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Heat conductivity and nature of the lunar surface material, by *A. J. Wesselink.*

A quantitative explanation is given for the variations in the lunar surface temperature that have been observed during an eclipse of the moon as well as throughout a lunation.

In the first eight sections the theory of heat conduction, with constant coefficient of conduction, k, and constant specific heat per unit volume, c, has been summarized and developed. Analytical as well as numerical methods are described. It is shown how the product kc can be found from theory and observation. In section 9 the observational data obtained by PETTIT and NICHOLSON are described. In section 10 the provisional result $(kc)^{1/2} = {\cdot}012$ cal cm^{-2} min$^{-1/2}$ (°C)$^{-1}$ is obtained from data outside an eclipse. The amount of heat conducted inwards per minute at the subsolar point is less than one percent of the amount of heat radiated and of the radiant energy which is absorbed in the same time.

A first approximation to the internal temperature distribution is given. In section 11 the more accurate result $(kc)^{1/2} = {\cdot}0084$ is obtained from PETTIT's data on the eclipse of October 28, 1939. It agrees with EPSTEIN's result: $(kc)^{1/2} = {\cdot}0083$ based on data from an earlier eclipse. Section 12 contains a prediction of the variation in surface temperature of an area near the centre of the lunar disc during a lunation (Figure 3). A second approximation to the internal temperature distribution is found. The constant temperature in the inner part of the moon is found to be 212° K. Section 13 contains a theoretical determination of the surface variation of T during the eclipse of October 28, 1939. In section 14 the variations during a lunation are compared with those during an eclipse from the point of view of homology transformations, of which a general description is given in section 8.

In section 15 we take $c = {\cdot}40$, which gives $k = 16 \times 10^{-5}$ cal cm^{-1} min^{-1} (°C)$^{-1}$. The penetration of the variations in temperature below the lunar surface is calculated to be only a few millimetres at an eclipse, whereas the wave-length of a harmonic heat wave with a period equal to the synodic month is 14·5 cm. In section 16 the difficulty of finding a terrestrial substance that has as low a heat conductivity as was found for the moon, is discussed. It is remarked that pumice, which had been proposed by EPSTEIN, does not give an adequate explanation. The solution is found in section 17, where it is shown that our result for the moon is in excellent agreement with the laboratory measures of SMOLUCHOWSKI on powders in vacuo. As the heat conductivity of a powder increases rapidly with the gas pressure in the interstices, PETTIT and NICHOLSON's temperature measures may be said to give a new and independent proof for the absence of a lunar atmosphere.

In section 18 it is shown that the grains of the lunar surface powder must be smaller than ·3 mm. If the grain size is between ·1 and ·3 mm, k varies with temperature. If it is smaller than ·1 mm, the variation of k with temperature is small.

Notations and definitions

T, absolute temperature in degrees Kelvin.
x, depth in cm below the lunar surface, reckoned positive inwards.
t, time in minutes.
c, specific heat per unit volume (cm^3), or the amount of heat in calories needed to raise one cm^3 one degree centigrade.
k, coefficient of heat conduction; k equals the amount of heat in calories flowing per minute across a square cm perpendicular to the direction of the flow, when the temperature gradient is one degree centigrade per cm. Unit: cal cm^{-2} min^{-1} (°C.cm^{-1})$^{-1}$ = cal cm^{-1} min^{-1} (°C)$^{-1}$.
P, period of temperature variation; in the application P equals the synodic month.
ω, $\omega = \frac{2\pi}{P}$; the value corresponding to a synodic month is $1{\cdot}4776 \times 10^{-4}$ min^{-1}.
Semi-infinite solid, a solid filling all space to one side of a plane homogeneously.
l, wave length of harmonic heat wave.
$\xi = x/l$.
F, flow of heat across a square cm per minute; F is reckoned positive outwards.
A, amount of heat absorbed by a square cm of the surface per minute. A_s is the value of A at the subsolar point. $A_s = 1{\cdot}55$ cal cm^{-2} min^{-1}; the value is derived in the text.
σ = Stefan's constant; $\sigma = {\cdot}82 \times 10^{-10}$ cal cm^{-2} min^{-1} (°C)$^{-4}$.

Introduction

1. The purpose of this article is to account for the temperatures observed on the lunar surface, during an eclipse of the moon and during a lunation, by means of the theory of heat conduction. Throughout the investigation we shall consider the situation in a limited region of the lunar surface. The depths involved in the problem are so small compared with the lunar radius, that it is perfectly legitimate to neglect the curvature of the surface. Moreover the roughness of the surface will be neglected. In our problem the moon thus behaves as a so called semi-infinite solid, a well known notion in the theory of heat conduction. At any time we have the same situation for all points on planes parallel to the plane surface. For a complete specification of the situation we therefore need only a single spatial co-ordinate: the depth below the surface, x, and the time, t, as independent variables. In general both the coefficient of heat conduction, k, and the specific heat per unit volume, c, have been taken as absolute constants. At the end of the article a possible variation of k and c with T is discussed and results of computations with variable k have been communicated in section 18.

The lunar surface is supposed to radiate as a black body, i.e. according to Stefan's law. All the absorption of the incident radiation is supposed to occur at the smooth surface. Otherwise stated, we assume infinite optical depth for every layer below the surface. In sections 1 — 8 we summarize and develop the theory of heat conduction for a semi-infinite solid so far as is necessary for the treatment of the observational data (k and c are taken constant).

The change in heat content of an internal element of volume equals the net amount of heat energy conducted through its walls. This statement (law of conservation of energy) is expressed mathematically by the well known equation of heat conduction:

$$c\frac{\partial T}{\partial t}=k\frac{\partial^2 T}{\partial x^2}. \tag{1}$$

The same condition, but now applied to an infinitesimal element of volume at the surface, leads to:

$$\sigma T_0^4 = A + F_0. \tag{2}$$

In (2) the index o indicates surface values (x = o).

At any depth we have further:

$$F = k\frac{\partial T}{\partial x}. \tag{3}$$

With given boundary conditions, (1), (2) and (3) determine a unique solution.

In the following sections some particular cases are considered.

2. Suppose the variation of temperature on the surface to be simply harmonic, and the constants of this variation to be known. It is required to find the variations in the solid when the periodic state has been established. In the periodic state the variations are periodic at any depth.

We have at the surface:

$$T(0,t) = a + b\cos(\omega t + \varepsilon). \tag{4}$$

When this harmonic surface variation of the temperature has lasted long enough, the state at any depth inside the solid will also be periodic. This periodic state is independent of the initial distribution of temperature; apart from the constants k and c, depending on the nature of the solid, it depends on the surface variation alone.

The expression for the temperature as a function of x and t in the periodic state is derived in several textbooks on general physics [1]).

It is:

$$T(x,t) = a + b\,e^{-2\pi x/l}\cos 2\pi\left(\frac{t}{P}-\frac{x}{l}+\varepsilon\right), \tag{5}$$

in which
$$l = 2\left(\pi\frac{k}{c}P\right)^{1/2}. \tag{6}$$

It is seen from (5) that the phase difference between the variation inside the solid and the variation at the surface is proportional to the depth. At $x = l$ this phase retardation is 2π, for which reason l has been called the wave-length of the heat wave. The oscillations are damped, the amplitude at $x = l$ being only $e^{-2\pi}$, or nearly 2°/₀₀, of the surface amplitude. If k/c is not known, we cannot determine l. The temperature is however a known function of ξ and t, where $\xi = x/l$.

Differentiation of (5) with respect to x and multiplication of the result with k gives an expression for F. At any depth F varies harmonically with period P: the maximum flow inwards occurs one eighth of a period before the moment of maximum temperature (at the same depth). The amplitude of F declines inwards exponentially, keeping a constant ratio $(kc\omega)^{1/2}$ to that of T. In particular we have at the surface:

$$\text{semi-amplitude of } F_0 = b\ (kc\omega)^{1/2}. \tag{7}$$

At large depths the temperature tends to a constant value a, which is also the average temperature over the time at any depth, in particular at the surface. The limiting value of F at large depths is zero.

Suppose that the variation (4) is due to a periodic variation in the incident radiation with period P. If the periodic variation of the amount of radiative energy absorbed is known as a function of the time, $A\ (t)$, we can find F_0 numerically from (2). Then $(kc)^{1/2}$ follows from (7).

3. Consider the more general case that the given variation of the surface temperature is periodic but not necessarily harmonic. Again we want to find the periodic state. The Fourier development of the given surface variation may be written:

$$T(0,t) = \sum_0^\infty b_n \cos(n\,\omega\, t + \varepsilon_n), \tag{8}$$

in which the coefficients b_n and the phase constants ε_n can be determined from the given function T (o, t). Each of the terms in (8) if present alone would lead to a periodic solution of the form (5), in which for the term of order n, for P and l, respectively P/n and $l/\sqrt{n}$ have to be substituted. As a consequence of the line-

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[1]) E.g. R. A. Houston, *An Introduction to Mathematical Physics*, p. 94.
G. Joos, *Lehrbuch der Theoretischen Physik*, p. 406.

arity of (1) we find the periodic state corresponding to (8) by simple addition of the various solutions, each of which corresponds to one particular term in (8). The periodic solution in terms of ξ and t is then:

$$T(\xi,t)=\sum_{0}^{\infty} b_n e^{-2\pi\xi\sqrt{n}}\cos 2\pi\left(\frac{t}{P/n}-\xi\sqrt{n}+\varepsilon_n\right),\ \text{where } \xi=x/l \text{ as before.} \qquad (9)$$

As in section 2, differentiation of (9) with respect to x and multiplication of the result with k according to (3) gives an expression for F. If F_n is the term in the expansion for F corresponding to the term T_n in (8), we have the same relations of amplitude and phase between F_n and T_n as noted in section 2, remembering that the period is P/n. At any depth, F is richer in higher harmonics than T, the ratio between the amplitudes of F_n and T_n being $(kc\omega n)^{1/2}$, which is proportial to $\sqrt{n}$. Both T and F approach to simple harmonic variation with increasing depth, the terms of higher order dying away more rapidly. At large depth T is constant and equals b_0, while the limiting value of F is zero.

Suppose again the surface variation (8) to be due to a periodic variation in the incident radiation. If A is known, F_0 may be found numerically from (2). The expression for F_0 deduced from (9) and (3) contains the factor $(kc)^{1/2}$, which can be found.

4. It is convenient to introduce ξ instead of x, not only in the solutions of the equations but also in these equations themselves. The following formulae are equivalent to (1), (2) and (3). Equation (2) is not altered; it is repeated as (11) for convenience.

$$\frac{\partial T}{\partial t}=(4\pi P)^{-1}\frac{\partial^2 T}{\partial \xi^2}, \qquad (10)$$

$$\sigma T_0^4 = A + F_0, \qquad (11)$$

$$F=(4\pi P)^{-1/2}.\,(kc)^{1/2}.\frac{\partial T}{\partial \xi}. \qquad (12)$$

Though the period P occurs in these equations, they are just as general as (1), (2) and (3) and are by no means restricted to periodic phenomena with period P. P has entered the equations as a consequence of the introduction of $\xi = x/l$ and l is related to P by (6). When the equations are applied to the temperature variations during a lunation, P will be taken to be the synodic month, which seems only natural. In our discussion of the variations at an eclipse of the moon, P is again taken equal to the synodic month, but now because it was found convenient to use the same definition of ξ as in the former case.

Equation (10) becomes very simple for numerical integration if the small but finite differences $\Delta\xi$ and Δt are chosen in the manner first proposed by SCHMIDT 1). In finite differences (10) may be written

$$\frac{(\Delta T)_t}{\Delta t}=(4\pi P)^{-1}\frac{(\Delta_2 T)_\xi}{(\Delta\xi)^2}. \qquad (13)$$

Put $\xi=\xi_0+m\Delta\xi$
$t=t_0+n\,\Delta t,$ m and n being integers.

Then $(\Delta T)_t \;= T_{n+1,m}-T_{n,m}$ (14)

$(\Delta_2 T)_\xi = T_{n,m+1}-2\,T_{n,m}+T_{n,m-1}.$ (15)

Choose $\Delta\xi$ and Δt such that

$$(4\pi P)^{-1}\frac{\Delta t}{(\Delta\xi)^2}=\tfrac{1}{2} \qquad (16)$$

and substitute (14), (15) and (16) in (13); we obtain SCHMIDT's form of (10):

$$T_{n+1,m}=\tfrac{1}{2}\,(T_{n,m-1}+T_{n,m+1}). \qquad (17)$$

By (17) the temperature distribution at $t+\Delta t$ can be calculated if it is already known at t. (17) is of course independent of ξ_0 and t_0. It is clear that the surface temperature cannot be found from (17), so that (17) is only sufficient for an integration when the surface variation is known. In other cases (11) and (12) have to be used in addition. (12) for the surface combined with (11) gives:

$$\sigma T_0^4 = A+(4\pi P)^{-1/2}(kc)^{1/2}.\left(\frac{\partial T}{\partial \xi}\right)_0. \qquad (18)$$

5. Suppose the temperature distribution at $t=0$, $T(\xi, 0)$, and the temperature variation at the surface, $T(0, t)$, to be given. It is required to find $T(\xi, t)$.

We take $\xi_0=t_0=0$. If $\Delta\xi$ and Δt are chosen according to (16), the problem is solved by continual application of (17).

When the surface variation of T is due to a variation in the incident radiation, F_0 follows from (11) when A is known. According to (12), F_0 must be proportional to $\left(\frac{\partial T}{\partial \xi}\right)_0$, which quantity can be found from the solution $T(\xi, t)$ just obtained. $F_0 \Big/ \left(\frac{\partial T}{\partial \xi}\right)_0$ then yields $(kc)^{1/2}$ by (12).

Consider the problem of a given periodic surface variation. It is required to find the periodic solution. The problem has been treated analytically in section 3; we shall now describe a numerical procedure.

Starting with some arbitrary initial distribution of temperature, equation (17) is applied continually until at any depth the variations have become periodic with sufficient approximation. However, unless the initial distribution of temperature is chosen with care, this task may well prove rather elaborate. If the initial distribution of temperature is taken constant

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1) SCHMIDT, *Foepples Festschrift*, p. 179, Julius Springer, Berlin 1924.

and equal to the time average of the known surface variation, and if the integration is started at any of the moments that this temperature is reached at the surface, the amount of work will be relatively small. The final state for the deeper parts of the solid, where the convergence is slowest, is then reached already at the start.

6. The initial temperature distribution and $(kc)^{1/2}$ are given. Let further A be known after $t = 0$. It is required to find the surface temperature for $t > 0$. We take $t_0 = 0, \xi_0 = -\frac{1}{2}\Delta\xi$. In the solid the integration proceeds according to (17), $\Delta\xi$ and Δt having been chosen in accordance with (16). A fictitious temperature at $-\frac{1}{2}\Delta\xi$ is added for convenience. It is determined by the requirement that $T(0, t)$ shall be the arithmetic mean of the simultaneous values of T at $-\frac{1}{2}\Delta\xi$ and $+\frac{1}{2}\Delta\xi$. The temperature at $+\frac{1}{2}\Delta\xi$ at time $t + \Delta t$ is found from (17), thus being the mean of the temperatures at $-\frac{1}{2}\Delta\xi$ and $+1\frac{1}{2}\Delta\xi$ at time t. (18) in a form suitable for numerical integration is written as follows:

$$\sigma T_0^4 = A + (\pi P)^{-1/2}.(kc)^{1/2}(\Delta\xi)^{-1}(T_{+1/2\Delta\xi} - T_0). \quad (19)$$

T_0 is found by solving the quartic equation (19). This is most easily done by trial and error. The computation has been facilitated by a table, giving σT_0^4 as a function of T_0 for every degree, which had been prepared once and for all. When T_0 is found, $T_{-1/2\Delta\xi}$ is determined, and so on.

It should be noticed that ξ_0 might have been taken equal to zero. A solution as proposed above with $\xi_0 = -\frac{1}{2}\Delta\xi$, however, with the same $\Delta\xi$, uses a more accurate approximation to $\left(\frac{\partial T}{\partial\xi}\right)_0$, without appreciable increase in the amount of computation to be done. The gain in accuracy of the method "$\xi_0 = -\frac{1}{2}\Delta\xi$" over "$\xi_0 = 0$" is often significant, since especially near the surface the variations are most rapid. It is clearly not necessary to keep to the same values of $\Delta\xi$ and Δt throughout the same solution. When at some stage the variations of T with ξ and t are relatively small, it is often advantageous to use larger values $\Delta\xi$ and Δt. If $\Delta\xi$ is taken twice as large as before, the new Δt is four times larger than the old one, and (16) is again satisfied. Hence an eightfold increase in the speed of the numerical work is obtained in the solid. The reduction of the work on (19) is obviously four times.

7. Given $(kc)^{1/2}$ and the periodic variation of A. It is required to find the periodic solution, in particular the surface variation. Starting from an arbitrary initial distribution of temperature the integration proceeds as in section 6. Since the periodic surface variation is still unknown, it is not now possible to represent, as in section 5, the constant temperature at large depths correctly at the start. A practical way to obtain the required result quickly is to derive the initial distribution for a second approximation from (9), when the surface variation has been derived for a complete period in the first integration.

8. *Homology transformation.*

Suppose $T(\xi, t)$ is the solution of the equations (10), (11) and (12) corresponding to a certain value $(kc)^{1/2}$, initial distribution $T(\xi, 0)$ and surface condition $A(t)$.

When we substitute 1):

$$\left.\begin{aligned}
T &= \alpha T_* \\
(kc)^{1/2} &= \beta (kc)_*^{1/2} \\
t &= \beta^2.\alpha^{-6}.t_* \\
\xi &= \beta\alpha^{-3}.\xi_* \\
A &= \alpha^4 A_* \\
F &= \alpha^4 F_*
\end{aligned}\right\} \quad (20)$$

in the equations, we see that the factors α and β cancel out and the equations are again satisfied in the starred quantities. It follows that T_* is the same function of ξ_* and t_* as T is of ξ and t. Hence, from one given solution a whole class of solutions can be derived by a simple change in the scale according to (20).

The equations are said 'to admit of a homology transformation', and the constants α and β are called 'homology constants'. As an example, consider the variation of the temperature in a semi-infinite solid, which initially is at a uniform temperature and that is instantly allowed to cool by radiation from its surface ($A = 0$), according to Stefan's law. For a particular value of the initial uniform temperature and of $(kc)^{1/2}$ the solution can be found by the numerical method described in the preceding sections. The solution corresponding to any other initial uniform temperature and substance can be obtained from this one without a fresh integration, by means of a homology transformation. If we confine ourselves to one and the same solid (same value of $(kc)^{1/2}$, hence $\beta = 1$), we see that the rate of cooling, e.g. of a semi-infinite solid initially at a uniform temperature a, is 10^{-6} times slower than that of the solid when initially at the uniform temperature $10\,a$.

In Figure 1 the variation of the surface temperature of a semi-infinite solid initially at the uniform temperature 370° K, with $(kc)^{1/2} = \cdot 0080$ is shown, the cooling having started at $t = 0$. The curve has been obtained by numerical integration in the manner described in section 6. The temperature 370° K is the temperature of the subsolar point adopted in this article (see section 9). The value $(kc)^{1/2} = \cdot 0080$ is

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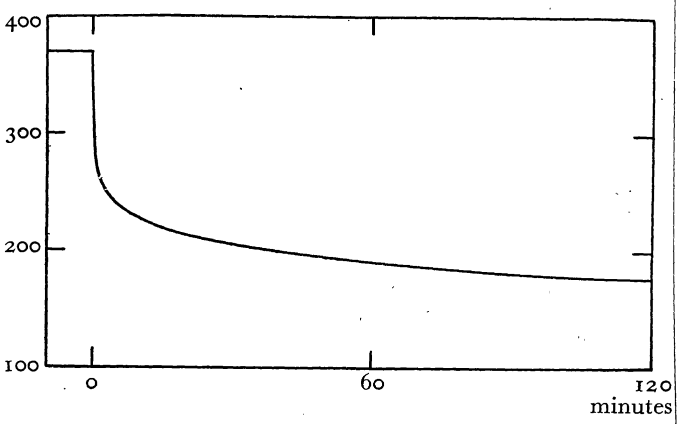
1) Compare S. CHANDRASEKHAR, *An Introduction to the Study of Stellar Structure*, Chicago 1939, p. 102.

FIGURE I



Calculated variation of surface temperature at subsolar point in the imaginary case of a sudden total eclipse, at $t = 0$.

the final result adopted for the lunar surface material in section 13. Though the temperature distribution at the subsolar point is not uniform, the inward decrease of the temperature is so slow that Figure 1 may be regarded to show the variation of the surface temperature there, if the solar radiation were suddenly cut off. We shall have occasion to use the curve of Figure 1 later on, when discussing the variations in the surface temperature during the total phase of an eclipse and during the lunar night.

9. *The data of observation.*

PETTIT and NICHOLSON [1]) determined the temperature of a small area near the border of the lunar disc during the eclipse of June 14, 1927. EPSTEIN [2])analysed these observations, and found $(kc)^{1/2} = \cdot 0083$. With an improved technique PETTIT [3]) observed the temperature of a small area near the centre of the lunar disc during the eclipse of October 28, 1939. These observations will be discussed in the present article. PETTIT and NICHOLSON also measured the variation of the temperature over the disc in a narrow band of constant declination at full moon. They measured the temperature of the subsolar point from its appearance on the western limb until its disappearance at the eastern limb. The same observers found the temperature of an area opposite the sun to be 120° K with an uncertainty of 13%, the recorded amount of radiation having a probable error of 50%. There does not yet exist a record of the temperature of a definite area throughout a lunation. Though, as a consequence of the extreme weakness of the moon's radiation during the lunar night, such an undertaking would be very difficult, it would undoubtedly be most valuable.

The temperature near the centre of the lunar disc at full moon, as found by PETTIT just before the beginning of the 1939 eclipse, was 370° K. We shall use this value throughout this article.

10. *Provisional estimate of $(kc)^{1/2}$ and first approximation to the periodic solution from the data during a lunation.*

The following, admittedly rough, treatment gives already considerable insight into the order of magnitude of the various quantities involved. The only data outside an eclipse of which we dispose are the temperature 370° K for the subsolar point and the result 120° K for the point opposite to this. Let us assume, as a first approximation, that the surface temperature during a lunation varies harmonically, and that the temperatures just mentioned are maximum and minimum values, respectively.

The theory as given in section 2 is then applicable. Neglecting the one eighth period difference in phase between F_0 and T_0, the semi-amplitude of F_0 is equal to the radiation corresponding to the minimum temperature 120° K; it is ·017 cal cm^{-2} min^{-1}. We then have by (7), since $b = \frac{1}{2}(370° - 120°) = 125°$,

$$125\,(kc\omega)^{1/2} = \cdot 017,$$

ω, corresponding to the synodic month, being equal to $1{\cdot}4776 \times 10^{-4}$ min^{-1}. Hence we find:

$$(kc)^{1/2} = \cdot 012 \text{ cal cm}^{-2} \text{ min}^{-1/2}\, (°\text{C})^{-1}.$$

Apart from the fact that we applied a rough theory, the main uncertainty in this provisional result is due to that of the semi-amplitude of F_0. We note that, within the margin of uncertainty, the present result agrees with EPSTEIN's, which was found from eclipse data.

At full moon the heat radiation is $\sigma(370)^4 = 1{\cdot}54$ cal cm^{-2} min^{-1}. F_0 is then ·017 cal cm^{-2} min^{-1} inwards. According to equation (2) we find provisionally $A_s = 1{\cdot}54 + \cdot 02 = 1{\cdot}56$ cal cm^{-2} min^{-1}. It thus appears that during most of the time that the surface receives solar heat F_0 is only a small fraction of A and $\sigma T_0{}^4$, which quantities are then nearly equal. Only near sunrise and sunset A, $\sigma T_0{}^4$ and F_0 become comparable. Throughout the lunar night $A = 0$ and $\sigma T_0{}^4 = F_0$, which is only one per cent of the values of $\sigma T_0{}^4$ and A at full moon near the centre of the disc.

The periodic solution can be calculated in terms of ξ and t from (5). In particular, the internal distribution of temperature at the subsolar point is in first approximation given by:

$$T(\xi) = 245° + 125°\, e^{-2\pi\xi} \cos 2\pi\xi. \qquad (21)$$

In Figure 2, (21) is shown graphically (upper curve). Though we need only A_s in our future derivations, it is of some interest to compare A_s with the incident energy. From the solar constant we find the amount of radiant energy falling on the subsolar point to be

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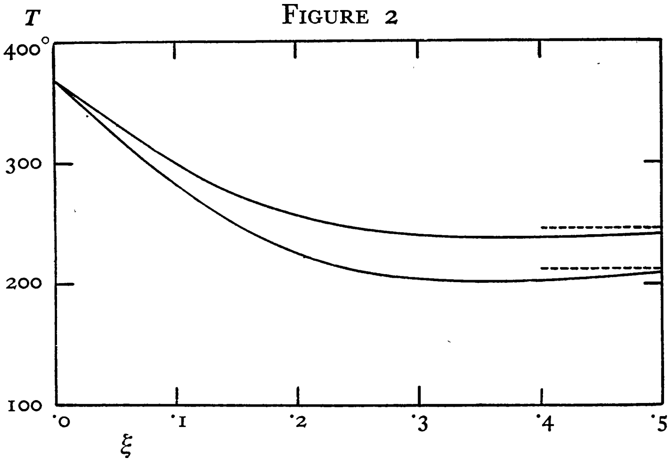
[1]) E. PETTIT and S. B. NICHOLSON, *Ap. J.* **71**, 102, 1930.

[2]) P. EPSTEIN, *Physical Review*, **33**, 269, 1929.

[3]) E. PETTIT, *Ap. J.* **91**, 408, 1940.

FIGURE 2



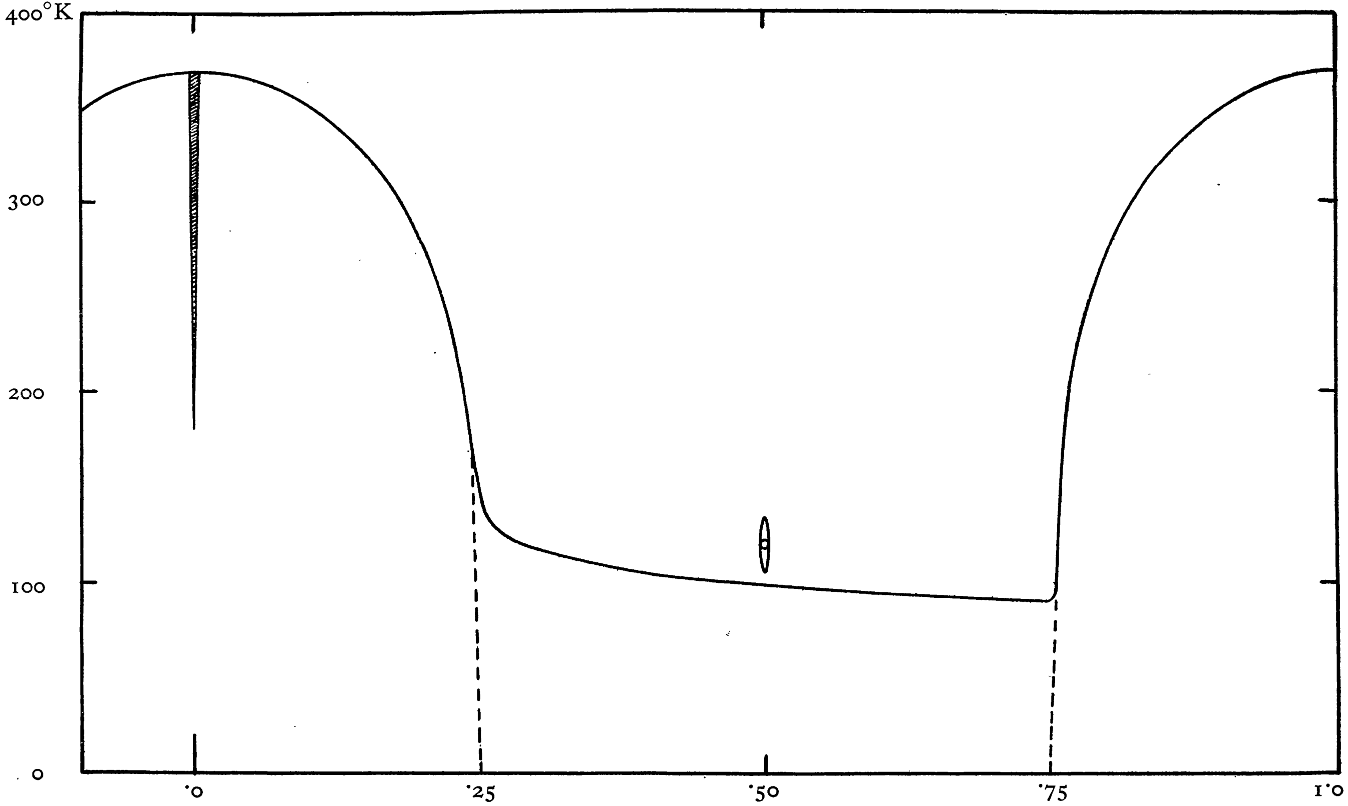
Temperature distributions as a function of ξ below the subsolar point.
Upper curve: first approximation.
Lower curve: second approximation.
Dotted lines show limiting values for large ξ.

1·90 cal cm^{-2} min^{-1}. Hence $A_s = 1{\cdot}56/1{\cdot}90 = 80$ per cent of the incident energy; 20 per cent is reflected. PETTIT obtains a reflected amount of 12 per cent from somewhat different data. It should be noted that this computation is rather uncertain; a change of one per cent in A_s causes a change of 5 per cent in the reflected amount. The situation is discussed in detail by PETTIT and NICHOLSON (*l.c.*).

11. *Determination of* $(kc)^{1/2}$ *from* PETTIT*'s temperature measurements made during the lunar eclipse of October* 28, 1939.

The numerical method of integration explained in section 5 for the non-periodic case has been applied. The initial distribution of temperature has been the provisional result (21) derived in the preceding section. t_0 corresponds to $4^{\mathrm{h}}04^{\mathrm{m}}$ G.M.T., which is a few minutes before the beginning of the partial phase. The surface temperature was taken from PETTIT's article (*l.c.*). We took $\xi_0 = 0$, $\Delta\xi = {\cdot}003$ and $\Delta t = 2{\cdot}407$ min. These values $\Delta\xi$ and Δt satisfy (16) so that (17) is valid. When the integration was completed, $\left(\frac{\partial T}{\partial \xi}\right)_0$ could be found as a function of the time. $F_0 = \sigma T_0^4$, during the total phase, has been plotted against $\left(\frac{\partial T}{\partial \xi}\right)_0$. The quantities proved to be reasonably well proportional to each other, as they should be according to (18), A being zero. From the factor of proportionality we found $(kc)^{1/2} = {\cdot}0084$, which is in per-

FIGURE 3



The calculated surface temperature of an area of the moon near the centre of the disc is shown during 1·1 lunation by the full-drawn curve. The dotted lines show the variation in the hypothetical case of zero conduction. The open circle represents PETTIT and NICHOLSON's observation during the lunar night. At full moon at left the variation during the 1939 eclipse is drawn on the same scale.

fect agreement with EPSTEIN's result $(kc)^{1/2} = \cdot 0083$ that he found from the lunar eclipse of June 14, 1927.

It might seem as if the result just derived is rather influenced by the uncertainty in the initial distribution of temperature, which had been derived from the uncertain data during a lunation by a rough theory. The eclipse phenomenon, however, is of relatively short duration as compared with a synodic month (compare Figure 3) and therefore penetrates relatively little into the moon. At full moon, in the absence of an eclipse, the inward decrease of temperature is very slight in the small interval of ξ, where during an eclipse the temperature undergoes noteworthy variation. It is therefore of little importance what initial temperature distribution is taken in the present solution, provided its decrease inwards is sufficiently slow. In fact, the result found above for $(kc)^{1/2}$ might have been obtained as well if we had started with a uniform initial distribution of temperature just before the beginning of the partial phase (370° K). The inclusion of the partial phases in the plot of F_0 against $\left(\frac{\partial T}{\partial \xi}\right)_0$ would add but little weight to the determination of $(kc)^{1/2}$. Outside totality F_0 and $\left(\frac{\partial T}{\partial \xi}\right)_0$ are smaller and F_0 has an absolute error that is larger than during the total phase.

$(kc)^{1/2}$ is found more accurately from eclipse observations than from observations made during a lunation, F_0 during total eclipse being about 11 times larger than during the lunar night, during which latter interval the heat radiation is at the limit of measurement.

We are now able to find the definitive value of A_s. Substitution of $(kc)^{1/2} = \cdot 0084$ in (7) with $b = 125°$, yields: semi-amplitude of F_0 during a lunation $= \cdot 01$ cal cm^{-2} min^{-1}.

Equation (2) gives $A_s = 1{\cdot}54 + \cdot 01 = 1{\cdot}55$ cal cm^{-2} min^{-1}, which differs but little from our provisional result 1·56, as could have been expected.

12. *Calculation of the variation of the surface temperature near the centre of the lunar disc during a lunation, and second approximation to the periodic solution.*

This problem has been treated in a general way in section 7.

The question which function $A(t)$ has to be used needs some consideration. If the lunar surface were smooth and if the amount of radiation absorbed were a constant fraction of the amount of energy striking a square cm of the surface, whatever the angle of incidence, A at a certain area would simply be $A_s \cos z$, where z is the selenographical zenith distance of the sun. If the small amount of heat conducted is neglected ($F_0 = 0$), the temperature at any point would simply follow from $\sigma T_0^{\,4} = A_s \cos z$ according to (2). Two deviations of this simple state of affairs have been noted by PETTIT and NICHOLSON (*l.c.*). In the first place they found the heat radiation at full moon to vary over the disc not like $\cos z$, but rather as $\cos^{2/3} z$, so that the lunar surface at any point is hotter than according to the simple cosine law [1]).

Secondly, they found the temperature of the subsolar point to be lower at quarter phase than at full moon, so that the temperature of an area has been found to depend on the direction from which it is viewed. PETTIT and NICHOLSON were able to explain these features of the lunar radiation by the roughness of the surface. Though we are aware of the approximate character of the cosine law, we believe that its use in the following calculation will give results that are nearer to the truth than the first approximation given in section (10).

Hence we took

$A = 1{\cdot}55 \cos \omega t$ during the lunar day,
$A = 0$ during the lunar night,
$t = 0$ at full moon.

We consider an area near the centre of the disc, and suppose the sun to pass through its zenith once in a synodic month. $(kc)^{1/2} = \cdot 0084$, the result obtained in section 11. As initial temperature distribution at $t_0 = 0$ (full moon) the first approximation to this function given by (21) and shown graphically in Figure 2, was taken. We choose $\Delta t = {}^1/_{60}$ synodic month and $\Delta\xi = \cdot 0515$; Δt and $\Delta\xi$ so chosen satisfy (16), so that (17) may be used. The numerical form of (19) used has been:

$$\sigma T_0^{\,4} = A + \cdot 0004477\,(T_{+\frac{1}{2}\Delta\xi} - T_0).$$

When the integration had been completed over a single period, we calculated a second approximation to the periodic solution by (9).

The corresponding distribution at full moon (shown in Figure 2) was then used as initial distribution for a second integration. The surface conditions and the value of $(kc)^{1/2}$ were the same as in the first integration. The second integration was continued for one complete period and the resulting surface variation was found practically identical with the result from the first integration. Both the periodic variation at the surface and the periodic solution obtained in this second integration may therefore be considered as final. The time average of the surface variation and hence also the constant temperature at large depths is 212° K.

The full-drawn curve in Figure 3 shows the periodic surface variation over 1·1 lunation. At phase ·0 (full

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1) In *Ap. J.* **71** on pages 102 and 118 the heat radiation is written $E = a \cos^{3/2} \theta$, θ corresponding to what I have called z. The exponent $3/2$ is not in accordance with the data as shown for instance in Figure 4, page 118, nor does it agree with the qualitative explanation of the effect that the authors give. There is little doubt, therefore, that the form $E = a \cos^{2/3} \theta$ is meant.

moon) we have inserted the variation during the 1939 eclipse on the same scale for comparison. The dotted line in Figure 3 shows the variation of the surface temperature in the hypothetical case of a complete absence of conduction ($k = 0$). This curve has been calculated from equation (2) with $F_0 = 0$, thus from the formula $\sigma T_0{}^4 = A$.

The curve drops to zero at sunset (phase ·25) and rises again at sunrise (phase ·75). It is seen that the curve is undistinguishable from the full-drawn curve almost during the complete lunar day. The curves separate and join respectively a short time before sunset and after sunrise. We may say with fair accuracy that the surface temperature depends on the incident radiation and is independent of kc and the internal temperature distribution during the lunar day, whereas it depends on kc and the internal temperature during the lunar night. The situation is shown mathematically in equation (2), in which F_0 may be neglected during the lunar day, but which reads $\sigma T_0{}^4 = F_0$ during the lunar night. As moments where the two cases pass into each other we may take the times at which $A = F_0$. This happens $2\frac{1}{2}$ hours = ·003 lunation before sunset and $\frac{1}{2}$ hour = ·0006 lunation after sunrise. This remark is of course only approximate, since a complete rising or setting of the sun takes ·0014 lunation, whereas the surface condition $A = 1{\cdot}55 \cos \omega t$ assumes a negligible angular size of the sun. Moreover the effect of the rough surface is most pronounced at these instants.

During the lunar night the temperature falls from 144° K to 90° K, the temperature at the middle being 98° K. This moderate decline over so long an interval as a fortnight is due to the low temperature, the loss of heat by radiation being very small. It is determined by the internal temperature distribution at sunset. For, suppose the temperature gradient were constant at sunset and equal to the actual value at the surface. The surface temperature in that case would remain constant (144° K) for an indefinite interval of time. As an alternative extreme we may consider the case that the temperature at sunset were uniform and equal to 144° K. The variation in surface temperature corresponding to this case is found by applying a homology transformation on the curve obtained in section 8 (Figure 1).

The interval in time in Figure 1 corresponding to 14·7 days equals $\left(\frac{144}{370}\right)^6 \times 14{\cdot}7 \text{ days} = 78 \text{ minutes}$. In this time interval in Figure 1 the temperature drops from 370° K to 185° K or to half the initial value. According to our assumption the temperature just before sunrise would be: $\frac{1}{2} \times 144° \text{K} = 72° \text{K}$.

The actual internal temperature at sunset rises with ξ whereas the gradient decreases; it is therefore intermediate between the simple cases, and so is the actual temperature (90° K) before sunrise.

13. *Theoretical determination of the variation of the surface temperature during the eclipse of the moon of October 28, 1939 for the area near the centre of the disc observed by* Pettit.

The small area of which Pettit measured the variation in temperature is ·17 lunar radii north of the centre of the disc. In section 11 these measures were used for the derivation of $(kc)^{1/2}$. Then, we did not use $A(t)$ during the partial phases. We now want to show that the complete curve can be calculated from $A(t)$ and the value of $(kc)^{1/2}$ then found. During the eclipse there is no appreciable change in direction neither of the solar rays nor in the line of sight relative to the area. The complications discussed in the beginning of section (12) therefore do not arise and A can be taken to be proportional to the incident energy. We have calculated the fraction α of the sun's radiant energy that is taken away by the earth for the area in question, as a function of the time. The darkening towards the limb of the integrated radiation was taken into account with a coefficient of darkening in the usual formula for limb-darkening, equal to ·6 [1]). It is easily verified that with this value for the coefficient of limb darkening we simply have:

$$\alpha = \tfrac{1}{2}\,(\alpha_U + \alpha_D)\ ^{2)}$$

where α_U is the fraction of the area of the solar disc that is obscured, while α_D is the fraction of the light that would have been taken away, if the coefficient of darkening were 1. α_U and α_D were taken from existing tables originally designed for the derivation of the orbital elements of eclipsing binaries [3]).

The apparent radii of sun and earth as seen from the area on the moon were °·2677 and °·9313 respectively. The variation of the angular distance between the centres of sun and earth was derived from the positions of sun and moon as given in the *Nautical Almanac* for 1939. $A(t) = 1{\cdot}55\,(1-\alpha)$ has been given in the 4th column of Table 1. The initial temperature distribution has been the second approximation to this function at the subsolar point as obtained in section 12 (Figure 2). We took $(kc)^{1/2} = {\cdot}008$; $\Delta\xi = {\cdot}003$, $\Delta t = 2{\cdot}407$ minutes. $\Delta\xi$ and Δt satisfy (16) and hence (17) is valid. $\xi_0 = -\frac{1}{2}\Delta\xi$, $t_0 = 4^{h}04^{m}$ G.M.T. The numerical form of (19) is $\sigma T_0{}^4 = A + {\cdot}007319\,(T_{+\frac{1}{2}\Delta\xi} - T_0)$. The integration at the surface has been explained in

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[1]) Abbot, Fowle and Aldrich, *Annals Astroph. Obs. Smithsonian Inst.* **111**. E. A. Milne, *Handbuch der Aph.*, Bd. III, 143. M. Minnaert, *B.A.N.* **2**, 78.

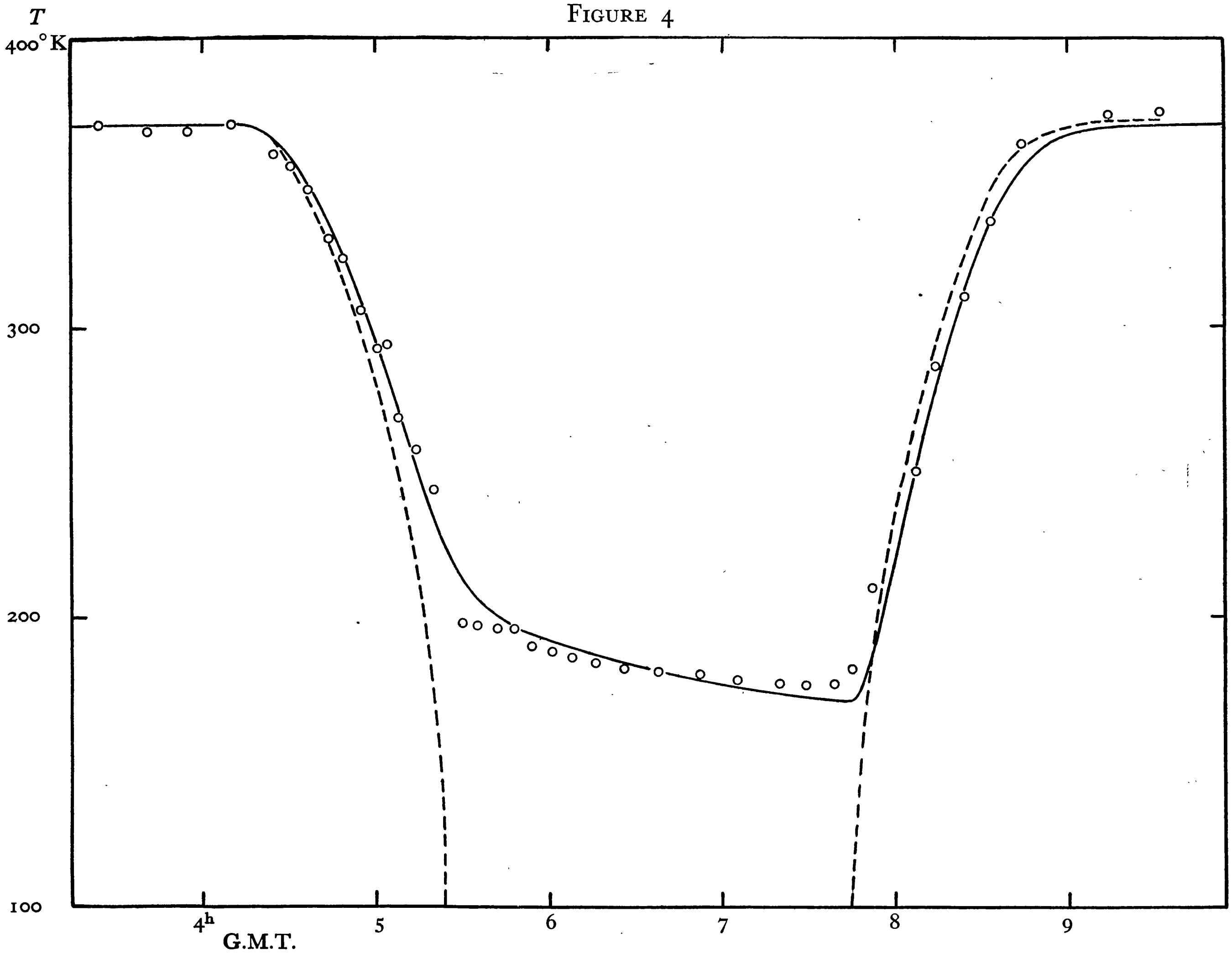
[2]) H. N. Russell, *Ap. J.* **36**, p. 70, 240.
A. Pannekoek and Elsa van Dien, *B.A.N.* **8**, 142.

[3]) E. Hetzer, *Beitrag zu H. N.* Russell's *Methode.* Diss. Leipzig (1931).
Zessewitsch, *Poulkovo Circ.* No. 24, 41, 1938.

TABLE 1

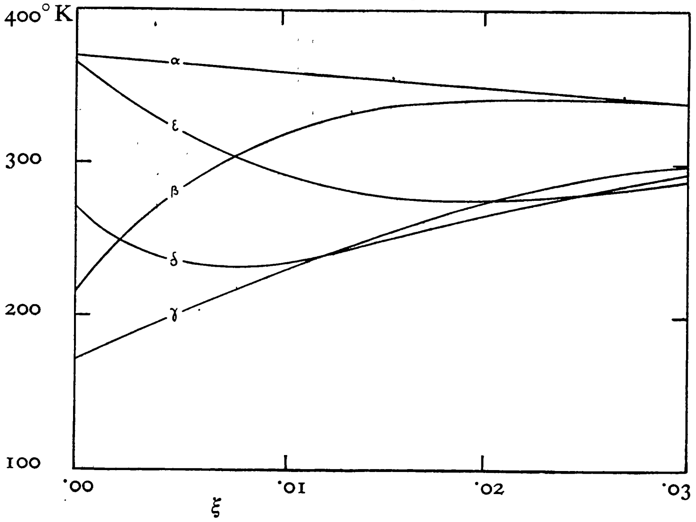
G.M.T.	α_U	α	A	observed T_0	$(kc)^{1/2} = {\cdot}008$ T_0'	$(kc)^{1/2} = {\cdot}007$ T_0''	$(kc)^{1/2} = {\cdot}000$ T_0'''
h m							
3 24	·000	·000	1·550	370	370	370	371
41	·000	·000	1·550	368	370	370	371
55	·000	·000	1·550	368	370	370	371
4 10	·000	·000	1·550	370	370	370	371
25	·090	·076	1·432	360	364	364	363
31	·168	·150	1·317	356	357	357	356
37	·248	·232	1·190	348	349	349	347
44	·372	·364	·986	331	336	336	331
49	·460	·460	·837	324	324	324	318
55	·561	·579	·653	306	309	309	299
5 01	·663	·682	·493	293	294	294	278
04	·716	·738	·406	294	285	285	265
08	·781	·804	·304	269	272	270	247
14	·867	·887	·175	258	254	250	215
20	·943	·953	·073	244	235	229	173
30	1·000	1·000	·000	198	213	205	0
35	1·000	1·000	·000	197	207	200	0
42	1·000	1·000	·000	194	201	195	0
48	1·000	1·000	·000	194	197	192	0
54	1·000	1·000	·000	190	194	190	0
h m							
6 01	1·000	1·000	·000	188	192	188	0
08	1·000	1·000	·000	186	189	184	0
16	1·000	1·000	·000	184	187	182	0
26	1·000	1·000	·000	182	184	179	0
38	1·000	1·000	·000	181	182	177	0
52	1·000	1·000	·000	180	179	174	0
7 05	1·000	1·000	·000	178	177	172	0
20	1·000	1·000	·000	177	174	169	0
29	1·000	1·000	·000	176	173	168	0
39	1·000	1·000	·000	177	171	167	0
45	·998	·999	·017	182	171	166	119
52	·921	·935	·101	210	189	185	188
8 07	·698	·720	·434	250	248	252	270
14	·581	·594	·629	286	279	281	296
23	·424	·421	·897	310	310	311	323
33	·258	·242	1·175	336	336	336	346
44	·103	·088	1·414	363	354	354	362
9 14	·000	·000	1·550	373	369	369	371
32	·000	·000	1·550	374	369	369	371

FIGURE 4



Temperature near centre of lunar disc during eclipse of October 28, 1939. Dots: observed temperatures. Full-drawn curve: theoretical variation (T_0') for $(kc)^{1/2} = {\cdot}0080$ cal cm^{-2} min$^{-1/2}$ (°C)$^{-1}$. Dotted line: T_0''', variation in the absence of conduction ($k = 0$).

FIGURE 5



Temperature distributions during the eclipse of October 28, 1939.
(α) at full moon, before eclipse.
(β) $5^{h}28^{m}$ G.M.T., at beginning of total phase.
(γ) 7 45 G.M.T., at end of total phase.
(δ) 8 12 G.M.T.
(ε) 8 55 G.M.T., at end of eclipse.

section 6. The agreement between theory (T_{0}') and observation may be judged from the data in Table 1 and from Figure 4 and is seen to be satisfactory in general. In Figure 5 the internal distribution of temperature at some selected instants is shown. The computed slope during totality (Figure 4) is slightly steeper than observed. It was therefore tried to obtain a better fit by varying $(kc)^{1/2}$. A new solution was made, which differed from the preceding one only in the value for $(kc)^{1/2}$, this being now ·007. Though the agreement with the observed temperatures was somewhat better near the beginning of the total phase, it became worse near the end, whereas the slope during totality had hardly changed. The computed values T_{0}'' of this latter solution are given in Table 1. We adopt $(kc)^{1/2} = {}$·0080 as our final result.

The dotted line in Figure 4 shows the variation in temperature (T_{0}''') in the hypothetical case $kc = 0$. It has been calculated from the formula $\sigma T_{0}^{4} = A$, which did not require a fresh integration. T_{0}''' is zero during totality and deviates but little from T_{0}' during the partial phases.

During the partial phases T_{0}' is generally later than the same value T_{0}'''. This time lag of T_{0}' with respect to T_{0}''' and the radiation is well shown by the observations. We found F_{0} during the lunar day negligibly small compared with A and σT_{0}^{4}. During the partial phases of the eclipse F_{0} is relatively small but not negligible, the curves T_{0}' and T_{0}''' being clearly separated. At the same value of A, F_{0} during an eclipse is more than 10 times larger than during a lunation. As in our discussion of the lunation, we have calculated for the eclipse the times at which $A = F_{0}$. We found this to happen 10 minutes before the beginning of totality and $2\frac{1}{2}$ minutes after the end. Between these moments F_{0} preponderates over A, whereas outside this interval A is larger than F_{0}. The relatively small decline of the temperature during totality has been checked similarly as in section 12 for the lunar night, by the aid of Figure 1.

Pettit has remarked that during the partial phase the energy radiated (σT_{0}^{4}) is nearly proportional to the amount of radiant energy received. This is a consequence of the relative smallness of F_{0} as compared with A and σT_{0}^{4}. If F_{0} were zero, the proportionality just mentioned would be exact, since A is a constant fraction of the incident energy. The fact shows moreover the accuracy of the observations and the reliability of the involved reduction from galvanometer reading to temperature.

The remark made in section 11 is of course again valid for the computation of T_{0}' described in this section: the initial temperature distribution is of very little consequence for the resulting computed surface variation, provided its decrease inwards is sufficiently slow. A uniform initial temperature distribution of 370° K would have led to practically the same results at the surface.

14. In this section we want to compare the solution for the eclipse discussed in section 13 with that for a lunation treated in section 12 from the standpoint of homology. It will be shown that the relationship between these solutions near the surface may be expressed concisely. We use the notation of section 8. Let starred quantities refer to the eclipse, whereas unstarred values shall indicate quantities during a lunation. It is easy to see that the solutions do not belong to the same homologous family.

A during a lunation is approximately similar to A during the eclipse, the ranges being equal (hence $\alpha = 1$) whereas the time scales are in the ratio 140. Hence $\beta^{2}\alpha^{-6}$ should be 140, whereas, since kc is the same in both cases ($\beta = 1$), $\beta^{2}\alpha^{-6} = 1$. Consequently the solutions are not homologous.

Let us call I and II, respectively, the solutions for eclipse and lunation with $(kc)^{1/2} = {}$·0080, which differ from those discussed in sections 12 and 13 in that the initial temperatures are taken uniform and equal to 370° K.

We have already remarked that the surface temperature variation in the eclipse solution is little dependent on the initial distribution of temperature, provided its decrease with ξ inwards is sufficiently slow. The same is true approximately for the surface temperature variation during a lunation and therefore we shall neglect the differences of the actual variations in the surface temperature with those derived from I and II. Consider an eclipse solution I′, which differs from I in that the value of $(kc)^{1/2}$ is taken $\sqrt{140}$ or nearly 12 times smaller. The relations (20) are seen to be satisfied for I′ and II. Hence the variation in surface temperature of I′ is like that of II on a 140 times smaller time scale. Similarly we can consider a solution II′, which differs from II in the value of $(kc)^{1/2}$, being $\sqrt{140}$ or nearly 12 times larger than the value $(kc)^{1/2} = \cdot 0080$ employed there. The relations (20) are again satisfied for II′ and I. Hence the variation in surface temperature of II′ is like that of I, except for the time scale, which is 140 times larger.

Hence approximately, the effect of multiplying (kc) with a factor p has the same effect on the surface temperature as the multiplication of the time scale with p^{-1}. This is a concise way of comparing the surface variations at an eclipse and during a lunation as found in sections 12 and 13, respectively.

15. The specific heats per gram of minerals are not widely different, a good average value being ·20 [1]. The average density of the moon is 3·33 [2]. The density of the surface material is unknown but is likely to be somewhat smaller than this average value. We tentatively adopt for the surface density $\rho = 2\cdot 0$. The specific heat per cm³ is then $c = 2 \times \cdot 20 = \cdot 40$.

Combining this value of c with the result $(kc)^{1/2} = \cdot 0080$, derived in preceding sections, we find

$$k = 16 \times 10^{-5} \text{ cal cm}^{-1} \text{ min}^{-1} (^\circ\text{C})^{-1}. \qquad (22)$$

The wave length of a harmonic heat wave with period equal to the synodic month can now be found by (6). We find $l = 14\cdot 5$ cm. This result shows how little the variations in temperature penetrate below the surface. The range in ξ in Figure 2 is half a wave length or only 7·2 cm. The total range in abscissa of Figure 5 (·03 in ξ) is only 4·4 mm.

16. We have seen that the behaviour of the lunar surface temperature throughout an eclipse and during a lunation can be understood on the theory of heat conduction (with constant values k and c), if $(kc)^{1/2} = \cdot 0080$ cal cm^{-2} min$^{-1/2}$ $(^\circ\text{C})^{-1}$. With the plausible value $c = \cdot 40$ we found $k = 16 \times 10^{-5}$ cal cm^{-1} min^{-1} $(^\circ\text{C})^{-1}$.

We have yet to investigate whether this value is in agreement with what is known about terrestrial substances. The smallest value [3] of k which I could find in an extensive list [4] of poor conductors of heat, containing data on 140 substances, is ·0044 cal cm^{-1} min^{-1} $(^\circ\text{C})^{-1}$. This is also about the lowest result occurring in the *Dictionary of Applied Physics* [5].

The results of Epstein and the writer for the moon are 28 times smaller than the laboratory results just mentioned. I do not think that the discrepancy can be removed by taking a much smaller value for the adopted density $\rho = 2\cdot 0$, used in the estimate of c; the volume of the interstices in terrestrial minerals never exceeds 50 percent of the total volume [1]. Pumice stone [2] is occasionally mentioned as furnishing an adequate explanation. It has the advantage of combining a low conductivity of heat with a low specific weight. If, as is done in litterature, the value $k = \cdot 001$ for pumice stone is combined with $\rho = \cdot 3$ and a specific heat per gram of ·20, we obtain a value $kc = 60 \times 10^{-6}$, which is in apparent agreement with the result found for the moon, 66×10^{-6} cal² cm^{-4} min^{-1} $(^\circ\text{C})^{-2}$. The agreement is however spurious since the unit for k in the value ·001 is different from that used for k in Epstein's and the writer's result for kc given above.

We find [3] for pumice stone a heat conductivity of $\cdot 92 \times 10^{-3}$ joule cm^{-1} sec^{-1} $(^\circ\text{C})^{-1}$. This is in numerical agreement with the value ·001 mentioned above, so that apparently there the unit is the same as that employed in the *I.C.T.*

Now 1 joule sec^{-1} = 14·34 cal min^{-1}. Hence k (pumice stone) in the units of this article is $14\cdot 34 \times \cdot 00092 = \cdot 0133$. With the same values for density (·3) and specific heat per gram (·20) as used above, we obtain a value kc for pumice stone that is 12 times the result for the moon. We therefore do not think that pumice stone furnishes an adequate solution for the low conductivity found for the lunar surface material.

17. M. Smoluchowski determined [4] the heat conductivity of a number of powders, the sizes of the grains of which varied from ·003 to ·26 mm. The average temperature during the experiment was about 45° C. The pressures of the gases in the interstices varied from very low values to one atmosphere. His results may be summarized as follows:

1. The heat conductivity depends on the nature of the powder and on that of the gas in the interstices.

2. For a given powder and gas it varies with the pressure of the gas so that the heat conductivity is smaller the lower the pressure.

3. In vacuum the heat conductivity is not zero, but is generally extremely small, viz., of the order of ·01 of the value at atmospheric pressure. In vacuum there remains the transport of heat through the contact places of the grains, through the grains themselves by ordinary conduction, and through the interstices by radiation.

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[1] B. Gutenberg, *Handbuch der Geophysik*, Bd II, 1, pag. 10.

[2] Russell, Dugan and Stewart, *Astronomy* I, (1st ed.), p. 168.

[3] This result was found for a sample of highly porous ebonite.

[4] H. J. Hamaker, *Technische warmtegeleidingsmetingen*, Thesis Utrecht 1939.

[5] Vol. I, p. 433; the article is by F. H. Schofield.

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[1] H. Reich, *Hdb. der Geophysik*, Bd. **6**, 1, p. 16.

[2] *Ap. J.* **71**, 129; *Phys. Rev.* **33**, 269; Russell, Dugan and Stewart I, 175.

[3] *International Critical Tables*, Vol. II, 313.

[4] *Bull. Acad. d. Sci.*, Cracovie, A, 1910, p. 129 and A 1911, p. 548. Compare also *International Critical Tables* vol. II, p. 315.

SMOLUCHOWSKI's results for vacuum do not show much variation. They may be summarized concisely by their average value and their dispersion. We find from the data on all 15 powders:

$$k = (20 \pm 11) \times 10^{-5} \text{ cal cm}^{-1} \text{ min}^{-1} \text{ }(^\circ\text{C})^{-1}. \qquad (23)$$

The contribution by the radiation to the conductivity of a powder can be shown to be proportional to the sizes of the grains and to the third power of the absolute temperature (compare section 18).

At the temperature of SMOLUCHOWSKI's experiments there are four powders, having the largest grains, for which the contribution to k by radiation is not negligible (up to 50%). For the remaining eleven powders with grain size smaller than ·1 mm, we find

$$k = (16 \pm 8) \times 10^{-5} \text{ cal cm}^{-1} \text{ min}^{-1} \text{ }(^\circ\text{C})^{-1}. \qquad (24)$$

The values (23) and (24) are in excellent agreement with EPSTEIN's and the writer's results for the moon (22). We may say that PETTIT and NICHOLSON's lunar temperature measurements afford an independent proof for the absence of an atmosphere on the moon.

If the grains of the powder on the lunar surface are smaller than for instance ·1 mm, the dependence on temperature of k will be small. The specific heats of many substances are known to vary with temperature, especially so at temperatures near absolute zero. The temperatures observed on the moon are not so extreme that a material change in c can be expected.

It follows that our analysis, which assumes constant values for k and c, is valid and gives a quantitative explanation for the observed variations of temperature on the lunar surface if the surface layers consist of a powder with grains smaller than ·1 mm.

It should be noted that our conclusion with regard to the powdery structure of the lunar surface has been reached before in an independent way from the observed phase function of the reflected light 1).

18. We have seen that powders, like those studied by SMOLUCHOWSKI in the laboratory, and in which the transfer of heat by radiation can be shown to be negligible, give an adequate explanation for the variations of the lunar surface temperature.

On the other hand it is instructive to study an imaginary powder in which the conduction through the contact places is negligible and the radiative transfer prominent. We want to show that in this way an upper limit to the dimensions of grains and interstices of the powder on the lunar surface may be derived.

Since the radiative "coefficient of heat conductivity" varies as the third power of the absolute temperature, the theory hitherto used, which assumes constant k, is not applicable. We proceed to derive a new set of equations in a form which takes the variation of k with T into account.

Instead of equation (1) we have generally for variable k:

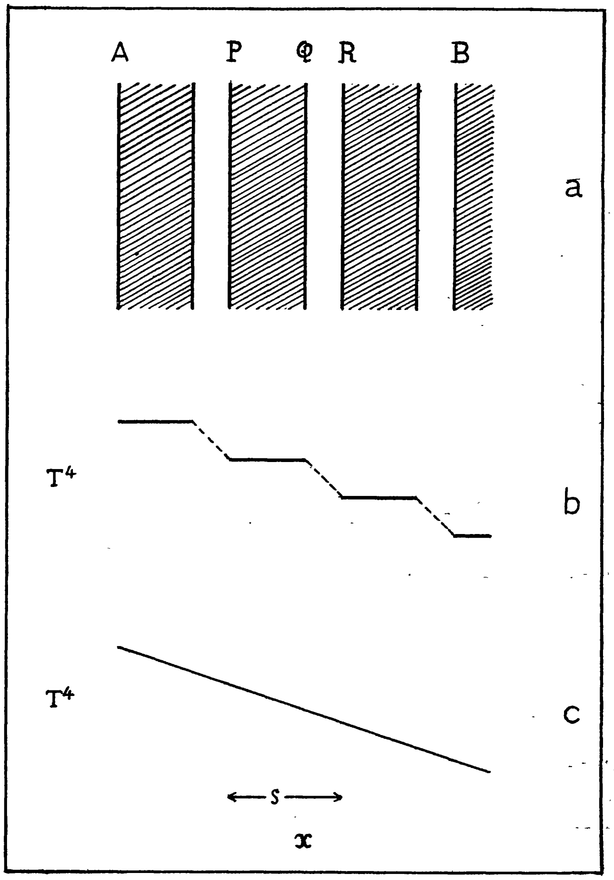
$$c\frac{\partial T}{\partial t}=\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right), \qquad (25)$$

whereas (2) and (3) are not altered.

In order to proceed, it seems necessary to simplify the true picture of grains and interstices of a powder in a drastic manner. We do not think that the order of magnitude of the dimensions of grains and interstices subsequently found for the model will be widely different from that in the true picture.

We imagine the lunar surface material stratified into horizontal parallel and solid slabs of equal thickness. Let they be separated by empty spaces, bounded by the parallel and plane surfaces of adjacent slabs, so that their distances are all equal. Denote the sum of the thicknesses of one slab and one space by s.

FIGURE 6



a). cross-section through model by plane perpendicular to the surface.

b). schematic run of T^4 with x.

c). smoothed run of T^4 with x.

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1) ROUGIER, *Ann. Strasbourg* **2**, fasc. 3, p. 328, 1933.
LYOT, *Ann. Paris-Meudon* **8**, fasc. 1.

A cross-section of this model by a plane normal to the surface is shown in Figure 6*a*. In the model s should correspond to the sum of the average dimensions of grain and interstice. Like in the actual powder, the heat energy in the model traverses alternatively matter and space.

Let the temperatures on the faces A and B be kept constant. When the temperatures have become stationary everywhere, there will be a constant flow of heat through any plane parallel to the surface, from A to B. In the solid the conductivity is relatively high, and the temperature gradient consequently relatively small. A finite drop in T^4 occurs at each space. The course of T^4 with depth is sketched in Figure 6*b*.

We have:

$$F = \sigma\,(T_Q{}^4 - T_R{}^4).$$

We neglect the small difference between T_P and T_Q, then:

$$F = \sigma\,(T_P{}^4 - T_R{}^4)$$

or

$$F = \sigma \frac{\partial T^4}{\partial x} s = 4 \sigma s\, T^3 \frac{\partial T}{\partial x}. \qquad (26)$$

In (26), $\partial T / \partial x$ stands for the average gradient from A to B. Comparison of (26) with (3) gives:

$$k = 4 \sigma s\, T^3. \qquad (27)$$

The equations which replace (1), (2) and (3) may be written:

$$c \frac{\partial T}{\partial t} = \frac{\partial F}{\partial x} \qquad (28)$$

$$\sigma\, T_0{}^4 = A + F_0 \qquad (29)$$

$$F = 4 \sigma s\, T^3 \frac{\partial T}{\partial x}. \qquad (30)$$

We now introduce new variables: z and η by the relations:

$$z = \sigma\, T^4$$

$$\eta = \tfrac{1}{2} \sigma^{-1/8} \left(\frac{c}{s}\right)^{1/2} \times x.$$

We then find for (28), (29) and (30) the equivalent set:

$$\frac{\partial z}{\partial t} = z^{3/4} \frac{\partial^2 z}{\partial \eta^2} \qquad (31)$$

$$z_0 = A + F_0 \qquad (32)$$

$$F = B \frac{\partial z}{\partial \eta} \qquad (33)$$

$$\text{where } B = \frac{(sc)^{1/2}}{2\sigma^{1/8}} \qquad (34)$$

Equation (31) was then integrated numerically with the following boundary conditions:

1. $z(\eta, 0) = A_s = 1·55$

2. $z(0, t)$ is the observed variation of the heat radiation during the eclipse of October 28, 1939.

When the solution had been completed, $(\partial z / \partial \eta)_0$ could be found from the known solution $z\ (\eta,\ t)$. During the total phase z_0 and $(\partial z / \partial \eta)_0$ were found reasonably well proportional, as they should be according to (32) and (33), since then $A = 0$. From the factor of proportionality we found:

$B = ·956$,

and from (34), with $c = ·40$,

we obtain:

$$s = ·028 \text{ cm} = ·28 \text{ mm}.$$

So ·3 mm is an upper limit to the size of the grains of the lunar surface powder.

It is a pleasure to express my gratitude to Professor Oort for his interest in this article. I am much indebted to Miss H. A. Kluyver for her assistance in looking up litterature inaccessible to me.

When this article had been written, my attention was called to a paper by W. G. Kannuluik and L. H. Martin 1) in which accurate results are given of k for powders at various gas pressures. Their result for the order of magnitude of k in vacuo is 60×10^{-5} cal cm^{-1} min^{-1} (°C)$^{-1}$, which is rather larger than Smoluchowski's result for powders of similar grain size. Kannuluik and Martin's extrapolation to vacuum, however, must be considered with reserve, their lowest pressure being as high as 3·5 mm Hg, whereas Smoluchowski's extreme lowest pressure is ·05 mm Hg. Additional measures on Kannuluik and Martin's method at pressures as low as a few hundredths of a mm Hg would be most valuable.

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1) *Proceedings of the Royal Society*, A **141**, 144, 1933.