$

In an absorbing medium $c \equiv 0$ and b has the expectation value

$$\langle b_n^\dagger(\omega) b_m(\omega')\rangle = \delta_{nm}\delta(\omega - \omega') f(\omega, T), \qquad T > 0 . \tag{9}$$

Conversely, in an amplifying medium $b \equiv 0$ and c has the expectation value

$$\langle c_n(\omega) c_m^\dagger(\omega')\rangle = -\delta_{nm}\delta(\omega - \omega') f(\omega, T), \qquad T < 0 . \tag{10}$$

The probability $p(n)$ that n photons are counted in a time t is given by[16]

$$p(n) = \frac{1}{n!}\langle : W^n e^{-W} : \rangle , \tag{11}$$

[b]This definition of P is equivalent to Eq. (1); In some papers the noise power is defined with an extra factor of 2.

where the colons denote normal ordering with respect to a^{out}, and

$$W = \int_0^t dt' \sum_{n=N+1}^{2N} a_n^{\text{out}\dagger}(t') a_n^{\text{out}}(t') , \tag{12}$$

$$a_n^{\text{out}}(t) = (2\pi)^{-1/2} \int_0^\infty d\omega\, e^{-i\omega t} a_n^{\text{out}}(\omega) . \tag{13}$$

Expectation values of a normally ordered expression are readily computed using the optical equivalence theorem.[16] Application of this theorem to our problem consists in discretising the frequency in infinitesimally small steps of Δ (so that $\omega_p = p\Delta$) and then replacing the annihilation operators $a_n^{\text{in}}(\omega_p)$, $b_n(\omega_p)$, $c_n(\omega_p)$ by complex numbers a_{np}^{in}, b_{np}, c_{np}. The coherent state of the incident radiation corresponds to a non-fluctuating value of a_{np}^{in}, such that $|a_{np}^{\text{in}}|^2 = \delta_{nm_0}\delta_{pp_0} 2\pi I_0/\Delta$ (with $\omega_0 = p_0\Delta$). The thermal state of the vacuum fluctuations corresponds to uncorrelated Gaussian distributions of the real and imaginary parts of the numbers b_{np} and c_{np}, with zero mean and variance $\langle |b_{np}|^2\rangle = -\langle |c_{np}|^2\rangle = f(\omega_p, T)$, in accordance with Eqs. (9) and (10).

To evaluate the moments of the photocount distribution we need to perform Gaussian averages. The first two moments determine $\bar{I}$ and P. The results are[3,7]

$$\bar{I} = I_0 [t^\dagger t]_{m_0 m_0} + \int_0^\infty \frac{d\omega}{2\pi} f(\omega, T)\, \text{Tr}(\mathbb{1} - rr^\dagger - tt^\dagger) , \tag{14}$$

$$P = \bar{I} + 2 I_0 f(\omega_0, T) [t^\dagger (\mathbb{1} - rr^\dagger - tt^\dagger) t]_{m_0 m_0} + \int_0^\infty \frac{d\omega}{2\pi} f(\omega, T)^2\, \text{Tr}(\mathbb{1} - rr^\dagger - tt^\dagger)^2 . \tag{15}$$

The mean photocurrent is the sum of two terms, a term $\propto I_0$ equal to the transmitted part of the incident current and a term $\propto f$ that represents the thermal emission of radiation. The noise power is the sum of three terms, the Poisson noise $\bar{I}$ plus two sources of excess noise. The term $\propto f^2$ is due to thermal emission while the term $\propto I_0 f$ is the excess noise due to beating of vacuum fluctuations with the incident radiation. For a unitary scattering matrix both terms vanish and $P = \bar{I}$ equals the Poisson value.

The contributions from thermal emission to $\bar{I}$ and P can be eliminated by filtering the output through a narrow frequency window around ω_0. Only the terms proportional to the incident current I_0 remain,

$$\bar{I} = I_0 [t^\dagger t]_{m_0 m_0} , \tag{16}$$

$$P = \bar{I} + 2 I_0 f(\omega_0, T) [t^\dagger (\mathbb{1} - rr^\dagger - tt^\dagger) t]_{m_0 m_0} . \tag{17}$$

This yields the optical shot noise formula (3) discussed in the introduction.

3. Absorbing Random Medium

We consider an ensemble of absorbing disordered waveguides, with different realizations of the disorder, and evaluate the ensemble averages of Eqs. (16) and (17).

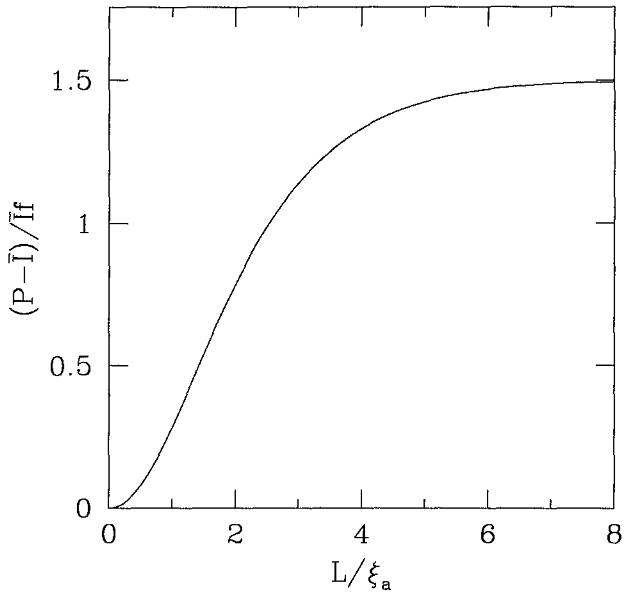

Fig. 2. Excess noise power for an absorbing disordered waveguide, computed from Eqs. (18) and (19). The ratio $P/\bar{I}$ tends to $1+\frac{3}{2}f$ for $L \gg \xi_a$.

For a random medium the dependence on the index m_0 of the incident radiation is insignificant on average, so we may replace the average of a matrix element $[\cdots]_{m_0 m_0}$ by the average of the normalized trace $N^{-1}\mathrm{Tr}$. Moments of $rr^\dagger$ and $tt^\dagger$ in the presence of absorption have been computed by Brouwer[24] using the methods of random-matrix theory, in the regime that both the length L of the waveguide and the absorption length ξ_a are much greater than the mean free path l but much less than the localization length Nl. This is the large-N regime $N \gg L/l, \xi_a/l \gg 1$. The ratio $L/\xi_a \equiv s$ is arbitrary.

The result is[7]

$$\bar{I} = \frac{4l}{3L} I_0 \frac{s}{\sinh s}, \tag{18}$$

$$5P = \bar{I} + \frac{2l}{3L} I_0 f s \left[\frac{3}{\sinh s} - \frac{2s + \operatorname{cotanh} s}{\sinh^2 s} - \frac{s \operatorname{cotanh} - 1}{\sinh^3 s} + \frac{s}{\sinh^4 s}\right]. \tag{19}$$

The ratio $P/P_{\mathrm{Poisson}} \equiv P/\bar{I}$ increases from 1 to $1+\frac{3}{2}f$ with increasing s, see Fig. 2. The limiting value $P/P_{\mathrm{Poisson}} \to 1+\frac{3}{2}f(\omega_0,T)$ for $L \gg \xi_a$ depends on temperature and frequency through the Bose–Einstein function, but is independent of the scattering or absorption rates. This might be seen as the optical analogue of the universal limiting value $P/P_{\mathrm{Poisson}} \to \frac{1}{3}$ for $L \gg l$ of the electronic shot noise.[14,15]

In the derivation of the limiting value of $P/\bar{I}$ we have assumed that the length L of the waveguide remains small compared to the localization length $\xi_{\mathrm{loc}} = Nl$. What happens in the localized regime $L \gtrsim \xi_{\mathrm{loc}}$? Using the results from Ref. 24 we find

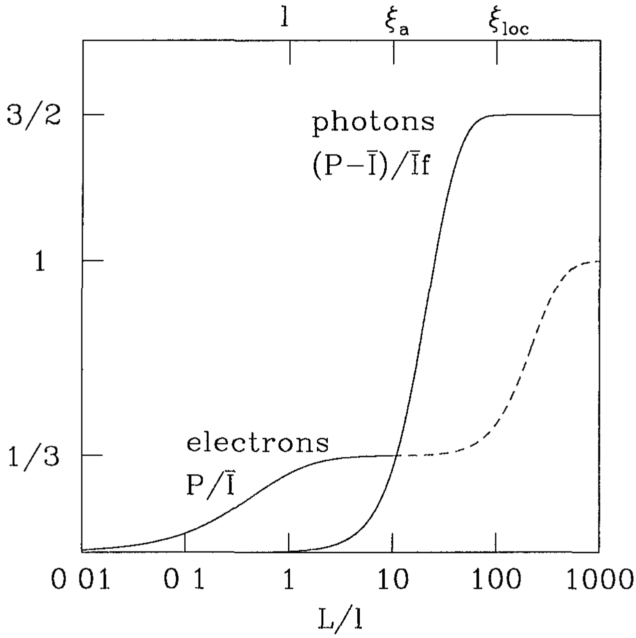

Fig 3 Different dependence for electrons and photons of the shot-noise power P on the length L of the waveguide The curve for photons is the same as in Fig 2, the solid curve for electrons has been calculated in Ref 25, the dashed curve is a qualitative interpolation We have assumed a factor of 10 between the mean free path l, the absorption length ξ_a, and the localization length $\xi_{loc} = Nl$ (For electrons, the absorption length should be interpreted as a dephasing length) The electronic P increases from 0 to $\frac{1}{3}$ of the Poisson value $\bar{I}$ when L becomes larger than l, and then increases further to full Poisson noise at $\xi_{loc} = Nl$ (Dephasing has no effect) The photonic P has only a single transition, at ξ_a, from $\bar{I}$ to $(1 + \frac{3}{2})\bar{I}$ Nothing happens at $L = l$ or $L = \xi_{loc}$ to the shot noise of coherent radiation

that the ensemble averages $\langle \bar{I} \rangle$ and $\langle P \rangle$ of current and noise in the localized regime are suppressed below the results Eqs. (18) and (19) in the diffusive regime,[c] but the ratio $\langle P \rangle / \langle \bar{I} \rangle$ remains equal to $1 + \frac{3}{2} f$. This is a remarkable difference with the electronic analogue, where P becomes equal to the Poisson noise $\bar{I}$ in the localized regime. The difference between shot noise for electrons and photons is summarized in Fig. 3.

4. Amplifying Random Medium

The results for an amplifying disordered waveguide in the large-N regime follow from Eqs. (18) and (19) for the absorbing case by the substitution $\tau_a \to -\tau_a$, or equivalenty $s \to is$. One finds

$$\bar{I} = \frac{4l}{3L} I_0 \frac{s}{\sin s}, \qquad (20)$$

[c]The precise result is $\langle \bar{I} \rangle = (1 + \frac{3}{2} f)^{-1} \langle P \rangle = \frac{8}{3} (l/\xi_a) I_0 \exp(-L/\xi_a - \frac{3}{8} L/Nl)$ in the regime $N, L/l \gg \xi_a/l \gg 1$, for any value of $L/Nl \lesssim Nl/\xi_a$

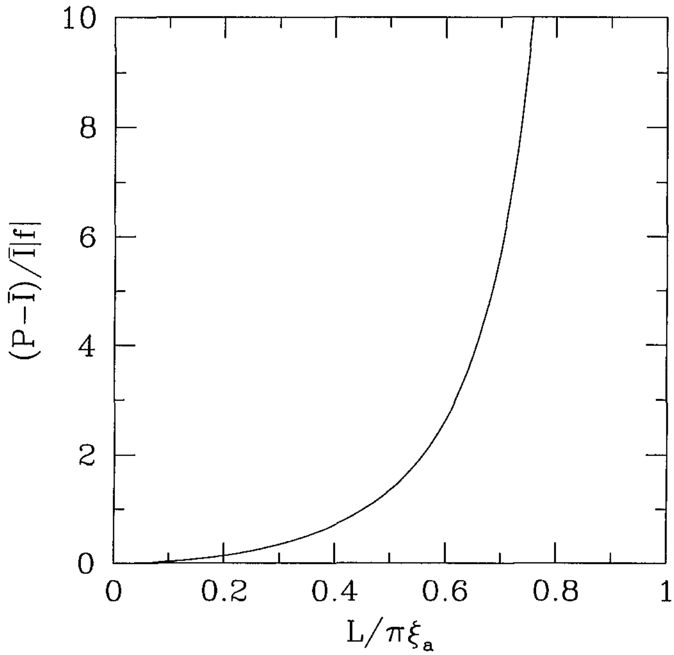

Fig. 4. Excess noise power for an amplifying disordered waveguide, computed from Eqs. (20) and (21). The ratio $P/\bar{I}$ diverges at the laser threshold $L = \pi\xi_a$.

$$P = \bar{I} + \frac{2l}{3L} I_0 f s \left[\frac{3}{\sin s} - \frac{2s - \operatorname{cotan} s}{\sin^2 s} + \frac{s \operatorname{cotan} s - 1}{\sin^3 s} - \frac{s}{\sin^4 s} \right] . \tag{21}$$

Recall that the Bose–Einstein function $f < -1$ in an amplifying medium. As shown in Fig. 4, the ratio $P/\bar{I}$ increases without bound as the length $L \to \pi\xi_a$ or, equivalently, the amplification rate $1/\tau_a \to \pi^2 D/L^2$. This is the laser threshold.

To understand better the behavior close to the laser threshold, we consider the scattering matrix $S(\omega)$ as a function of complex frequency ω. In the absence of amplification all poles (resonances) of S are in the lower half of the complex plane, as required by causality. Amplification shifts the poles upwards by an amount $1/2\tau_a$. The laser threshold is reached when the first pole hits the real axis, say at resonance frequency Ω. For ω near Ω the scattering matrix has the generic form

$$S_{nm} = \frac{\sigma_n \sigma_m}{\omega - \Omega + (i\Gamma/2) - (i/2\tau_a)} , \tag{22}$$

where σ_n is the complex coupling constant of the resonance to the nth mode in the waveguide and Γ is the decay rate. The laser threshold is at $\Gamma\tau_a = 1$. We will now show that, while P and $\bar{I}$ diverge at the laser threshold, the signal-to-noise ratio $\mathcal{S} = \bar{I}^2/P$ has a finite limit — independent of σ_n, Γ, or τ_a.[7]

We assume that the incident radiation has frequency $\omega_0 = \Omega$. Substitution of Eq. (22) into Eqs. (16) and (17) gives the simple result

$$\mathcal{S} = \frac{I_0|\sigma_{m_0}|^2}{2|f|\Sigma}, \qquad \Sigma = \sum_{n=1}^{2N} |\sigma_n|^2 . \tag{23}$$

The total coupling constant $\Sigma = \Sigma_l + \Sigma_r$ is the sum of the coupling constant $\Sigma_l = \sum_{n=1}^{N} |\sigma_n|^2$ to the left end of the waveguide and the coupling constant $\Sigma_r = \sum_{n=N+1}^{2N} |\sigma_n|^2$ to the right. The ensemble average $\langle |\sigma_{m_0}|^2/\Sigma \rangle$ is independent of $m_0 \in [1, N]$, hence

$$\langle \mathcal{S} \rangle = \frac{I_0}{2|f|N} \langle \Sigma_l/\Sigma \rangle = \frac{I_0}{4|f|N}, \tag{24}$$

since $\langle \Sigma_l/\Sigma \rangle = \langle \Sigma_r/\Sigma \rangle \Rightarrow \langle \Sigma_l/\Sigma \rangle = 1/2$. The signal-to-noise ratio of the incident coherent radiation (with noise power $P_0 = I_0$) is given by $\mathcal{S}_0 = I_0^2/P_0 = I_0$. The ratio $\mathcal{S}/\mathcal{S}_0$ is the reciprocal of the noise figure of the amplifier. The signal-to-noise ratio of the transmitted radiation is maximal for complete population inversion, when $|f| = 1$ and $\langle \mathcal{S} \rangle$ is smaller than $\mathcal{S}_0$ by a factor $4N$. This universal limit $\langle \mathcal{S}/\mathcal{S}_0 \rangle \to 1/4N$ does not require large N, but holds for any $N = 1, 2, \ldots$. It is the multi-mode generalization of a theorem for the minimal noise figure of a single-mode linear amplifier.[17,26]

5. Outlook

We conclude by mentioning some directions for future research. In the electronic case it is known that the result $P/\bar{I} = 1/3$ for the Fano factor of a diffusive conductor can be either computed from the scattering matrix[14] (using random-matrix theory) or from a kinetic equation known as the Boltzmann–Langevin equation.[15] Here we have shown using the former approach that the optical analogue is a Fano factor of $1 + \frac{3}{2}f$ for a disordered waveguide longer than the absorption length. To obtain this result from a kinetic equation one needs a Boltzmann–Langevin equation for bosons. Work in this direction is in progress.[27]

The effect of localization on the Fano factor is strikingly different for electrons and photons. In the electronic case the average $\langle P/\bar{I} \rangle$ goes to 1 in the localized regime, but we have found for the optical case that the ratio $\langle P \rangle / \langle \bar{I} \rangle$ of average noise and average current is unchanged as the length of the waveguide becomes longer than the localization length. We surmise that the same applies to the average $\langle P/\bar{I} \rangle$ of the ratio, but this remains to be verified. (In the diffusive regime the difference between the two averages can be neglected.)

In the case of an amplifying disordered waveguide we have restricted ourselves to the linear regime below the laser threshold. Above threshold the fluctuations in the amplitude of the electromagnetic field are strongly suppressed and only phase fluctuations remain.[16] These determine the quantum-limited linewidth of the radiation. A theory for this linewidth in a random medium is under development.[28] The application to a disordered waveguide would require a knowledge of the statistics of the poles of the scattering matrix in such a system, which is currently lacking.[d]

The recent interest in the Hanbury-Brown and Twiss experiment for electrons in a disordered metal[1] suggests a study of the optical case. The formalism presented here for auto-correlations of the photocurrent can be readily extended to cross-correlations,[30] but it has not yet been applied to a random medium.

We do not know of any experiments on photon shot noise in a random medium, and hope that the theoretical predictions reviewed here will stimulate work in this direction. The universal limits of the Fano factor in the absorbing case and the signal-to-noise ratio in the amplifying case seem particularly promising for an experimental study.

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[d]Since in Sec. 4 the laser threshold was found to be at $1/\tau_a = \pi^2 D/L^2$ in the large-N limit, we conclude that $\Gamma = \pi^2 D/L^2$ is the minimal decay rate in that limit. In other words, the density of S-matrix poles for a disordered waveguide without amplification should vanish for $\mathrm{Im}\,\omega > -\pi^2 D/2L^2$ if $N \to \infty$. This density is unknown, but a similar gap in the density of poles has been found for the scattering matrix of a chaotic cavity.[29]

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