SCATTERING IN A PLANETARY ATMOSPHERE

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ABSTRACT

In this paper the problem of multiple scattering in a planetary atmosphere, both with and without a diffusely reflecting bottom surface, is discussed. We assume that the atmospheric scattering is isotropic, with an albedo a, and that the ground surface reflects the radiation according to Lambert's law, with an albedo b.

Sections II–IV are devoted to the *standard problem*, i.e., to the problem of a finite atmosphere without a ground surface. After the exact expressions for single and double scattering have been obtained in Section II by direct integration, we show in Section III that an indirect method gives the same result in a more elegant way. This method is based on the use of Chandrasekhar's equations by which the intensities of the reflected and transmitted radiation can be expressed in terms of two functions, $X(s)$ and $Y(s)$. A new feature in the present discussion is that physical meanings are assigned to these functions. With these meanings the general relations are newly derived; and the expressions for single and double scattering are obtained in a simple manner. Section IV completes these calculations by adding an approximate expression for the third-order scattering. A numerical example shows that in this way an accuracy of 0.1 per cent is reached for an atmosphere with $\tau = 0.1$.

Section V discusses the *planet problem*, i.e., the problem of an atmosphere with a bottom surface. The solution of this problem is expressed in a general way in terms of the solution of the standard problem. The exact solution for the case of an isotropically scattering atmosphere involves the functions $X(s)$ and $Y(s)$ and their moments. Formulae suitable for the case of a thin atmosphere are also given and illustrated by a numerical example.

A comprehensive discussion of the *mathematical functions* occurring in these calculations is given in the appendix. They are of three types: The functions $E_n(x)$ are the ordinary exponential integrals. The functions $F_n(b, x)$ are definite integrals, of which the integrand is the product of an E-function and an exponential function. The functions $G_{nm}(x)$ and $G'_{nm}(x)$ are definite integrals, of which the integrand is the product of two E-functions. Only the last type of functions cannot be expressed in terms of known functions. For their calculation a new elementary function, the exponential integral of the second order, is introduced. Numerical values of all functions needed are given in four tables.

Quantitative knowledge about the composition and depth of planetary atmospheres can be gained only from photometric observations. The term "photometry" has here to be taken in its most general sense: the intensity being specified according to the position on the disk of the planet, the wave length, and the state of polarization. The atmosphere of the earth can be investigated by other than optical methods, but general photometry of the sky light still provides an important source of information.

The interpretation of any observation in this general field has to deal with scattering problems. The most well known of these problems is that of explaining the brightness and polarization of the blue daylight sky. L. V. King[1] and others have made approximate calculations on this subject. Much attention has further been given to the limb darkening of the outer planets, in particular by E. Schoenberg[2] and V. C. Fessenkoff.[3] Other allied problems are those of the polarization of the planets, the escape of the heat radiation of the planet through its atmosphere, and the formation of absorption lines in the process of diffuse reflection by the planetary atmospheres. Finally, the theory of reflection nebulae and undoubtedly various problems occurring in physical experiments may also be treated by the same formalism.

The analysis of the problems that we have mentioned presents difficulties of two sorts. First, the formulae describing the *elementary processes* of scattering by a single air mole-

[1] *Phil. Trans. R. Soc. London, A*, **212**, 375, 1913.

[2] *Handb. d. Ap.*, **2**, Part I, 221, 1929.

[3] *Astr. J. Sovjet Union*, **21**, 257, 1944.

cule, or fog droplet, and of diffuse reflection by an area of the ground may already have a fairly complex form. Second, the *multiple scattering* in the atmosphere, combined with the diffuse reflection by the ground, introduces further mathematical difficulties of an essentially different nature.

In the present paper we shall concentrate on the second difficulty. We shall, therefore, make, throughout this paper, the simplest possible assumptions about the elementary processes. In particular we shall assume that (1) the scattering by a volume element of the atmosphere is *isotropic scattering with an albedo* a and (2) the reflection by the ground is *reflection according to Lambert's law with an albedo* b. Both a and b are defined in the usual way:

$$\text{Albedo} = \frac{\text{Energy scattered in all directions}}{\text{Energy removed from original direction}}.$$

The lost fraction, 1-albedo, will be called the "absorbed energy." Of course, it may reappear as radiation in some other wave length. For the present calculation, however, this does not make any difference.

A calculation based on these simple assumptions will bring out most clearly what kind of mathematical difficulties occur in the theory of multiple scattering. The problems that arise if the scattering is of a more complex type, for instance, Rayleigh scattering with exact account of the polarization of the incident and scattered light, are mostly of a similar nature. The calculations of S. Chandrasekhar[4] have shown that the solution in that case will consist of a larger number of terms but that the forms of the separate terms closely resemble the form for isotropic scattering.

I. METHODS OF SOLUTION

We may first remark that the presence of a diffusely reflecting ground beneath the atmosphere does not introduce any new difficulties. We shall refer to the problem in which the bottom surface is absent as the "standard problem" and to the case in which it is present as the "planet problem." It will be shown in Section V that the second problem can be reduced to the first one. Most of our discussion will therefore be devoted to the standard problem.

The methods by which these problems may be solved can be divided into three types. In methods of type I, one tries to follow the career of an incident quantum of radiation. It may be scattered 0, 1, 2, or any number of times. The emerging intensities, including all such possibilities, would together give the required result. However, the calculations are too complex to be pursued beyond $n = 2$. Therefore, this method is useful only in cases in which the series converges very rapidly.

In methods of type II the radiation field inside the atmosphere is introduced as an unknown function of the optical depth and the angles. This function has to satisfy an integrodifferential equation, *the equation of transfer*, and certain boundary conditions. Approximate solutions of this equation may be obtained by various methods. The most widely known method is the Milne-Eddington approximation. A more effective one is the method of Gaussian subdivisions and characteristic roots recently used extensively by Chandrasekhar.[5]

In methods of type III, only the reflected and transmitted intensities are introduced as unknown functions. Using certain invariances that must hold if, for instance, a layer $d\tau$ is removed from the top and added to the bottom of the atmosphere, one can show that these functions must satisfy certain *functional equations*. These equations have the

[4] A series of papers entitled "On the Radiative Equilibrium of a Stellar Atmosphere," starting in *Ap. J.*, **99**, 180, 1944. These papers will be referred to as "Paper I," etc. Chandrasekhar has given a comprehensive account of these investigations in his Gibbs lecture, *Bull. Amer. Math. Soc.*, **53**, 641, 1947.

[5] Paper II, *Ap. J.*, **100**, 76, 1944, and following papers.

form of nonlinear integral equations. This method was introduced by Ambarzumiam and has been made into a very powerful tool by Chandrasekhar.[6] In particular, Chandrasekhar has proved that the solutions can be separated into products of a function of the angle of incidence only *times* a function of the angle of emergence only. These functions satisfy separate functional equations. The equations can be easily solved with an accuracy of five or more decimals, provided that a fairly good approximation is already known to start with. If, however, such a starting solution was not known, the nonlinearity of these equations would virtually forbid their solution by trial and error.

In choosing the appropriate method for our present problem we may be guided by the following considerations. The devices of type III will prove useful in any case, both as a check on the general form of the solution and as the only means of reaching the utmost accuracy. But, first, an approximate solution has to be found by means of either method I or method II. Now in problems of planetary illumination we are primarily interested in thin atmospheres. Thus the optical thickness, τ, of the terrestrial atmosphere at λ 5000 A is 0.14. Since the sources of scattered radiation are about evenly distributed in such an atmosphere, the intensities in vertical direction are of the order of τ times the intensities in horizontal direction. It has been found that the methods of type II that assume an intensity distribution expressible in a few spherical harmonics are not always satisfactory in this case. Fortunately, the thin atmospheres just provide a case in which methods of type I are useful. For the flux of $n + 1$ times scattered light is of the order of $a\tau$ times the flux of n times scattered light, so that the series converges rapidly.

It should be noted that the idea of applying a method of this kind is not new. The well-known law of Lommel-Seeliger is, for example, the single-scattering term for a semi-infinite atmosphere. King's calculations,[1] though presented in the form of type II, are actually of type I. Further, L. S. Ornstein[7] and A. Hammad and S. Chapman[8] have suggested this method. However, it gains sufficient power only in the present context, where the result can be used in conjunction with the formulae obtained by method III and thus be transformed into a completely rigorous solution, if so desired.

II. THE STANDARD PROBLEM

Definitions.—Let the atmosphere be bounded by two parallel planes that for convenience we shall call "horizontal." The cosine of the angle between any direction and the downward vertical will be denoted by μ. The radiation that is incident on the top of the atmosphere has a direction for which μ has a positive value, μ'. Let $s' = 1/\mu'$. The radiation field inside the atmosphere will contain both upward- and downward-directed radiations. For the upward radiation, $\mu < 0$, we shall put $s = -1/\mu$; for the downward radiation, $\mu > 0$, we shall put $s = 1/\mu$; s is therefore always positive.

The intensities of upward- and downward-directed radiations at a depth x below the top boundary will be denoted by $I^-(s, s', x)$ and $I^+(s, s', x)$, respectively. We shall denote the reflected and transmitted intensities by

$$R(s, s') = I^-(s, s', 0)$$

and

$$T(s, s') = I^+(s, s', \tau),$$

respectively. Here τ is the total optical thickness of the atmosphere.

We shall further normalize the functions R and T by requiring that the flux incident on a unit area of the top boundary be π. Diffuse reflection according to Lambert's law, would then make $R(s, s')$ identically 1. We may therefore say that $R(s, s')$ is the *Re-*

[6] Paper XIV, *Ap. J.*, **105**, 164, 1946, and following papers.

[7] *M.N.*, **97**, 207, 1937.

[8] *Phil. Mag.*, **28**, 99, 1939.

flected intensity expressed in terms of the intensity that would obtain in the case of Lambert reflection.

Let πr and πt denote the flux of radiation that emerges from a unit area of the top boundary or the bottom boundary, respectively. Then

$$r(s') = 2\int_1^\infty R(s, s')\frac{ds}{s^3}, \qquad t(s') = 2\int_1^\infty T(s, s')\frac{ds}{s^3}. \tag{1}$$

Recursion formulae for the intensities of n *times scattered light.*—We shall now use a subscript n to denote the various quantities pertaining to the light which has been

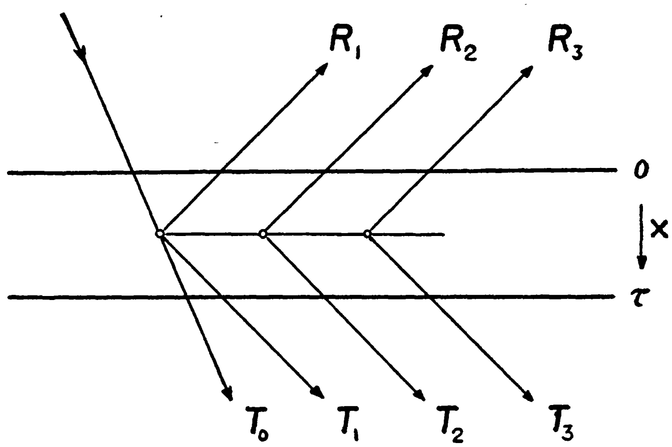

FIG. 1.—Schematic diagram of multiple scattering in the standard problem

scattered exactly n times (cf. Fig. 1). Scattering of the intensity, I_n, will then give rise to a source function, J_{n+1}, of the $n+1$ times scattered light as follows:[9]

$$J_{n+1}(x) = \frac{a}{2}\int_1^\infty \{I_n^+(s, x) + I_n^-(s, x)\}\frac{ds}{s^2}. \tag{2}$$

This source function, in turn, gives rise to the intensities,

$$I_{n+1}^-(s, x) = s\int_x^\tau J_{n+1}(z)\, e^{(x-z)s} dz, \qquad I_{n+1}^+(s, x) = s\int_0^x J_{n+1}(z)\, e^{(z-x)s} dz. \tag{3}$$

Equations (2) and (3) form a recursion scheme from which the entire radiation field may, in principle, be derived. Eliminating the I's, we find the recursion formula,

$$\begin{aligned} J_{n+1}(x) &= \frac{a}{2}\int_0^\tau J_n(z) E_1(|x-z|)\, dz \\ &= \frac{a}{2}\int_0^x J_n(x-y) E_1(y)\, dy + \frac{a}{2}\int_0^{\tau-x} J_n(x+y) E_1(y)\, dy, \end{aligned} \tag{4}$$

[9] The additional arguments, s' and τ, on which all these functions depend, need not be written down.

where $E_1(y)$ denotes the exponential integral (appen., sec. 2).[10] The corresponding emergent intensities are

$$R_n = s \int_0^\tau J_n(x)\, e^{-xs} dx\,, \qquad T_n = s \int_0^\tau J_n(x)\, e^{-(\tau-x)s} dx\,. \tag{5}$$

Further, according to equations (1), the emerging fractions of the incident flux are

$$r_n = 2 \int_0^\tau J_n(x)\, E_2(x)\, dx\,, \qquad t_n = 2 \int_0^\tau J_n(\tau - x)\, E_2(x)\, dx\,. \tag{6}$$

Finally, we shall denote by f_n the fraction of the incident flux that is scattered *at least* n times. Then

$$f_n = 4 \int_0^\tau J_n(x)\, dx\,. \tag{7}$$

The factor 4 in this equation arises from the fact that a unit point source emits the flux 4π, i.e., 4 times the incident flux. If no radiation is lost by absorption ($a = 1$) we must further have

$$f_{n+1} = f_n - r_n - t_n\,. \tag{8}$$

The correctness of this equation can be verified by means of equations (4), (6), and (7).

Exact expressions for first- and second-order scattering.—After these preliminaries we can write down the formulae for the light that is scattered not at all, once, twice, etc., by straightforward integrations.

n = *0.*—The incident radiation is confined to a small solid angle, $\Delta\omega$, about one direction. In order that the flux on a unit area of the top of the atmosphere may be π, the specific intensity inside this solid angle must be $s'\pi/\Delta\omega$. This gives for the direct radiation the intensities,

$$I_0^-(x) = \frac{s'\pi}{\Delta\omega}\, e^{-s'x} \quad \text{and} \quad T_0 = \frac{s'\pi}{\Delta\omega}\, e^{-s'\tau}\,.$$

The transmitted fraction of the total flux is

$$t_0 = e^{-s'\tau}\,.$$

Further, the source density for primary scattering is

$$J_1(x) = \frac{a\,s'}{4}\,(1 - e^{-s'\tau})\,,$$

and the fraction of the flux that is scattered once is

$$f_1 = a\,(1 - e^{-s'\tau})\,.$$

n = *1.*—Equations (5) give to the reflected and transmitted intensities the expressions

$$R_1 = a\,\frac{s\,s'}{4\,(s + s')}\,\{1 - e^{-\tau(s+s')}\}$$

and

$$T_1 = a\,\frac{s\,s'}{4\,(s - s')}\,\{ e^{-\tau s'} - e^{-\tau s}\,.$$

[10] The mathematical formulae needed in these integrations have been collected in an appendix (pp. 240–46).

The corresponding flux integrals are

$$r_1 = a\frac{s'}{2}F_2(-s', \tau)$$

and

$$t_1 = a\frac{s'}{2}F_2(s', \tau)\, e^{-s'\tau}.$$

Further, the new source density derived from equation (4) is

$$J_2(x) = a^2\frac{s'}{8}e^{-xs'}\{F_1(s', x) + F_1(-s', \tau - x)\}.$$

The functions $F_n(b, x)$ that occur in these expressions are defined by the integral,

$$F_n(b, x) = \int_0^x e^{bt}E_n(t)\,dt.$$

The properties of these functions have been discussed in the appendix (secs. 3 and 4).

Finally, using equation (7), we find the flux of radiation scattered twice to be (appen., sec. 5):

$$f_2 = \frac{a^2}{2}[(1 - e^{-\tau s'})\{1 - E_2(\tau)\} + F_1(-s', \tau) - e^{-\tau s'}F_1(s, \tau)].$$

Using the recursion formula for the F-functions, we can easily verify equation (8),

$$f_2 = f_1 - r_1 - t_1,$$

for the case $a = 1$ as a check.

n = *2.*—The reflected and transmitted intensities of light that is scattered twice follow from equations (5) by substituting for $J_2(x)$ the expression just derived and by using the definite integrals of the appendix, section 5. The resulting expressions are

$$R_2 = \frac{a^2 s s'}{8(s+s')}[F_1(-s, \tau) + F_1(-s', \tau) - e^{-(s+s')\tau}\{F_1(s, \tau) + F_1(s', \tau)\}]$$

and

$$T_2 = \frac{a^2 s s'}{8(s-s')}[e^{-\tau s'}\{F_1(-s, \tau) + F_1(s', \tau)\} - e^{-\tau s}\{F_1(-s', \tau) + F_1(s, \tau)\}].$$

At this stage in our calculation the same thing happens as has happened in all earlier investigations of this kind: the integrations become too complex. Neither the new source function, $J_3(x)$, nor the flux integrals, r_2, t_2, and f_3, can be expressed in terms of simple functions involving the exponential integrals. However, we shall presently see that the terms with $n = 1$ and $n = 2$ already form a suitable approximation that can be introduced into the general equations of Chandrasekhar and thus lead to significant results.

Separation of the solution.—Chandrasekhar has proved that the rigorous solution must have the form (Paper XXII, § 7)

$$R = \frac{a s s'}{4(s+s')}\{X(s)X(s') - Y(s)Y(s')\},$$
$$T = \frac{a s s'}{4(s-s')}\{X(s)Y(s') - Y(s)X(s')\} + T_0. \tag{9}$$

These equations follow from the paper referred to[11] by noting that the s and s' used here are the inverse values of the μ and μ_0 used there and that the functions derived there are, by definition, $F\mu\mu_0$ times the R and T used in the present paper.

[11] Paper XXII, *Ap. J.*, **107**, 48, 188, 1948.

The results derived above can be re-written in the form,

$$R = R_1 + R_2 + \text{terms of the order of } (a\tau)^3 ,$$
$$T = T_0 + T_1 + T_2 + \text{terms of the order of } (a\tau)^3 . \qquad (10)$$

By an inspection of the formulae for R_1, T_1, R_2, and T_2, we find that equations (10) can indeed be written in the form (9), if we assume that $X(s)$ and $Y(s)$ have the following values:

$$X(s) = 1 + \tfrac{1}{2} a F_1(-s, \tau) ,$$
$$Y(s) = e^{-\tau s}\{1 + \tfrac{1}{2} a F_1(s, \tau)\} . \qquad (11)$$

This solution is still an approximate one. Terms of the order of $(a\tau)^2$ and higher are still lacking. However, for any small value of $a\tau$ equations (11) already give a satisfactory approximation. Depending on the accuracy required, this approximation may be used as it stands, or it may be used as a "starting solution" in Chandrasekhar's functional equations and in that way lead to a numerical solution of any desired degree of accuracy.

III. ALTERNATIVE DEDUCTION; PHYSICAL EXPLANATION OF THE FUNCTIONS $X(s)$ AND $Y(s)$

Although the results of the preceding section are satisfactory, the method by which they were derived is not elegant. In particular, we combined two entirely different points of view, between which there is no apparent relation. The recursion formulae and the consequent expressions for R_1, R_2, T_1, and T_2 were based on a visualization of the multiple scattering; the final separation of the solution, however, was based on the general equations derived by Chandrasekhar from the equations of transfer and certain invariances characterizing the angular distribution of the emergent radiation.

It appeared worth while, therefore, to make an effort to resolve this duality and place the entire deduction on the basis of a visual understanding of the problem. We shall show that it is, in fact, possible to assign definite physical meanings to the functions $X(s)$ and $Y(s)$. In terms of these meanings the formulae become very transparent and the relevant equations can be written down almost immediately.

Physical meaning of X(s) *and* Y(s).—Basic to the present deduction is the physical meaning to be attached to these functions. We shall start by defining these functions as follows: Let a point source of unit brightness (emitting the total flux 4π) be placed at some distance above the plane-parallel atmosphere. Part of the light of this light-source will suffer multiple scattering, resulting in transmission and reflection by the atmosphere. Now the same point source together with the illuminated part of the atmosphere will from a very large distance again appear as a point source. The *brightness of this combined source* will be denoted by $X(s)$ for directions on the same side as the illuminating source and by $Y(s)$ for directions on the opposite side (see Fig. 2). In either case, s is the positive secant of the angle with the normal.

The following, equivalent, definition is a little more convenient. Let a homogeneously emitting, but completely transparent, layer be placed above the atmosphere. Then $X(s)$, respectively $Y(s)$, is the ratio of the intensity emitted by the luminous layer together with the atmosphere to the intensity that would be emitted by the luminous layer alone.

We shall now use this definition in order to express X and Y in terms of the reflection and transmission functions, $R(s, s')$ and $T(s, s')$, defined in the preceding section. Earlier we adopted the normalization that the flux reaching a unit area of the top of the atmosphere is π. If the incident radiation comes from various directions, s', its intensity, $I(s')$, must therefore satisfy the equation

$$2\int_1^\infty I(s') \frac{ds'}{s'^3} = 1 . \qquad (12)$$

Accordingly, the intensities of reflected and transmitted radiation emerging in a direction s are

$$2\int_1^\infty R(s, s')\, I(s')\frac{ds'}{s'^3} \quad \text{and} \quad 2\int_1^\infty T(s, s')\, I(s')\frac{ds'}{s'^3}, \tag{13}$$

respectively. Only the function $T_0(s, s')$, representing the direct transmission, has to be defined more precisely than we did before. Equation (13) shows that the correct intensity, $I(s)\, e^{-s\tau}$, of the directly transmitted light is obtained if $T_0(s, s')$ has the form

$$T_0(s, s') = \tfrac{1}{2} s^3 e^{-s\tau} \delta(s, s'), \tag{14}$$

where $\delta(s, s')$ is the usual δ-function.

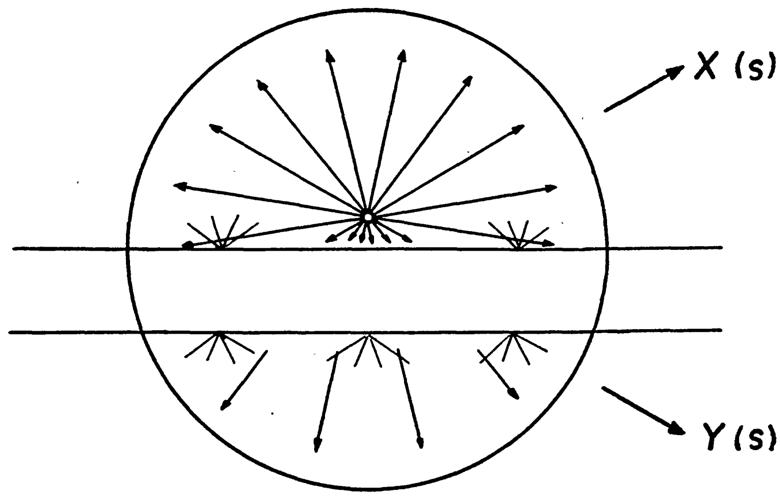

FIG. 2.—Physical meaning of $X(s)$ and $Y(s)$. The combined brightness of the light-source and the illuminated atmosphere in a given direction is $X(s)$, or $Y(s)$, times the brightness of the source alone.

Now the intensity of the thin luminous layer is proportional to s'. According to equation (12), we must therefore put

$$I(s') = \frac{s'}{2}. \tag{15}$$

Substituting this value in equation (13) and dividing the result by the intensity, $s/2$, of the layer alone, we obtain at once

$$\begin{aligned} X(s) &= 1 + \frac{2}{s}\int_1^\infty R(s, s')\frac{ds'}{s'^2}, \\ Y(s) &= \frac{2}{s}\int_1^\infty T(s, s')\frac{ds'}{s'^2}. \end{aligned} \tag{16}$$

The term 1 on the right-hand side for $X(s)$ stands for the unmodified radiation that is emitted by the luminous layer in directions away from the atmosphere. Using equation (14), we find that a similar term, $e^{-s\tau}$, is included in $Y(s)$ as the term due to the directly transmitted light.

Reciprocal meaning of X(s) *and* Y(s).—Now it should be noted that any function occurring in scattering problems has two reciprocal meanings, corresponding to the different senses of direction of the light-rays. The reciprocal meaning of $X(s)$ and $Y(s)$, suggested by inverting the direction of the radiation in Figure 2, is the following:

Considering again the standard problem, wherein the radiation is incident from one direction, s', we observe that the *density of radiation* above the atmosphere divided by the density of radiation of the incident light is $X(s')$. The corresponding density below the atmosphere in terms of the same units is $Y(s')$; for the integrals that define these radiation densities are identical with those in equations (16).

Expression of R *and* T *in terms of* X *and* Y.—We shall now derive some equations by which $R(s, s')$ and $T(s, s')$ can be expressed in terms of $X(s)$ and $Y(s)$. Let us consider again the standard case, with incident radiation in the direction s' and emergent radiation in the direction s. Now we add a layer with thickness $d\tau$ to the top of the atmosphere. The source density of scattered radiation inside this layer is proportional to the radiation density. In fact, equation (2) shows that

$$J=\frac{a s'}{4} X(s')$$

inside this layer. If this layer were alone, it would emit the intensities $sJd\tau$, according to equation (5). Because of the presence of the atmosphere, however, these intensities have to be multiplied by $X(s)$ or $Y(s)$. Further, we have to allow for the fact that the layer scatters, or absorbs, a small part of the radiation that would be reflected, or transmitted, otherwise. We find therefore that the addition of this thin layer increases the intensities, $R(s, s')$ and $T(s, s')$, by the amounts,

$$dR(s, s') = -(s'+s)R(s, s')\,d\tau+\tfrac{1}{4}a s s' X(s')\,X(s)\,d\tau$$

and

$$dT(s, s') = -s'T(s, s')\,d\tau+\tfrac{1}{4}a s s' X(s')\,Y(s)\,d\tau\,.$$

Instead of adding a layer to the top, we may equally add a layer to the bottom of the atmosphere. In the latter case the source density in this layer becomes

$$J=\frac{a s'}{4} Y(s')\,,$$

and the reflected and transmitted intensities are increased by

$$dR(s, s') = \tfrac{1}{4}a s s' Y(s')\,Y(s)\,d\tau$$

and

$$dT(s, s') = -sT(s, s')\,d\tau+\tfrac{1}{4}a s s' Y(s')\,X(s)\,d\tau\,.$$

Since the net effect of an additional layer at top or bottom must be the same, we may equate the two expressions for $dR(s, s')$ and those for $dT(s, s')$. The result is that, identically with equations (9),

$$\begin{aligned} R(s, s') &= \frac{a s s'}{4(s+s')}\{X(s)\,X(s') - Y(s)\,Y(s')\} \\ T(s, s') &= \frac{a s s'}{4(s-s')}\{X(s)\,Y(s') - Y(s)\,X(s')\} + T_0(s, s')\,. \end{aligned} \tag{17}$$

The value of the transmission term, $T_0(s, s')$, of the direct light is not found by means of this deduction because it vanishes except for $s - s' = 0$. Therefore, we have added this term as a separate term.

Functional equations for X *and* Y.—Substituting equations (17) in equations (16), we obtain the functional equations,

$$\begin{aligned} X(s) &= 1+\frac{a}{2s}\int_1^\infty \frac{s s'}{s+s'}\{X(s)\,X(s') - Y(s)\,Y(s')\}\frac{ds'}{s'^2}, \\ Y(s) &= e^{-\tau s}+\frac{a}{2s}\int_1^\infty \frac{s s'}{s-s'}\{X(s)\,Y(s') - Y(s)\,X(s')\}\frac{ds'}{s'^2}. \end{aligned} \tag{18}$$

These relations are the basic equations in the mathematical treatment given in Chandrasekhar's paper (Paper XXII, eqs. [172] and [173]). For a detailed discussion we may refer to that paper.

Integrodifferential equations for X *and* Y.—By similar reasoning we can derive integrodifferential equations for $X(s)$ and $Y(s)$. Let us place a luminous layer, emitting the intensity s', above an atmosphere of optical thickness τ. The intensity emerging at the bottom is then, by definition, $s\,Y(s)$. Now add to the bottom a layer of thickness $d\tau$. The source density in this layer, according to equation (2), will be

$$\frac{a}{2}\int_1^\infty Y(s)\frac{ds}{s}\equiv y_{-1}. \tag{19}$$

The emergent intensities due to this additional source density will be, again by definition, $y_{-1}sd\tau Y(s)$ for upward radiation and $y_{-1}sd\tau X(s)$ for downward radiation. Further, the radiation transmitted by the original atmosphere loses a fraction $sd\tau$ of its amount because of the additional layer. Equating the new intensities thus found to $s(X + dX)$ and $s(Y + dY)$, we obtain at once the equations

$$\begin{aligned}\frac{dX(s)}{d\tau}&=y_{-1}Y(s)\,,\\ \frac{dY(s)}{d\tau}+sY(s)&=y_{-1}X(s)\,.\end{aligned}\tag{20}$$

These equations have also been derived by Chandrasekhar (*ibid.*, eqs. [175] and [176]).

Series expansion of X(s) *and* Y(s).—The physical interpretation of the functions $X(s)$ and $Y(s)$ has the great advantage of suggesting a rapid method for calculating them. Thus, let us consider the scattering of light emitted by a luminous layer on top of the atmosphere. We may use the same formulae for multiple scattering that were used in Section II.

Let the source density inside the luminous layer be

$$J_0=\delta(x,0)\,.$$

The source density for the scattering of any order may then be computed by means of equation (4). In particular, we obtain

$$J_1=\frac{a}{2}E_1(x)\,.$$

The next term will be

$$J_2=\frac{a^2}{4}\int_0^x E_1(x-y)E_1(y)\,dy+\frac{a^2}{4}\int_0^{\tau-x}E_1(x+y)E_1(y)\,dy\,,$$

in which the first integral is identical to $G'_{11}(x)$ (appen., sec. 9) and the second integral has a similar nature. It does not seem profitable to go further into the calculations of these integrals, except by means of approximations or by numerical methods.

The corresponding intensities of emergent light follow from equations (5). Their values, divided by the intensities, s, that the luminous layer itself would emit, give the functions $X_n(s)$ and $Y_n(s)$, so that

$$\begin{aligned}X_n(s)&=\int_0^\tau J_n(x)\,e^{-xs}dx\,,\\ Y_n(s)&=e^{-\tau s}\int_0^\tau J_n(x)\,e^{xs}dx\,.\end{aligned}\tag{21}$$

By adding the functions due to scattering of any order we have

$$X(s) = X_0(s) + X_1(s) + X_2(s) + \ldots ,$$
$$Y(s) = Y_0(s) + Y_1(s) + Y_2(s) + \ldots . \tag{22}$$

These functions follow from the total source density,

$$J(x) = J_0(x) + J_1(x) + J_2(x) + \ldots , \tag{23}$$

by

$$X(s) = \int_0^\tau J(x)\, e^{-xs} dx ,$$
$$Y(s) = e^{-\tau s} \int_0^\tau J(x)\, e^{xs} dx . \tag{24}$$

Thus, writing only the terms of order 0 and 1 and using the expressions for $J_0(x)$ and $J_1(x)$ given above, we find

$$X(s) = 1 + \frac{a}{2} F_1(-s, \tau) ,$$
$$Y(s) = e^{-s\tau} \left\{ 1 + \frac{a}{2} F_1(s, \tau) \right\} . \tag{25}$$

These equations are identical with equations (11), found in Section II only after a lengthy calculation.

Equations (24) further explain the existence of one more formula derived by Chandrasekhar:

$$Y(s) = e^{-\tau s} X(-s) . \tag{26}$$

This relation, however, is only a formal one, because only the values of $s > 1$ have physical significance. The actual functions $X(s)$ and $Y(s)$ to be computed are two entirely different functions.

It should be noted that the indices, $n = 0, 1, 2, \ldots$, occurring in the present deduction do not exactly correspond to the indices n used in Section II. There the index n denoted the nth-order term in a series expansion of $R(s, s')$ and $T(s, s')$ in powers of a. Here, however, they denote the same in an expansion of $X(s)$ and $Y(s)$. The exact relation, equations (24), shows that there is no simple correspondence. It is easily seen, however, that formulae for $X(s)$ and $Y(s)$ complete to the nth order, when substituted in equations (24), will give $R(s, s')$ and $T(s, s')$ complete up to the $n + 1$th order plus further incomplete terms. Thus the first-order approximation of X and Y, equations (11) or equations (25), was sufficient to obtain the second-order approximation of R and T.

IV. CALCULATION OF THE THIRD-ORDER TERM

It is not the object of the present paper to go into the numerical computations that are needed for reaching the utmost accuracy. The relative simplicity of the formulae prompts us, however, to calculate at least one further term of the solutions. This calculation will give us the second-order terms of $X(s)$ and $Y(s)$ and, accordingly, the third-order terms of $R(s, s')$ and $T(s, s')$.

It is not feasible to calculate the exact values of these terms; for we have seen, both in Section II and in Section III, that the source functions from which these terms have to be derived already involve integrals of too complex a type. It is possible, however, to find satisfactory approximations if $s\tau$ is not too large.

New terms of X *and* Y.—We shall first derive the values that would hold in the (physically impossible) case of $s = 0$. Equations (21) become for that case

$$X_2(0) = Y_2(0) = \int_0^\tau J_2(x)\, dx .$$

This integral can be calculated exactly. Expressing $J_2(x)$ by an integral according to equation (4) and inverting the order of integration, we obtain

$$\int_0^\tau J_2(x)\,dx = \frac{a}{2}\int_0^\tau \{2 - E_2(z) - E_2(\tau - z)\} J_1(z)\,dz .$$

With the substitution $J_1(z) = \frac{1}{2}aE_1(z)$ in the foregoing equation, the result becomes

$$X_2(0) = Y_2(0) = \frac{a^2}{4} g(\tau) , \tag{27}$$

where

$$g(\tau) = 2 - 2E_2(\tau) - G_{12}(\tau) - G'_{12}(\tau) . \tag{28}$$

The functions

$$G_{mn}(\tau) = \int_0^\tau E_m(z) E_n(z)\,dz \quad \text{and} \quad G'_{mn}(\tau) = \int_0^\tau E_m(z) E_n(\tau - z)\,dz$$

occurring in this expression are discussed in the appendix, sections 7–10.

The same result may be derived in several other ways. For example, we may use the differential equations (20). We shall denote the approximate solution expressed by equations (25) by $X^*(s)$ and $Y^*(s)$. This solution may be used as a starting solution. The value of y_{-1} corresponding to this solution is (cf. appen., sec. 6):

$$\begin{aligned} y^*_{-1} &= \frac{a}{2}\int_1^\infty \left\{ e^{-\tau s} + \frac{a}{2} e^{-\tau s} F_1(s, \tau) \right\} \frac{ds}{s} \\ &= \frac{a}{2} E_1(\tau) + \frac{a^2}{4} G'_{11}(\tau) . \end{aligned} \tag{29}$$

After substituting this expression and the expression for $Y^*(s)$ in the right-hand side of equation (20), we obtain an expression for $dX(s)/d\tau$ which is correct up to terms of the order of $a^2\tau$, inclusive. Again we find that integration is feasible only for the case $s = 0$. The integration for that case can be made after some reduction and gives, indeed, a result identical with equation (27).

In the derivation of equation (27) we have been forced to assume that $s = 0$. We shall now derive an approximate expression, useful for any value of s, by assuming that the source function $J_2(x)$ is approximately constant. It follows, then, from equations (21) that $X_2(0)$ and $Y_2(0)$ have to be multiplied by

$$\frac{1}{\tau}\int_0^\tau e^{-sx}dx = \frac{1}{s\tau}(1 - e^{-s\tau})$$

in order to obtain the values for $X_2(s)$ and $Y_2(s)$. So we have

$$X - X^* = Y - Y^* = \frac{a^2}{4} g(\tau) \frac{1}{s\tau}(1 - e^{-s\tau}) . \tag{30}$$

It should be noted that the approximate values of X and Y defined by this equation exactly satisfy the relation expressed by equation (26).

Final expressions for R *and* T.—We may finally investigate the consequences of this additional term for the reflected and transmitted intensities, $R(s, s')$ and $T(s, s')$. Let the values of R and T obtained by substituting the approximate solutions X^* and Y^* in equations (17) be denoted by R^* and T^*. The improved values of R and T will then be found by substituting in the same equation the improved solutions,

$$X(s) = X^*(s) + U(s)$$

and

$$Y(s) = Y^*(s) + U(s) .$$

Here $U(s)$ has been written for the right-hand member of equation (30).

For convenience of notation we shall for a moment omit the asterisks from X^* and Y^* and denote the argument s' by adding a prime to the symbols X, Y, and U. The quantities in braces in equations (17) can then be reduced as follows:

$$(X+U)(X'+U') - (Y+U)(Y'+U') = XX' - YY' + U'(X-Y) + U(X'-Y'),$$

and

$$(X+U)(Y'+U') - (Y+U)(X'+U') = XY' - YX' + U'(X-Y) - U(X'-Y').$$

The first two terms on the right-hand side of these equations give rise to the original values, R^* and T^*. Now we may neglect terms that are of the order of $a\tau$ times the term we seek to derive. Therefore,

$$X(s) - Y(s) = 1 - e^{-s\tau}.$$

Thus it appears that the remaining terms of the expressions just derived have the factor

$$(1 - e^{-s\tau})(1 - e^{-s'\tau})$$

in common. A simple reduction now gives:

$$R - R^* = T - T^* = \frac{a^3}{16\tau} g(\tau)(1 - e^{-\tau s})(1 - e^{-\tau s'}). \tag{31}$$

The result that the correction terms of R and T turn out to be the same is not unexpected; for the derivation of equation (31) holds strictly only if we assume that both τs and $\tau s'$ are small compared to 1. This assumption amounts to saying that we consider only angles for which the atmosphere is transparent. In that case the intensity emerging at either side of the atmosphere must be the same function of the angle of emergence. If, on the other hand, τs is large, the factor $1 - e^{-\tau s}$ gives correctly the effective depth of the atmosphere in the direction s, but the assumption that the average source density in that layer equals the average source density in the entire atmosphere is no longer correct.

It should be noted that the earlier approximation, R^* and T^*, is useful in a much wider range than is the correction term just derived; for the terms of X^* and Y^* are the rigorous expressions for the first two terms in series expansions of X and Y in powers of a. Therefore, these expressions provide a useful approximation as soon as $a\tau \ll 1$. The correction term given by equation (31), however, is subject to the further restriction that $s\tau \ll 1$, i.e., that τ itself must be $\ll 1$. Thus, in the case of a thick atmosphere with heavy absorption (so that $a\tau$ is small), the approximations R^* and T^* are still useful, but the expression for the correction term cannot be trusted.

Practical computation.—Recapitulating the relevant formulae, we may state, first, that the values of X^* and Y^* have to be computed according to equations (25). Then, in accordance with equations (17), the values of R^* and T^* follow from

$$\begin{aligned} R^*(s, s') &= \frac{a s s'}{4(s+s')}\{X^*(s) X^*(s') - Y^*(s) Y^*(s')\}, \\ T^*(s, s') &= \frac{a s s'}{4(s-s')}\{X^*(s) Y^*(s') - Y^*(s) X^*(s')\} + T_0. \end{aligned} \tag{32}$$

Finally, the correction term to be added to both these values can be computed from equation (31), where $g(\tau)$ is given by equation (28).

One further remark of a practical nature may be made. Equation (32) leaves T^* indeterminate for $s = s'$. We can easily find a correct formula for this case by putting $s' = s - ds$ and letting ds approach 0. The formula thus obtained is

$$T(s, s) = \frac{a s^2}{4} \left\{ \frac{dX}{ds} Y(s) - \frac{dY}{ds} X(s) \right\}. \tag{33}$$

The derivatives corresponding to equations (25) are (appen., sec. 3)

$$\frac{dX}{ds} = \frac{a}{2}\left[\frac{1}{s+1}\{e^{-(s+1)\tau} - 1\} + F_2(-s, x)\right]$$

and

$$\frac{dY}{ds} = -\tau Y(s) + \frac{a}{2} e^{-s\tau}\left[\frac{1}{s-1}\{e^{(s-1)\tau} - 1\} - F_2(s, x)\right]. \tag{34}$$

The substitution of these expressions, and of equations (25) themselves, in equation (33) does not lead to any particularly simple result. It would therefore seem that formula (33) is best used directly for numerical computations.

Numerical example.—The use of the foregoing formulae may be illustrated by a numerical example. Let $\tau = 0.10$, $s = s' = 2$, and $a = 1$. Equations (25) then give

$$X^* = 1.1280 \quad \text{and} \quad Y^* = 0.9423\,.$$

The corresponding derivatives, found from equations (34), are

$$\frac{dX^*}{ds} = -0.0051 \quad \text{and} \quad \frac{dY^*}{ds} = -0.08891\,.$$

The approximate values of R and T are then found from equations (32) and (33), respectively:

$$R^* = 0.09611 \quad \text{and} \quad T^* = 0.09548\,.$$

The correction term, computed by means of equation (31), is equal to 0.00184; so the values are:

$$R = 0.09795 \quad \text{and} \quad T = 0.09732\,.$$

In this computation the tables of the E-, F-, and G-functions given at the end of this paper have been used.

It is thus seen that, even without any process of further numerical improvement, the formulae give results that are satisfactory for most purposes. A rough estimate of the higher-order terms by means of the rule that subsequent terms are in the ratio $ac(\tau) = 0.163$ (eq. [60]) shows that the values just computed are still too low by about 0.00045. This final correction is 0.5 per cent; the remaining uncertainty is less than 0.1 per cent.

V. THE PLANET PROBLEM

We shall now discuss the problem of scattering by a homogeneous atmosphere under which is a diffusely reflecting surface with an albedo b. We shall assume that the diffuse reflection follows Lambert's law, i.e., that the reflected radiation has the same intensity in all (outward) directions. The scattering properties of the atmosphere need not be specified beyond the general requirement that its reflection and transmission in the absence of a ground surface be described by the symmetrical functions $R(s, s')$ and $T(s, s')$. The symmetry of these functions is required because of the "reciprocity principle" of Helmholtz.

General formulae.—We may first compute the angular distribution with which the diffuse radiation that comes from the ground surface is transmitted or reflected to the bottom again. In this case the intensity, $I(s')$, of the radiation incident on the atmosphere from below is a constant, C, say. The intensities of transmitted and reflected light are, then, according to equations (13),

$$2C\int_1^\infty R(s,\,s')\,\frac{ds'}{s'^3}=C\,r(s)$$

and (35)

$$2C\int_1^\infty T(s,\,s')\,\frac{ds'}{s'^3}=Ct(s)\,,$$

respectively. Here $r(s)$ and $t(s)$ are the same integrals as occurred in the calculation of the flux in the standard problem; for their definition by means of equations (35) is equivalent to their definition by means of equations (1), because of the symmetry of R and T.

Further, the total flux of the transmitted and reflected light will be denoted by

$$2\pi C\int_1^\infty r(s)\,\frac{ds}{s^3}=\pi C\,r_c$$

and (36)

$$2\pi C\int_1^\infty t(s)\,\frac{ds}{s^3}=\pi C t_c\,,$$

where r_c and t_c are constants.

Equations (35) and (36) are all we need in order to solve the planet problem entirely. In the following we shall treat two slightly different cases. In case A we assume that the radiation is incident from outside; this case corresponds to the illumination of the planet by the sun. In case B we assume diffuse emission by the ground surface; this case corresponds to the scattering and absorption of the planet's own radiation.

A. Let the radiation be incident from one direction, s'. We assume, as in the standard case, that the flux incident on a unit area of the top of the atmosphere is π. The constant surface brightness, C, of the bottom surface can then be computed by considering the flux equation at this level.

The flux incident on the bottom surface consists of two parts: a part, $\pi t(s')$, of radiation that has never reached this surface before and a part, $\pi C r_c$, of radiation that has been reflected by this surface before and is now reflected to the bottom again. The flux emitted by the bottom surface is simply πC. Since the emitted flux must be b times the incident flux, we have

$$\pi C=\pi b\,\{C\,r_c+t(s')\}\,.$$

Hence

$$C=\frac{bt(s')}{1-b\,r_c}\,. \qquad (37)$$

The intensity reflected by the entire system of atmosphere plus ground surface consists also of two parts. The part that comes from the atmosphere will be denoted by $R(s,s')$, as in the standard problem. The part that comes, directly or indirectly, from the bottom is given by equation (35), where C has the value given by equation (37). Thus we obtain

$$R_{\text{planet}}(s,\,s')=R(s,\,s')+\frac{b}{1-b\,r_c}\,t(s)\,t(s')\,. \qquad (38)$$

This is the final formula by means of which any problems of phase-function, limb darkening, etc., of a planet may be solved, once a satisfactory solution of the standard problem has been obtained.

There is no "transmitted light" in the present case. However, we may inquire what the angular intensity distribution, measured by an observer at the base of the atmosphere, would be. The intensity in any outward direction is equal to C, as stated above. In any downward direction it is obviously

$$I^{+}_{\text{planet}}(s, \tau) = T(s, s') + \frac{b}{1 - b\, r_c}\, t(s')\, r(s)\,. \tag{39}$$

The first term of this expression is simply the transmission function for the standard case; it includes the term T_0 of the direct sunlight. The second term is the light from the ground scattered back by the air.

If all our assumptions were correct, equation (39) would give the intensity distribution of the daylight sky. Several modifications have to be made, however, before this is true. First, the standard problem for a finite atmosphere with Rayleigh scattering has to be solved. Chandrasekhar (Paper XXII, Sec. V) has obtained the solution of this problem; the functions occurring in this solution (equivalent to $X[s]$ and $Y[s]$ in the case of isotropic scattering) can probably be determined with sufficient accuracy by means of a method similar to that used in the present paper. Next, the solution should be substituted in equation (39). Because of the polarization, however, we have scattering matrices R and T instead of scattering functions; equation (39) will have to be modified accordingly. Finally, measurements might show that the ground reflects light according to a law different from Lambert's law. This would necessitate a further modification of equation (39), which is also of a fairly simple nature. All these modifications would make the calculations longer but not much more difficult. It would seem, therefore, that the calculation of brightness and polarization of the daylight sky, correctly within 0.1 per cent, is now distinctly within reach of the theory.

B. The converse problem can be solved in a similar way. Let the bottom surface be self-luminous and emit an intensity 1 in all outward directions. The atmosphere will then change both the angular distribution and the amount of the emitted radiation. Again we shall put the *total* surface brightness of the bottom equal to C. This surface brightness consists partly of the original emission and partly of radiation scattered back by the atmosphere and then reflected by the bottom surface again. Therefore,

$$C = 1 + bC\, r_c\,;$$

whence

$$C = \frac{1}{1 - b\, r_c}\,. \tag{40}$$

The intensity emerging in a direction s from the top of the atmosphere is then

$$I^{-}(s, 0) = \frac{t(s)}{1 - b\, r_c}\,. \tag{41}$$

The energy balance.—In some applications one may want to know what happens to the incident *energy*. There are three possibilities: part of the energy may escape into space, another part may be absorbed by the atmosphere, and a third part may be absorbed by the bottom surface. We shall denote these fractions of the incident energy by f_s, f_a, and f_b, respectively.

A. In the case of incident radiation from a direction s' we find by substituting equations (38) in equations (1) that

$$f_s = r(s') + \frac{b\, t_c}{1 - b\, r_c}\, t(s')\,. \tag{42}$$

Further, the amount absorbed by the bottom is $(1 - b)/b$ times the amount reflected by the bottom, so

$$f_b = \frac{1 - b}{1 - b\, r_c}\, t(s')\,. \tag{43}$$

And, finally, by the conservation of energy we have

$$f_a = 1 - f_s - f_b \,. \tag{44}$$

If the albedo of the atmospheric particles is 1 (this is called a "conservative case"), we have in the standard problem $r(s') + t(s') = 1$ and therefore also $r_c + t_c = 1$. It can be easily verified that in that case $f_a = 0$.

B. A similar calculation for the case of emission by the ground surface leads to the following expression for the distribution of the energy:

$$f_s = \frac{t_c}{1 - o\, r_c}\,, \qquad f_b = \frac{(1-b)\, r_c}{1 - b\, r_c}\,, \qquad f_a = 1 - f_s - f_b \,. \tag{45}$$

Again we can verify that $f_a = 0$ in conservative cases.

Isotropic scattering.—The solution of the planet problem given above holds for any scattering properties of the atmosphere. We shall now assume that the atmosphere consists of isotropically scattering particles with an albedo a. The solution of the standard problem for such an atmosphere, given in Sections II–IV, has resulted in the determination of the two functions, $R(s, s')$ and $T(s, s')$. We now have to perform the integrations shown by equations (35) and (36) in order to determine also the functions $r(s)$ and $t(s)$ and the constants r_c and t_c.

Let the *moments* of the functions $X(s)$ and $Y(s)$ be defined by

$$\begin{aligned} x_p &= \frac{a}{2} \int_1^\infty X(s)\, s^{-p-2} ds\,, \\ y_p &= \frac{a}{2} \int_1^\infty Y(s)\, s^{-p-2} ds\,. \end{aligned} \tag{46}$$

The factor $a/2$ here takes the place of the more general "characteristic function," $\Psi(\mu)$, in Chandrasekhar's formulae.

Now we shall show that the functions $r(s)$ and $t(s)$ can be expressed in terms of $X(s)$ and $Y(s)$ and their moments. By substituting in equations (35) equations (17) for $R(s, s')$ and $T(s, s')$, we obtain:

$$r(s) = \frac{a}{2} \int_1^\infty \frac{s}{s+s'} \{X(s)\, X(s') - Y(s)\, Y(s')\} \frac{ds'}{s'^2}$$

and

$$t(s) = \frac{a}{2} \int_1^\infty \frac{s}{s-s'} \{X(s)\, Y(s') - Y(s)\, X(s')\} \frac{ds'}{s'^2} + t_0(s)\,.$$

The integrands can be separated by means of the relations,

$$\frac{1}{s'(s+s')} = \frac{1}{s s'} - \frac{1}{s(s+s')}$$

and

$$\frac{1}{s'(s-s')} = \frac{1}{s s'} + \frac{1}{s(s-s')}\,.$$

The first terms on the right-hand side of these equations give rise to products containing the moments x_0 and y_0. The second terms give integrals identical with those in equations (18). Thus we obtain (cf. Paper XXII, eqs. [13] and [14])

$$\begin{aligned} r(s) &= x_0 X(s) - y_0 Y(s) - X(s) + 1\,, \\ t(s) &= y_0 X(s) - x_0 Y(s) + Y(s)\,. \end{aligned} \tag{47}$$

The term $t_0 = \exp(-\tau s)$ that is contained in $t(s)$ cancels out against the same term with negative sign occurring in the other part of the integral.

By substituting this result in equations (37) we further find directly:

$$r_c = \frac{4}{a}(x_0 - 1)\, x_1 - \frac{4}{a}\, y_0 y_1 + 1\,,$$
$$t_c = \frac{4}{a}\, y_0 x_1 - \frac{4}{a}(x_0 - 1)\, y_1\,. \tag{48}$$

Equations (47) and (48) give the exact expressions for $r(s)$, $t(s)$, r_c, and t_c in terms of the exact functions $X(s)$ and $Y(s)$. However, since we have only the approximate solutions of $X(s)$ and $Y(s)$ derived in the preceding sections, the following calculation is only approximate.

The expressions we derived for $X(s)$ and $Y(s)$ consist of three terms: the first two terms are given by equations (25) and the third one, which is the same for X and Y, by equation (30). Integrating term by term, we find that the corresponding moments are

$$x_p = \frac{a}{2}\frac{1}{p+1} + \frac{a^2}{4} G_{p+2,\,1}(\tau) + \frac{a^3}{8}\frac{g(\tau)}{p+1}\,,$$
$$y_p = \frac{a}{2} E_{p+2}(\tau) + \frac{a^2}{4} G'_{p+2,\,1}(\tau) + \frac{a^3}{8}\frac{g(\tau)}{p+1}\,. \tag{49}$$

Here the first terms follow from an elementary integration. The second terms have been found as shown in section 6 of the appendix. A rigorous expression for the third term cannot be derived, because we do not have expressions for the corresponding terms of $X(s)$ and $Y(s)$ that are correct for large values of $s\tau$. Equation (30) would give

$$\frac{a^3}{8}\frac{g(\tau)}{\tau}\left\{\frac{1}{p+2} - E_{p+3}(\tau)\right\} \tag{50}$$

for the third terms of x_p and y_p. The more convenient form shown in equations (49) was found by integrating equation (27) instead of equation (30). This amounts to the assumption that $X_2(s)$ and $Y_2(s)$ may be put equal to $X_2(0)$ and $Y_2(0)$ in the entire integration interval. Since the third-order term is already a small correction term, the difference between the forms shown by equations (49) and (50) is not important.

In particular we find that the moments of order 0 are

$$x_0 = \frac{a}{2} + \frac{a^2}{4} G_{12}(\tau) + \frac{a^3}{8} g(\tau)\,,$$
$$y_0 = \frac{a}{2} E_2(\tau) + \frac{a^2}{4} G'_{12}(\tau) + \frac{a^3}{8} g(\tau)\,. \tag{51}$$

Further, the moments of order 1 are

$$x_1 = \frac{a}{4} + \frac{a^2}{4} G_{13}(\tau) + \frac{a^3}{16} g(\tau)\,,$$
$$y_1 = \frac{a}{2} E_3(\tau) + \frac{a^2}{4} G'_{13}(\tau) + \frac{a^3}{16} g(\tau)\,. \tag{52}$$

Chandrasekhar has shown that there are various relations between the moments x_p and y_p. These relations can be used for a check on the approximate expressions just derived. For example, if $a \neq 1$, x_0 and y_0 satisfy the relation (Paper XXII, eq. [37])

$$(1 - x_0)^2 = 1 - a + y_0^2\,. \tag{53}$$

When equations (51) are substituted in this equation, the coefficients of 1, a, and a^2 satisfy the equation rigorously, but the coefficients of a^3 show a discrepancy. This illustrates once more the fact that the third-order terms in these equations are not exact. The relation for $a = 1$ is

$$x_0 + y_0 = 1 . \tag{54}$$

Using the definition of $g(\tau)$, equation (28), we find that this equation is exactly satisfied. A further consequence of this relation is, for $a = 1$,

$$r(s) + t(s) = 1 ; \tag{55}$$

this equation expresses the conservation of energy in the standard problem.

Practical computation.—Although we have now a complete set of equations for solving the planet problem, we have not yet arrived at the formulae that are most suitable for numerical computation. First, we may remark that, if the atmosphere is thick or if a solution of very high accuracy is required, the method outlined above needs only little change. This change consists in the first step: the moments x_0, y_0, x_1, and y_1 have to be computed by direct integration from the exact X- and Y-functions rather than from the approximate equations (51) and (52). The further steps remain the same: by substituting the moments in equations (47) and (48), we find $r(s)$, $t(s)$, r_c, and t_c. Finally, these functions and constants, when substituted in any of the equations (38)–(45), will give the solution of some problem in planetary illumination.

If, however, the atmosphere is thin, accurate results may be obtained in a much more rapid way. In particular, it is then possible to avoid the moments entirely. In fact, in Section II we have already found the exact forms for the terms of $r(s)$ and $t(s)$ that correspond to the direct light and to single scattering. These formulae together give the equations,

$$\begin{aligned} r^*(s) &= \frac{a\,s}{2} F_2(-s, \tau) , \\ t^*(s) &= e^{-\tau s} + \frac{a\,s}{2} e^{-\tau s} F_2(s, \tau) . \end{aligned} \tag{56}$$

Here the asterisk is used to indicate that these expressions are incomplete, the higher-order terms being still absent. Integrating these equations according to equations (36), we further find

$$\begin{aligned} r_c^* &= aG_{22}(\tau) , \\ t_c^* &= 2E_3(\tau) + aG'_{22}(\tau) . \end{aligned} \tag{57}$$

Equivalent formulae may be derived by substituting equations (51) and (52) in equations (47) and (48). If we use recursion formulae for the F- and G-functions, the expressions thus found can be greatly simplified; the result is that the terms of order 0 and 1 become identical with those in equations (56) and (57).

The higher-order terms cannot be expressed in elementary functions. The derivation of formulae suitable for finding the approximate values of these terms is, therefore, to some extent a matter of taste. It appears most practicable to go back to the calculations in Section II, in which the physical meaning is most evident.

The method proceeds by calculating successively the amounts of radiation that are scattered 0, 1, 2, etc., times. Let the incident amount be 1. Briefly repeating the first steps, we have an amount, t_0, which is directly transmitted. A fraction $1 - a$ of the remainder is absorbed, and the fraction

$$f_1 = a(1 - t_0) \tag{58}$$

is scattered. The amounts of radiation emerging after this first scattering are r_1 and t_1. Again, a fraction $1 - a$ of the remainder is absorbed, and the further fraction, which equals

$$f_2 = a\,(f_1 - r_1 - t_1)\,, \tag{59}$$

is scattered for the second time. This is as far as the rigorous calculation goes.

Now we shall dispose of the remaining energy by assuming that the source density of second- and higher-order scattering may be considered constant through the atmosphere. According to equations (6) and (7), we then have

$$r_n = t_n = \frac{1}{2\tau}\{\tfrac{1}{2} - E_3(\tau)\}\,f_n\,,$$

so that

$$f_{n+1} = a\,c\,(\tau)\,f_n\,, \tag{60}$$

where

$$c\,(\tau) = \frac{1}{2\tau}\{2\tau - 1 + 2E_3(\tau)\}\,. \tag{61}$$

It is seen that in every step $ac(\tau)$ may be described as "the fraction of the scattered flux which will be scattered another time." It is now easy to add all the emergent amounts of radiation in a geometrical series. Thus we obtain

$$r_2 + r_3 + \ldots = t_2 + t_3 + \ldots = \frac{1}{2}\,\frac{1 - c\,(\tau)}{1 - a\,c\,(\tau)}\,f_2\,. \tag{62}$$

Likewise, the total amount of radiation that is absorbed by the atmosphere after being scattered one or more times is

$$\frac{(1-a)\,c\,(\tau)}{1 - a\,c\,(\tau)}\,f_2\,. \tag{63}$$

These equations can be used for computing the functions $r(s)$ and $t(s)$ as well as the constants r_c and t_c. The exact expressions of t_0, r_1, and t_1 for either case have already been given in equations (56) and (57). Equations (58) and (59) then give f_2; and equations (62) and (63) show how much of this remainder emerges and how much is absorbed.

Numerical example.—These formulae may be illustrated by a numerical example. Let $\tau = 0.1$, $a = 1$, and $s = 2$. We then have

$$\left.\begin{aligned} r^*(s) &= 0.0762 \\ t^*(s) &= 0.8187 + 0.0755 = 0.8942 \end{aligned}\right\}\ \text{with}\ \left\{\begin{aligned} f_1 &= 0.1813\,, \\ f_2 &= 0.0296\,; \end{aligned}\right.$$

and, further,

$$\left.\begin{aligned} r_c^* &= 0.0707 \\ t_c^* &= 0.8326 + 0.0695 = 0.9021 \end{aligned}\right\}\ \text{with}\ \left\{\begin{aligned} f_1 &= 0.1674\,, \\ f_2 &= 0.0272\,. \end{aligned}\right.$$

The ratio f_2/f_1 in both cases is 0.163, i.e., practically equal to $c(\tau) = 0.1630$. This is a sufficient guaranty that the computation of the further terms by means of equation (62) is correct.

Since $a = 1$, we have to divide f_2 equally between the reflected and the transmitted light. The final results are

$$r\,(s) = 0.0910\,, \qquad r_c = 0.0843\,,$$

$$t\,(s) = 0.9090\,, \qquad t_c = 0.9157\,.$$

Combining these results with the results of Section IV, we find that equation (38) for $s = s' = 2$, $\tau = 0.1$, $a = 1$, and $b = 0.1$, gives

$$R_{\text{planet}}(s,\ s') = 0.09795 + 0.08333 = 0.18128 .$$

This result shows that, under the specified conditions, 54 per cent of the planetary light comes from the atmosphere and 46 per cent directly or indirectly from the bottom surface; the entire surface brightness is 0.180 times the brightness of a surface reflecting according to Lambert's law. Problems of limb darkening, visibility of surface markings, formation of absorption lines, etc., can be investigated by means of similar computations.

Under the same conditions, equation (39) yields the following result:

$$I^{+}_{\text{planet}}(s,\ \tau) = 0.09732 + 0.00834 = 0.10566 .$$

This means that, under the specified conditions, about 8 per cent of the brightness of the daylight sky at 30° above the horizon would arise from reflection by the ground.

This investigation was made in connection with the Symposium on Planetary Atmospheres held at Yerkes Observatory, September 8–10, 1947. With pleasure I recall the stimulating discussions with several members of the Yerkes staff. In particular, I wish to record my thanks to Dr. Chandrasekhar, without whose encouraging interest this paper would not have been written.

Appendix: On Certain Functions Related to the Exponential Integral

1. The exponential integral,

$$Ei\,(x) = \int_{-\infty}^{x} \frac{e^{t}}{t}\, dt ,$$

is the elementary function to which most of the following functions can be reduced. Only one further elementary function will be needed; this function will be introduced in section 9.

2. It has been customary for a long time to introduce in problems of radiative transfer the set of functions

$$E_n\,(x) = \int_{1}^{\infty} \frac{e^{-xt}}{t^{n}}\, dt \qquad (x \geq 0,\ n = \text{integer} \geq 1) ,$$

which satisfy the recurrence relation,

$$nE_{n+1}\,(x) = e^{-x} - xE_n\,(x)$$

and the differential relation,

$$\frac{d}{dx} E_{n+1}\,(x) = -E\ \ (x) .$$

An easy substitution shows that

$$E_1\,(x) = -Ei\,(-x) ,$$

so that all the E-functions can be computed from tables of the exponential integral.

3. Further, we define the set of functions,

$$F_n\,(b,\ x) = \int_{0}^{x} e^{bt} E_n\,(t)\, dt \qquad (n \text{ and } x \text{ as above, } b \text{ any real value}) .$$

By using the recurrence relation for the E-functions, we find the relation

$$bF_{n+1}\,(b,\ x) = e^{bx} E_{n+1}\,(x) - \frac{1}{n} + F_n\,(b,\ x) .$$

The partial derivatives are

$$\frac{\partial F_n(b,x)}{\partial x}=e^{bx}E_n(x)\,,$$

which follows directly from the definition, and

$$\frac{\partial F_n(b,x)}{\partial b}=\frac{1}{b-1}\{e^{(b-1)x}-1\}-nF_{n+1}(b,x)\,,$$

which is obtained by differentiating the integrand and using the recursion formulae for the E-functions.

The calculation of $F_1(b, x)$ is not so obvious. Replacing $E_1(t)$ by its definition and inverting the order of integration, we find

$$F_1(b,x)=\int_1^\infty\frac{du}{u}\int_0^x e^{(b-u)t}dt=\frac{1}{b}\int_1^\infty\left(\frac{1}{u}+\frac{1}{b-u}\right)\{e^{(b-u)x}-1\}\,du\,.$$

In this way the function is reduced to exponential integrals, but it is necessary to distinguish carefully between the various cases. The various formulae are:

$$F_1(b,x)=\frac{1}{b}\{e^{bx}E_1(x)-E_1(x-bx)-\ln(1-b)\}\qquad(b<1\ \text{but}\neq 0)\cdot$$

$$F_1(b,x)=\frac{1}{b}\{e^{bx}E_1(x)+Ei(bx-x)-\ln(b-1)\}\qquad(b>1)\,;$$

$$F_1(0,x)=1-E_2(x)\,;$$

$$F_1(1,x)=e^{x}E_1(x)+l\,,\qquad\text{where } l=\gamma+\ln x\,.$$

In the last formula $\gamma = 0.577216$ is Euler's constant. Essentially the same formulae have been derived by King.[1]

4. If $x = \infty$, the F-integrals converge only for $b < 1$. The recursion formula for that case is

$$bF_{n+1}(b,\infty)=-\frac{1}{n}+F_n(b,\infty)\cdot$$

and the formulae for F_1 are

$$F_1(b,\infty)=-\frac{1}{b}\ln(1-b)\qquad(b<1\ \text{but}\neq 0)$$

$$F_1(0,\infty)=1\,.$$

5. The following integral can be reduced by replacing F_n by its definition and inverting the order of integration as follows:

$$\int_0^x e^{-at}F_n(b,t)\,a\,dt=\int_0^x e^{-at}a\,dt\int_0^t e^{bu}E_n(u)\,du$$

$$=\int_0^x e^{bu}E_n(u)\,du\int_u^x e^{-at}a\,dt=\int_0^x e^{bu}(e^{-au}-e^{-ax})E_n(u)\,du$$

$$=F_n(b-a,x)-e^{-ax}F_n(b,x)\,.$$

6. In a similar way we can derive the following equations:

$$\int_1^\infty F_n(-s,x)\frac{ds}{s^m}=\int_0^x E_n(u)E_m(u)\,du\equiv G_{nm}(x)$$

and

$$\int_1^\infty e^{-xs}F_n(s,x)\frac{ds}{s^m}=\int_0^x E_n(u)E_m(x-u)\,du\equiv G'_{nm}(x)\,.$$

The functions $G_{nm}(x)$ and $G'_{nm}(x)$ are symmetrical in n and m; they will be further discussed in sections 7, 8, and 10.

7. The integral,

$$G_{mn}(x) = \int_0^x E_m(t) E_n(t)\, dt\,,$$

can be reduced to known functions. By using the recursion formula for E_m and E_n, we obtain

$$(m-1) G_{mn}(x) = F_n(-1, x) - \int_0^x tE_{m-1}(t) E_n(t)\, dt$$

and

$$(n-1) G_{mn}(x) = F_m(-1, x) - \int_0^x tE_{n-1}(t) E_m(t)\, dt\,;$$

further, one partial integration gives

$$G_{mn}(x) = xE_m(x) E_n(x) + \int_0^x tE_{m-1}(t) E_n(t)\, dt + \int_0^x tE_{n-1}(t) E_m(t)\, dt\,.$$

By adding these expressions, we obtain

$$(m+n-1) G_{mn}(x) = xE_m(x) E_n(x) + F_m(-1, x) + F_n(-1, x)\,.$$

In the particular case of $m = n-1$ this formula admits of a further reduction. Using the recursion formulae for the E- and F-functions, we find

$$2G_{n,\,n-1}(x) = \frac{1}{(n-1)^2} - \{E_n(x)\}^2\,;$$

this result can also be found directly by partial integration.

8. The integral,

$$G'_{mn}(x) = \int_0^x E_m(t) E_n(x-t)\, dt\,,$$

is *not* expressible in terms of the ordinary exponential integral. By reducing it in a manner similar to that adopted for $G_{mn}(x)$, we first obtain

$$(m-1) G'_{mn}(x) = e^{-x} F_n(1, x) - \int_0^x tE_{m-1}(t) E_n(x-t)\, dt$$

and

$$(n-1) G'_{mn}(x) = e^{-x} F_m(1, x) + \int_0^x tE_m(t) E_{n-1}(x-t)\, dt - xG_{m,\,n-1}(x)\,,$$

and then

$$G'_{mn}(x) = \frac{xE_m(x)}{n-1} + \int_0^x tE_{m-1}(t) E_n(x-t)\, dt - \int_0^x tE_m(t) E_{n-1}(x-t)\, dt\,.$$

Hence

$$(m+n-1) G'_{mn}(x) = \frac{xE_m(x)}{n-1} - xG'_{m,\,n-1}(x) + e^{-x}\{F_n(1, x) + F_m(1, x)\}\,.$$

The recurrence formula thus derived can serve to reduce any integral $G'_{mn}(x)$ to the integral $G'_{11}(x)$, discussed in section 10.

By differentiating $G'_{mn}(x)$ under the integral sign, we further obtain

$$\frac{dG'_{mn}(x)}{dx} = \frac{E_m(x)}{n-1} - G'_{m,\,n-1}(x)\,,$$

so that the recurrence formula gives rise to the differential equation,

$$(m+n-1) G'_{mn}(x) = x\frac{d}{dx} G'_{mn}(x) + e^{-x}\{F_n(1, x) + F_m(1, x)\}\,.$$

9. In preparation of the next section we shall define a new set of functions, $E_1^{(p)}(x)$, for which the name "exponential integrals of the pth order" would seem suitable. They are defined by

$$E_1^{(p+1)}(x) = \int_x^\infty E_1^{(p)} \frac{dx}{x} \qquad \text{and} \qquad E_1^{(0)}(x) = e^{-x}.$$

None of these functions can be reduced to the other ones, so that for each of them a separate numerical table has to be constructed. Their series expansions are

$$E_1^{(p)}(x) = P - \sum_{n=1}^{\infty} \frac{(-x)^n}{n^{p-1} n!},$$

where P is an even, or odd, polynomial in $l = \ln x + \gamma$, the highest term of which is $(-l)^p/p!$. I have not investigated what rule the further coefficients of these polynomials obey.

In the present calculations we need only the exponential integrals of the orders $p = 1$ and $p = 2$. The first one, $E_1^{(1)}(x)$, is identical with the function $E_1(x)$ discussed above. The second one,

$$E_1^{(2)}(x) = \int_x^\infty E_1(x) \frac{dx}{x},$$

seems to be new; I have found no reference to any such function in the literature. Its series expansion,

$$E_1^{(2)}(x) = \tfrac{1}{2}(\ln x + \gamma)^2 + \frac{\pi^2}{12} - x + \frac{x^2}{2^2 \cdot 2!} - \frac{x^3}{3^2 \cdot 3!} + \cdots,$$

is very suitable for numerical computation. The integration constant, $\pi^2/12$, appears in the form, $-\frac{1}{2}\gamma^2 + \frac{1}{2}\Gamma''(1)$. A check computation of $E_1^{(2)}(5)$, using twenty terms of this expansion, gave 0.0001728, while an asymptotic expansion gave 0.000172.

10. Now we can express the integral,

$$G'_{11}(x) = \int_0^x E_1(t) E_1(x-t)\, dt,$$

and, accordingly, all integrals $G'_{mn}(x)$ in terms of known functions. Let us briefly denote this integral by y. The differential equation derived in section 7 takes for $m = n = 1$ the form,

$$y = x\frac{dy}{dx} + 2e^{-x} F_1(1, x).$$

After dividing by x^2 and replacing $F_1(1, x)$ by its expression derived in section 3, we obtain

$$-\frac{d}{dx}\left(\frac{y}{x}\right) = \frac{2}{x^2}\{E_1(x) + l e^{-x}\}.$$

The function that satisfies this differential equation and vanishes for $x = \infty$ is

$$y = G'_{11}(x) = 2\{E_1(x) + lE_2(x) - xE_1^{(2)}(x)\}.$$

The correctness of this formula can be verified by direct differentiation. I do not see any obvious way of deriving this formula; I actually found it by using the series expansions for $E_1(t)$ and $E_1(x-t)$ in the integrand of the defining integral and integrating term by term. The result of a fairly tedious reduction[12] then is

$$G'_{11}(x) = \left(l^2 - 2l + 2 + \frac{\pi^2}{6}\right)x + 2\sum_{n=1}^{\infty} \frac{(-x)^{n+1}}{n(n+1)!}\left(-l + \frac{1}{n} + \frac{1}{n+1}\right),$$

which is identical with the series expansion of the formula given above.

[12] In this reduction I used integrals from Tables 1, 30, and 106 of D. Bierens de Haan, *Nouvelles tables d'intégrales définies* (Leyden: P. Engels, 1867).

It should further be noted that the differential equation from which we started does not follow strictly from the equations of section 8, because the demonstration given there does not hold for $m = n = 1$. Again, series expansion shows that the equation is indeed correct. On the whole, the functions discussed in sections 9 and 10 would seem to require a more thorough investigation. The discussion given here has not been extended beyond what was strictly needed for making the numerical computations.

11. All functions discussed above can be expanded in series in powers of x, with the understanding that some of the coefficients may contain

$$l = \ln x + \gamma \qquad (x > 0) ,$$

besides constant terms. The following formulae present the first few terms of those functions that occurred in the preceding calculations. All have been derived in several ways, e.g., by integration of separate terms, by expansion of recurrence formulae, etc.

$$Ei(x) = l + x + \tfrac{1}{4}x^2 + \tfrac{1}{18}x^3 ;$$

$$E_1(x) = -l + x - \tfrac{1}{4}x^2 + \tfrac{1}{18}x^3 ,$$

$$E_2(x) = 1 + (l-1)\,x - \tfrac{1}{2}x^2 + \tfrac{1}{12}x^3 ,$$

$$E_3(x) = \tfrac{1}{2} - x + \tfrac{1}{2}(-l + \tfrac{3}{2})\,x^2 + \tfrac{1}{6}x^3 ;$$

$$F_1(b, x) = (-l+1)\,x + \tfrac{1}{2}(-bl + \tfrac{1}{2}b + 1)\,x^2 ,$$

$$F_2(b, x) = x + \tfrac{1}{2}(l + b - \tfrac{3}{2})\,x^2 ;$$

$$G_{11}(x) = (l^2 - 2l + 2)\,x + (-l + \tfrac{1}{2})\,x^2 ,$$

$$G_{12}(x) = (-l+1)\,x - \tfrac{1}{2}(l^2 - 2l)\,x^2 ,$$

$$G_{13}(x) = \tfrac{1}{2}(-l+1)\,x + \tfrac{1}{2}lx^2 ,$$

$$G_{22}(x) = x + (l - \tfrac{3}{2})\,x^2 ;$$

$$G'_{11}(x) = \left(l^2 - 2l + 2 - \frac{\pi^2}{6}\right)x + (-l + \tfrac{3}{2})\,x^2 ,$$

$$G'_{12}(x) = (-l+1)\,x - \tfrac{1}{2}\left(l^2 - 3l + \tfrac{5}{2} - \frac{\pi^2}{6}\right)x^2 ,$$

$$G'_{13}(x) = \tfrac{1}{2}(-l+1)\,x + \tfrac{1}{2}(l-1)\,x^2 ,$$

$$G'_{22}(x) = x + (l - \tfrac{3}{2})\,x^2 .$$

These formulae are useful both for approximate computation and for checking the correctness of any equation involving these functions.

Finally, the function

$$g(x) = 2 - 2E_2(x) - G_{12}(x) - G'_{12}(x)$$

has as its dominant term

$$g(x) = \tfrac{1}{2}\left(2l^2 - 5l + \tfrac{9}{2} - \frac{\pi^2}{6}\right)x^2 ;$$

and the function

$$c(x) = \frac{x - \frac{1}{2} + E_3(x)}{x}$$

has the expansion

$$c(x) = \tfrac{1}{2}(-l + \tfrac{3}{2})\, x + \tfrac{1}{6} x^2 + \ldots .$$

12. *Tables.*—Numerical values of the functions needed have been computed for fifteen selected values of x and—in the case of the F-functions—for four selected values of s. These values are given in Tables 1–4. Table 1 gives the E-functions: the functions $\ln x$, e^{-x}, and $E_1(x)$ were taken from the *New York Mathematical Tables Project* tables; E_2 and E_3 were computed by means of the recurrence relation; $E_1^{(2)}(x)$ was computed from

TABLE 1

EXPONENTIAL INTEGRALS OF THE FIRST AND SECOND ORDER

x	l	e^{-x}	$E_1(x)$	$E_2(x)$	$2E_3(x)$	$E_1^{(2)}(x)$
0.01	−4.027954	0.990050	4.037930	0.949671	0.98055	8.924688
0.02	−3.334807	.980199	3.354708	.913105	.96194	6.362987
0.05	−2.418517	.951229	2.467898	.827834	.90984	3.697389
0.1	−1.725369	.904837	1.822924	.722545	.83259	2.212149
0.2	−1.032222	.818731	1.222651	.574201	.70389	1.160064
0.3	−0.626757	.740818	0.905677	.469115	.60008	0.729650
0.4	−0.339075	.670320	0.702380	.389368	.51457	0.498831
0.6	+0.066390	.548812	0.454380	.276184	.38310	0.265984
0.8	+0.354072	.449329	0.310597	.200852	.28865	0.156636
1.0	+0.577216	.367879	0.219384	.148496	.21938	0.097843
1.5	+0.982681	.223130	0.100020	.073101	.11348	0.034517
2.0	+1.270363	.135335	0.048901	.037534	.06027	0.014265
2.5	+1.493506	.082085	0.024915	.019798	.03259	0.006250
3.0	+1.675828	0.049787	0.013048	0.010642	0.01786	0.002879
∞	∞	0	0	0	0	0

TABLE 2

THE FUNCTIONS $F_1(b, x)$

x	$F_1(-s, x)$					$e^{-sx}F_1(s, x)$			
	$s=10$	$s=5$	$s=2$	$s=1$	$s=0$	$s=1$	$s=2$	$s=5$	$s=10$
0.01	0.0482	0.0492	0.0499	0.0501	0.0503	0.0501	0.0498	0.0490	0.0477
0.02	.0797	.0832	.0854	.0861	0.0869	.0859	.0849	.0823	.0779
0.05	.1404	.1551	.1650	.1685	0.1722	.1673	.1627	.1496	.1308
0.1	.1913	.2281	.2560	.2663	0.2775	.2617	.2471	.2091	.1612
0.2	.2270	.3001	.3667	.3945	0.4258	.3775	.3359	.2417	.1500
0.3	.2362	.3309	.4309	.4766	0.5309	.4414	.3698	.2283	.1200
0.4	.2387	.3450	.4707	.5329	0.6106	.4751	.3747	.2005	.0935
0.6	.2397	.3551	.5132	.6022	0.7238	.4908	.3432	.1428	.0583
0.8	.2398	.3576	.5321	.6399	0.7991	.4697	.2913	.0987	.0386
1.0	.2398	.3582	.5409	.6613	0.8515	.4317	.2379	.0685	.0266
1.5	.2398	.3584	.5479	.6839	0.9269	.3193	.1322	.0294	.0118
2.0	.2398	.3584	.5491	.6903	0.9625	.2208	.0698	.0138	.0057
2.5	.2398	.3584	.5493	.6922	0.9802	.1475	.0363	.0068	.0029
3.0	.2398	.3584	.5493	.6929	0.9894	0.0965	0.0188	0.0035	0.0015
∞	0.2398	0.3584	0.5493	0.6931	1.0000	0	0	0	0

its series expansion. Tables 2 and 3 give the F-functions. Since $F_1(s, x)$ occurs always with the factor e^{-sx}, we have tabulated in Table 2 the functions $F_1(-s, x)$ and $e^{-sx}F_1(s, x)$. The central column shows their common value for $s = 0$. Table 3 is arranged in the

TABLE 3

THE FUNCTIONS $F_2(b, x)$

x	$F_2(-s, x)$					$e^{-sx}F_2(s, x)$			
	$s=10$	$s=5$	$s=2$	$s=1$	$s=0$	$s=1$	$s=2$	$s=5$	$s=10$
0.01..........	0.0093	0.0095	0.0096	0.0097	0.0097	0.0097	0.0096	0.0095	0.0092
0.02..........	.0173	.0181	.0187	.0188	.0190	.0188	.0186	.0181	.0172
0.05..........	.0358	.0400	.0430	.0440	.0451	.0439	.0428	.0397	.0352
0.1...........	.0543	.0667	.0762	.0799	.0837	.0795	.0755	.0650	.0516
0.2...........	.0695	.0977	.1242	.1354	.1481	.1330	.1199	.0896	.0589
0.3...........	.0740	.1129	.1558	.1759	.2000	.1697	.1450	.0948	.0539
0.4...........	.0754	.1205	.1772	.2061	.2427	.1941	.1574	.0909	.0464
0.6...........	.0760	.1262	.2018	.2462	.3084	.2182	.1591	.0738	.0332
0.8	.0760	.1277	.2136	.2699	.3557	.2212	.1452	.0563	.0239
1.0...........	.0760	.1281	.2195	.2840	.3903	.2123	.1256	.0421	.0175
1.5...........	.0760	.1283	.2242	.2998	.4433	.1693	.0778	.0204	.0084
2.0...........	.0760	.1283	.2251	.3046	.4699	.1230	.0445	.0103	.0043
2.5...........	.0760	.1283	.2253	.3061	.4837	.0852	.0247	.0053	.0023
3.0...........	.0760	.1283	.2253	.3066	.4911	0.0573	0.0134	0.0028	0.0012
∞............	0.0760	0.1283	0.2253	0.3069	0.5000	0	0	0	0

TABLE 4

THE FUNCTIONS $G_{mn}(x)$, $G'_{mn}(x)$, $g(x)$, AND $c(x)$

x	$G_{11}(x)$	$G_{12}(x)$	$G_{22}(x)$	$G'_{11}(x)$	$G'_{12}(x)$	$G'_{22}(x)$	$g(x)$	$c(x)$
0.01..........	0.26325	0.04906	0.00946	0.24690	0.04882	0.00945	0.0028	0.0277
0.02..........	0.39734	.08312	.01812	.36484	.08229	.01810	0.0084	0.0484
0.05..........	0.64160	.15735	.04076	.56180	.15329	.04054	0.0337	0.0984
0.1...........	0.86500	.23896	.07065	.71010	.22624	.06951	0.0897	0.1630
0.2...........	1.08798	.33515	.11223	.79587	.29795	.10709	0.2185	0.2597
0.3...........	1.19924	.38997	.13926	.78552	.32354	.12766	0.3483	0.3335
0.4...........	1.26318	.42420	.15760	.74165	.32676	.13777	0.4703	0.3932
0.6...........	1.32825	.46186	.17942	.62625	.30294	.14011	0.6828	0.4859
0.8...........	1.35697	.47983	.19066	.51281	.26329	.13083	0.8552	0.5554
1.0...........	1.37081	.48897	.19670	.41451	.22448	.11624	0.9896	0.6097
1.5...........	1.38276	.49733	.20255	.24016	.13917	.07980	1.2173	0.7045
2.0...........	1.38540	.49930	.20401	.13611	.08472	.05056	1.3409	0.7651
2.5...........	1.38605	.49980	.20441	.07772	.05036	.03134	1.4102	0.8065
3.0...........	1.38622	.49994	.20452	.04449	.02975	.01912	1.4490	0.8363
∞............	1.38629	0.50000	0.20457	0.00000	0.00000	0.00000	1.5000	1.0000

same way. Table 4 gives the G-functions computed from the values in the preceding tables. However, six-place values instead of four-place values of the functions $F_n(-1, x)$ and $F_n(1, x)$ were used in this computation.

The computations were made with a Marchant machine. Apart from possible computational errors, all given decimals should be correct. Only the F-functions for $s = \pm 2$ may be in error by a full unit of the last decimal.