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Fluctuating phase rigidity for a quantum chaotic system with partially broken time-reversal symmetry

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The functional $\rho=|\int d\vec{r}\psi^2|^2$ measures the phase rigidity of a chaotic wave function $\psi(\vec{r})$ in the transition between Hamiltonian ensembles with orthogonal and unitary symmetry Upon breaking time reversal symmetry, ρ crosses over from one to zero We compute the distribution of ρ in the crossover regime and find that it has large fluctuations around the ensemble average These fluctuations imply long-range spatial correlations in ψ and non-Gaussian perturbations of eigenvalues, in precise agreement with results by Fal'ko and Efetov [Phys Rev Lett **77**, 912 (1996)] and by Taniguchi *et al* [Europhys Lett **27**, 335 (1994)] As a third implication of the phase-rigidity fluctuations we find correlations in the response of an eigenvalue to independent perturbations of the system [S1063 651X(97)50201-7]

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Wave functions of billiards with a chaotic classical dynamics have been measured both for classical [1,2] and quantum mechanical waves [3,4] The experiments are consistent with a χ^2_β distribution of the squared modulus $|\psi(\vec{r})|^2$ of a wave function at point $\vec{r}$, the index $\beta=1$ or 2 depending on whether time-reversal symmetry is present or completely broken These two symmetry classes are the orthogonal and unitary ensembles of random-matrix theory [5] For a complete description of the experiments one also needs to know what spatial correlations exist between $|\psi(\vec{r}_1)|^2$ and $|\psi(\vec{r}_2)|^2$ at two different points and how these correlations are affected by breaking of time-reversal symmetry In the orthogonal and unitary ensembles it is known that the correlations decay to zero if the distance $|\vec{r}_2-\vec{r}_1|$ greatly exceeds the wavelength λ [6]

Recently, Fal'ko and Efetov [7] managed to compute the two-point distribution $P_2(p_1,p_2)$ in the crossover regime between the orthogonal and unitary ensembles (We abbreviate $p_i\equiv V|\psi(\vec{r}_i)|^2$, with V the volume of the system) They found that the two-point distribution does not factorize into one-point distributions, $P_2(p_1,p_2)\neq P_1(p_1)P_1(p_2)$, even if $|\vec{r}_2-\vec{r}_1|\gg\lambda$ The existence of long-range correlations in a chaotic wave function came as a surprise

Two years earlier, in an apparently unrelated paper, Taniguchi *et al* [8] had studied the response of an energy level $E(X)$ to a small perturbation of the Hamiltonian (parameterized by the variable X) They discovered a non-Gaussian distribution of the level "velocity" dE/dX in the orthogonal to unitary crossover This was remarkable, since the distribution is Gaussian in the orthogonal and unitary ensembles

It is the purpose of the present paper to show that these two crossover effects are two different manifestations of one fundamental phenomenon, which we identify as *phase-rigidity fluctuations* The phase rigidity is the real number $\rho=|\int d\vec{r}\psi^2|^2$ in the interval [0,1], which equals 1 (0) in the orthogonal (unitary) ensemble The possibility of fluctuations in ρ was first noticed by French *et al* [9], but the distribution $P(\rho)$ was not known We have computed $P(\rho)$ in the crossover regime, building on work by Sommers and Iida [10], and find a broad distribution Previous theories for the crossover by Zyczkowski and Lenz [11], by Kogan and Kaveh [12], and most recently by Kanzieper and Freilikher [13] amount to a neglect of fluctuations in ρ, and thus imply the absence of long-range correlations in $\psi(\vec{r})$ and a Gaussian distribution of dE/dX. Conversely, once the fluctuations of the phase rigidity are properly accounted for, we recover the distant correlations and non-Gaussian distribution of Refs. [7,8], and find a correlation between level velocities for independent perturbations of the Hamiltonian.

We start from the Pandey-Mehta Hamiltonian [5,14] for a system with partially broken time-reversal symmetry,

$$H=S+i\alpha(2N)^{-1/2}A, \tag{1}$$

where α is a positive number, and S (A) is a symmetric (antisymmetric) real $N\times N$ matrix. The matrix S has the Gaussian distribution

$$P(S)\propto\exp(-\tfrac{1}{4}Nc^{-2}\mathrm{Tr}SS^{\dagger}), \tag{2}$$

and the distribution of A is the same. The real parameter c determines the mean level spacing Δ at the center of the spectrum for $N\gg 1$, by $c=N\Delta/\pi$. The distribution of H crosses over from the orthogonal to the unitary ensemble at $\alpha\simeq 1$. The wave function ψ_k of the kth energy level at widely separated points ($|\vec{r}_i-\vec{r}_j|\gg\lambda$) is represented by the unitary matrix U that diagonalizes H:

$$V^{1/2}\psi_k(\vec{r}_i)\rightarrow N^{1/2}U_{ik}. \tag{3}$$

Consider now an eigenvector $|u\rangle=(U_{1k},U_{2k},\ldots,U_{Nk})$. (Since we deal with a single eigenstate, we suppress the level index k.) Following Ref. [9] we decompose $|u\rangle$ in the form

$$|u\rangle=e^{i\phi}(t|R\rangle+i\sqrt{1-t^2}|I\rangle), \tag{4}$$

where $|R\rangle$ and $|I\rangle$ are real orthonormal N-component vectors, and $\phi\in[0,\pi/2)$ and $t\in[0,1]$ are real numbers. This decomposition exists for any normalized vector $|u\rangle$ and is unique for $t\neq 0,1$. The phase rigidity ρ is related to the parameter t by

$$\rho=\left|\int d\vec{r}\,\psi_k^2\right|^2\rightarrow\left|\sum_i U_{ik}^2\right|^2=(2t^2-1)^2. \tag{5}$$

In the orthogonal ensemble $t=0$ or 1, hence $\rho=1$, while in the unitary ensemble $t=\sqrt{1/2}$ hence $\rho=0$. In the crossover between these two ensembles the parameter ρ does not take on a single value but fluctuates.

To compute the distribution $P(\rho)$ we use a result of Sommers and Iida [10], for the joint probability distribution of an eigenvalue E and the corresponding eigenvector $|u\rangle$ of the Hamiltonian (1). Substitution of the decomposition (4), and inclusion of the Jacobian for the change of variables from $|u\rangle$ to ρ, gives the expression

$$P(\rho)\propto\frac{(1-\rho)^{N/2-3/2}}{D^{N/2-1}\sqrt{\Lambda}}\left[\frac{c^2}{N\Lambda}+\rho\left(\frac{2b_-}{D}\right)^2\frac{\partial}{\partial b_-}+\left(\frac{2b_+}{D}\right)^2\left(\frac{1}{2}\frac{\partial^2}{\partial E^2}+\frac{\partial}{\partial b_+}\right)\right]Z_{N-2}(E)\bigg|_{E=0}, \tag{6a}$$

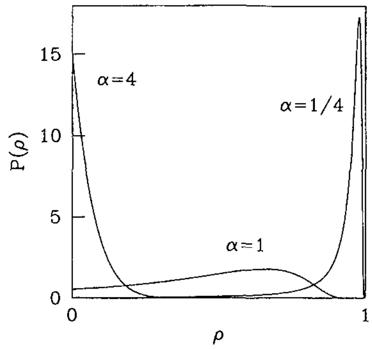

FIG. 1. Distribution of the phase rigidity ρ for $\alpha=1/4$, 1, and 4, computed from Eq. (9). The crossover from the orthogonal to unitary ensemble occurs when $\alpha\approx 1$, and is associated with large fluctuations in ρ around its ensemble average.

$$b_{\pm}=\frac{c^2}{N}\left(1\pm\frac{\alpha^2}{2N}\right),\quad D=4+\frac{2N}{\alpha^2}(1-\rho)\left(1-\frac{\alpha^2}{2N}\right)^2,$$

$$\Lambda=2+(1-\rho)\left(\frac{2N}{\alpha^2}-1\right), \tag{6b}$$

$$Z_N(E)=\frac{1}{N!}\left(b_+\frac{\partial}{\partial\omega}\right)^N(1-\omega b_-/b_+)^{-1}(1-\omega)^{-3/2}\times(1+\omega)^{-1/2}\exp\left(\frac{-\omega E^2}{(1+\omega)b_+}\right)\bigg|_{\omega=0}. \tag{6c}$$

We have set $E=0$, corresponding to the center of the spectrum. We still have to take the limit $N\rightarrow\infty$. Expansion of $Z_N(0)$ in a series,

$$Z_N(0)=b_+^N\sum_{k=0}^{N}a_k\left(\frac{b_-}{b_+}\right)^{N-k}, \tag{7a}$$

$$a_k=\frac{1}{k!}\frac{\partial^k}{\partial\omega^k}(1-\omega)^{-(3/2)}(1+\omega)^{-(1/2)}|_{\omega=0}\overset{k\gg 1}{\rightarrow}\sqrt{\frac{2k}{\pi}}, \tag{7b}$$

and replacement of the summation by an integration, yields

$$Z_N(0)=\frac{c^{2N}\sqrt{2/\pi}}{\alpha^2N^{N-3/2}}\left(e^{\alpha^2/2}+\frac{ie^{-\alpha^2/2}\sqrt{\pi}}{2\alpha}\mathrm{erf}(i\alpha)\right) \tag{8}$$

for $N\gg 1$. Here $\mathrm{erf}(i\alpha)\equiv 2i\pi^{-1/2}\int_0^{\alpha}e^{y^2}dy$. The double energy derivative of $Z_N(E)$ is computed similarly, but turns out to be smaller by a factor N and can thus be neglected. The derivatives with respect to $b_{\pm}$ can be found by differentiation of Eq. (8). Collecting all terms, we find

$$P(\rho)=(1-\rho)^{-2}\exp\left(\frac{\alpha^2}{\rho-1}\right)\left[\frac{\alpha^2-1+\rho}{1-\rho}\times\left(e^{\alpha^2}+\frac{i\pi^{1/2}}{2\alpha}\mathrm{erf}(i\alpha)\right)-\frac{i\alpha\pi^{1/2}}{2}\mathrm{erf}(i\alpha)\right]. \tag{9}$$

In Fig. 1 the distribution of ρ is plotted for three values of the crossover parameter α. It is very broad for $\alpha=1$, and narrows to a delta function at 1 (0) for $\alpha\rightarrow 0$ ($\alpha\rightarrow\infty$).

It remains to show that the long-range wave-function correlations and non-Gaussian level-velocity distributions of Refs. [7,8] follow from the distribution $P(\rho)$ that we have computed. We begin with the wave-function correlations, and consider the n-point distribution function

$$P_n(p_1,p_2,\ldots,p_n)=\left\langle \prod_{i=1}^{n} \delta(p_i-N|U_{ik}|^2)\right\rangle. \tag{10}$$

We substitute the decomposition (4) and do the average in two steps: First over $|R\rangle$ and $|I\rangle$, and then over t. Due to the invariance of $P(H)$ under orthogonal transformations of H, the vectors $|R\rangle$ and $|I\rangle$ can be integrated out immediately. In the limit $N\to\infty$, the components of the two vectors are $2N$ independent real Gaussian variables with zero mean and variance $1/N$. Doing the Gaussian integrals we find a generalization of results in Refs. [9,11] to $n>1$:

$$P_n(p_1,p_2,\ldots,p_n)=\int_0^1 d\rho P(\rho)\prod_{i=1}^{n} F(p_i,\rho), \tag{11a}$$

$$F(p,\rho)=(1-\rho)^{-(1/2)}\exp\left(\frac{p}{\rho-1}\right)I_0\left(\frac{p\sqrt{\rho}}{1-\rho}\right). \tag{11b}$$

Here I_0 is a Bessel function. We see that long-range spatial correlations exist only if the distribution $P(\rho)$ of ρ has a finite width. For example, the two-point correlator $\langle p_1^2p_2^2\rangle-\langle p_1^2\rangle\langle p_2^2\rangle$ equals the variance of ρ. The approximation of Ref. [11] (implicit in Refs. [12,13]) was to take ρ fixed at each α. If ρ is fixed, $P_n(p_1,\ldots,p_n)\to P_1(p_1)\cdots P_1(p_n)$ factorizes, and hence spatial correlations are absent. If instead we substitute for $P(\rho)$ our result (9), we recover exactly the results of Fal'ko and Efetov [7,15].

We now turn to the level-velocity distributions. We consider perturbations of the Hamiltonian (1) by a real symmetric (antisymmetric) matrix S' (A'),

$$H'=H+x_oS'+x_uiA'. \tag{12}$$

Here x_u, x_o are real infinitesimals, which parameterize, respectively, a perturbation that breaks or does not break time-reversal symmetry. The corresponding level velocities

$$v_o=\frac{\partial E_k}{\partial x_o},\quad v_u=\frac{\partial E_k}{\partial x_u}, \tag{13}$$

have distributions

$$P(v_o)=\left\langle \delta\left(v_o-\sum_{i,j} U_{ik}U_{jk}^*S'_{ji}\right)\right\rangle, \tag{14a}$$

$$P(v_u)=\left\langle \delta\left(v_u-\sum_{i,j} U_{ik}U_{jk}^*iA'_{ji}\right)\right\rangle. \tag{14b}$$

We substitute the decomposition (4) for the eigenvector U_{ik} of H and average first over S' and A', assuming a Gaussian distribution for these perturbation matrices. After averaging over S' and A', the eigenvector enters only via the parameter ρ. One finds

$$P(v_o)=\int_0^1 d\rho P(\rho)G_{1+\rho}(v_o), \tag{15a}$$

$$P(v_u)=\int_0^1 d\rho P(\rho)G_{1-\rho}(v_u), \tag{15b}$$

where $G_{1\pm\rho}$ is a Gaussian distribution with zero mean and variance $1\pm\rho$. We have normalized the velocities such that $\overline{v_o^2}=\overline{v_u^2}=1$ in the unitary ensemble. Substitution of Eq. (9) for $P(\rho)$ shows that the distribution of v_o coincides with the result of Ref. [8]. However, our $P(v_u)$ is different. This is because we have chosen A and A' to be independent random matrices, whereas they are identical in Ref. [8]. Independent matrices A and A' are appropriate for a system with a perturbing magnetic field in a random direction. Identical A and A' correspond to a system in which only the magnitude but not the direction of the field is varied. Equation (15) demonstrates that $P(v_o)$ and $P(v_u)$ are Gaussians in the orthogonal and unitary ensembles, since then $P(\rho)$ is a delta function. In the crossover regime the distributions are non-Gaussian, because of the finite width of $P(\rho)$. The relation (15) between the distributions of v and ρ for the GOE–GUE transition is reminiscent of a relation obtained by Fyodorov and Mirlin for the metal-insulator transition [16]. The role of the parameter ρ is then played by the so-called inverse participation ratio $I=\int d\vec{r}|\psi|^4$. In our system $NI\to\rho+2$ for $N\to\infty$. A difference from Ref. [16] is that our perturbation matrices are drawn from orthogonally invariant ensembles, whereas their perturbation is band diagonal.

As a final example of the importance of the phase-rigidity fluctuations in the crossover regime, we consider the response of the system to two or more independent perturbations,

$$H'=H+\sum_{i=1}^{m} x_{oi}S'_i+\sum_{j=1}^{n} x_{uj}iA'_j. \tag{16}$$

For example, one may think of the displacement of m different scatterers, or the application of a localized magnetic field at n different sites. Proceeding as before, we obtain the joint probability distribution of the level velocities $v_{oi}=\partial E_k/\partial x_{oi}$ and $v_{uj}=\partial E_k/\partial x_{uj}$,

$$P(v_{o1},v_{o2},\ldots,v_{om},v_{u1},v_{u2},\ldots,v_{un})$$
$$=\int_0^1 d\rho P(\rho)\prod_{i=1}^{m} G_{1+\rho}(v_{oi})\prod_{j=1}^{n} G_{1-\rho}(v_{uj}). \tag{17}$$

We see that as a result of the finite width of $P(\rho)$, the joint distribution of level velocities does not factorize into the individual distributions (15), implying that the response of an energy level to independent perturbations of the Hamiltonian is correlated.

To summarize, we have introduced the phase rigidity, defined as the squared modulus of the spatial average of the wave function squared, and computed its distribution for a chaotic system with partially broken time-reversal symmetry. Fluctuations of the phase rigidity from one wave function to another exist if time-reversal symmetry is partially broken. We have shown that these fluctuations imply long-range wave-function correlations and non-Gaussian eigenvalue perturbations, thereby unifying two previously unrelated discoveries [7,8] A manifestation of the phase-rigidity fluctuations is the existence of level-velocity correlations for independent perturbations of the system

Note added We have learned that Y Alhassid, J N Hormuzdiar, and N D Whelan have been working on this same problem, with some overlap of results

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[1] J Stein, H -J Stockmann, and U Stoffregen, Phys Rev Lett **75,** 53 (1995)
[2] V N Prigodin, N Taniguchi, A Kudrolli, V Kidambi, and S Sridhar, Phys Rev Lett **75,** 2392 (1995)
[3] A M Chang, H U Baranger, L N Pfeiffer, K W West, and T Y Chang, Phys Rev Lett **76,** 1695 (1996)
[4] J A Folk, S R Patel, S F Godijn, A G Huibers, S M Cronenwett, C M Marcus, K Campman, and A C Gossard, Phys Rev Lett **76,** 1699 (1996)
[5] M L Mehta, *Random Matrices* (Academic, New York, 1991)
[6] V N Prigodin and N Taniguchi, Mod Phys Lett B **10,** 69 (1996)
[7] V I Fal'ko and K B Efetov, Phys Rev Lett **77,** 912 (1996)
[8] N Taniguchi, A Hashimoto, B D Simons, and B L Altshuler, Europhys Lett **27,** 335 (1994)
[9] J B French, V K B Kota, A Pandey, and S Tomsovic, Ann Phys (N Y) **181,** 198 (1988)
[10] H -J Sommers and S Iida, Phys Rev E **49,** R2513 (1994)
[11] K Zyczkowski and G Lenz, Z Phys B **82,** 299 (1991)
[12] E Kogan and M Kaveh, Phys Rev B **51,** 16400 (1995)
[13] E Kanzieper and V Freilikher, Phys Rev B **54,** 8737 (1996)
[14] A Pandey and M L Mehta, Commun Math Phys **87,** 449 (1983)
[15] V I Fal'ko and K B Efetov, Phys Rev B **50,** 11267 (1994)
[16] Y V Fyodorov and A D Mirlin, Phys Rev B **51,** 13403 (1995)