

# BULLETIN OF THE ASTRONOMICAL INSTITUTES OF THE NETHERLANDS

1952 JULY 28

VOLUME XI

NUMBER 436

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 COMMUNICATION FROM THE OBSERVATORY AT LEIDEN
 

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## THE VELOCITY DISTRIBUTION OF THE INTERSTELLAR CALCIUM CLOUDS

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ADAMS' measures of the radial velocities of interstellar components of  $\text{Ca}^+$  are analysed on the basis of the simple picture of interstellar clouds with random peculiar motions. Special regard is paid to the effect of blending of neighbouring absorption lines. The resolving power on the spectrograms is derived from statistics of the differences between velocities of consecutive components (section 3). It is found that very few lines differing less than 9 km/sec are separated, and it appears reasonable to assume that half of the velocity differences between 9 and 18 km/sec are separated. This holds for spectrograms with dispersion 2.9 Å/mm, on which the analysis is exclusively based.

Section 4 deals with the division of the stars into groups according to their distances and with the general distribution of stars with multiple lines on the sky (Figure 2). In section 5 the distributions of velocities of the components measured by ADAMS (shown in Figures 3 and 4 and in Tables 5 and 6) are compared with theoretical distributions. These are based on two different assumptions with regard to the form of the true velocity distribution: the gaussian distribution and the form  $e^{-v/\eta}$ ,  $\eta$  being the mean speed of the clouds. It is shown that the second assumption fits the observations quite satisfactorily, which is not the case with the gaussian law. The mean speed is  $\eta = 5$  km/sec with a mean error of about 1 km/sec. The number of clouds intersected by the line of sight is found to be between 8 and 12 per 1000 parsecs. Section 6 gives details about the computations of the theoretical distributions.

A new determination of the solar motion with respect to the interstellar calcium is described in section 2.

### 1. Introduction.

ADAMS' <sup>1)</sup> measures of the radial velocities of components of interstellar H and K lines offer an opportunity to study the velocity distribution of the interstellar calcium. ADAMS' first publication, containing data for 50 stars, was followed by discussions by WHIPPLE <sup>2)</sup> and MELNIKOV <sup>3)</sup>. The present paper analyses the data in ADAMS' second paper, which contains measurements for 300 stars. In his own discussions of these measurements including his Russell Lecture <sup>4)</sup>, ADAMS has confined himself to a general description of the occurrence of multiple lines in relation to the distribution in galactic longitude and latitude, with special regard to regions of particular interest like those in Orion, Cygnus and Sagittarius.

The present analysis is based on the very simple assumption that the interstellar gas is distributed over separate clouds with random individual motions. This is undoubtedly a rather crude description of the real spatial distribution and state of motion. It is true that the concept of individual clouds has proved to be useful in many investigations dealing with the interstellar matter, like analyses of star counts and of equivalent widths of interstellar absorption lines, and

statistics of colour excesses. But the properties attributed to the clouds have a definite sense only in connection with the type of observations in question and the procedure adopted in their interpretation. Therefore comparisons between the numbers of clouds per 1000 parsecs derived from different investigations, or between the average line absorption and the average extinction per cloud, are only of limited value.

In the case of the analysis of the observations on interstellar K lines, the concept of interstellar clouds is, of course, introduced as a consequence of the appearance of separate components in many stars. This suggests a division of the interstellar calcium in separate units with individual peculiar motions. The principal quantities to be determined are, then, the mean speed and the number of clouds along the line of sight. Also, information about the form of the velocity distribution will be obtained. In addition, we shall try to estimate the mean amount of K-line absorption per cloud.

The derivation of these quantities would be a simple matter if the lines due to the individual clouds were well separated. However, neighbouring lines are often combined into blends. The blending effect decreases the separable number of lines, especially at the low velocities where the frequency of the velocities is highest and the chance for lines of two or more clouds to be combined into one blend accordingly is largest. The shape of the true velocity distribution is thus different from that of the distribution of the observed velocities

<sup>1)</sup> *Ap. J.* **97**, 105; *Mt Wilson Cont.* No. 673 (1943).

*Ap. J.* **109**, 354; *Mt Wilson Cont.* No. 760 (1949).

<sup>2)</sup> *Harvard Obs. Monographs* No. 7, 109 (1948).

<sup>3)</sup> *A. J. U.S.S.R.* **24**, 73 (1947).

<sup>4)</sup> *P.A.S.P.* **60**, 174 (1948).

of the blends. In the present attempt to determine the true velocity distribution and the number of clouds along the line of sight we shall pay special regard to this effect.

WHIPPLE, in the investigation mentioned above, has referred to the effect of blending in an attempt to explain the observed frequency distribution of the residual velocities of the 50 stars first observed by ADAMS. However, it seems to us that the effect must be more important than was realized by WHIPPLE. This author concluded that up to distances of 500 parsecs the individual clouds are fairly well separated. This conclusion was based on the observation that the numbers of components are proportional to the distances of the stars. It cannot, however, be reconciled with some direct observations concerning ADAMS' data. The limit of resolution on the spectrograms is at least 9 km/sec, and probably more for the average case. There is, on the other hand, in these stars within 500 parsecs a very strong concentration of the residual velocities of the components around zero; 27 of 58 velocities being between  $-6$  and  $+6$  km/sec. If this distribution of velocities represents the true form of the velocity distribution and there are three clouds per star on the average, the chance that two clouds, observed in the same star, will differ by less than the limit of resolution will be more than one half. This means that there must be many cases where two or more clouds were combined into a blend.

## 2. The solar motion with respect to the interstellar Ca clouds.

ADAMS has corrected the observed radial velocities for the solar motion, using the standard velocity 20 km/sec towards the apex  $\alpha = 270^\circ$ ,  $\delta = +30^\circ$ . The dispersion of the residual velocities is very small and, hence,

depends appreciably on the adopted elements of the solar motion. Therefore we have redetermined the solar motion with respect to the calcium clouds on the basis of the new data. The new solution was confined to the following stars: a) those with distance modulus  $m_0 - M$ , corrected for interstellar absorption, smaller than 7.0; b) those with  $m_0 - M$  between 7.0 and 10.0 and near the nodes of the galactic rotation effect, i.e. the stars in the intervals of galactic longitude  $50^\circ$  to  $65^\circ$ ,  $140^\circ$  to  $155^\circ$ ,  $230^\circ$  to  $245^\circ$  and  $320^\circ$  to  $335^\circ$ . The small amount of galactic-rotation effect left in the directions thus selected was estimated on the basis of the known distances of the stars and eliminated from the observed cloud velocities. Separate components in the spectrum of the same star were combined with weights proportional to their intensities and these mean values were used in two different solutions. In the first solution all stars were given equal weight; in the second the velocities were weighted according to the line intensities. The intensities used are ADAMS' estimates reduced to the scale of equivalent widths as described below. The two solutions agree within the limits of their uncertainty. The second one, which gave the most accurate results, is given here. A  $K$ -term was included in the solution. The results for the galactic component of the solar motion,  $S \cos B$ , the longitude of the apex  $L$ , and the  $K$ -term are in the third line of Table I, where they are compared with previous solutions from nearby stars by PLASKETT and PEARCE<sup>1)</sup> and by OORT<sup>2)</sup>. PLASKETT and PEARCE solved for the quantities mentioned in the table as well as for the elements  $l_0$  and  $A$  of galactic rotation. OORT used the Victoria data in a provisional form, assuming  $l_0 = 327^\circ$ . Both authors adopted  $K = 0$ . These previous solutions have been reduced here to the system of galactic co-ordinates based on OHLSSON's galactic pole.

TABLE I  
The elements of the solar motion

|                        | $L$                       | $S \cos B$                    | $K$                             |
|------------------------|---------------------------|-------------------------------|---------------------------------|
| OORT                   | $22^\circ \pm 2.0$ (p.e.) | $17.7 \text{ km/sec} \pm 0.6$ |                                 |
| PLASKETT and PEARCE    | $22.4 \pm 1.4$ „ „        | $19.9 \pm 0.56$               |                                 |
| From ADAMS' velocities | $19.1 \pm 1.7$ „ „        | $18.6 \pm 0.36$               | $+0.12 \text{ km/sec} \pm 0.37$ |

The new solution includes also the  $Z$ -component although this, owing to the strong concentration of the stars towards the galactic plane, is only inaccurately determined. The total velocity thus derived is

$$S = 20.1 \text{ km/sec} \pm 0.7 \text{ (p.e.)}$$

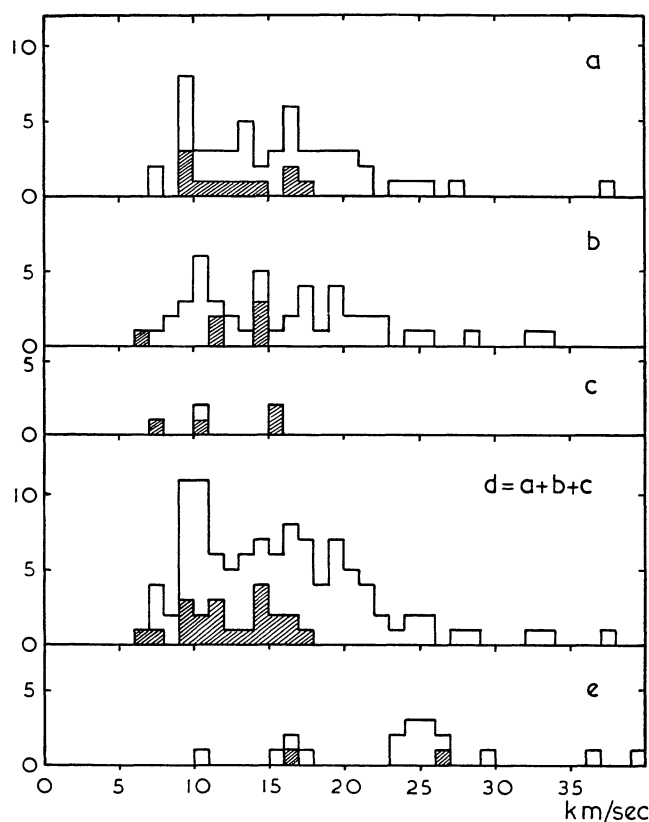
towards  $L = 19^\circ.1 \pm 1^\circ.7$ ,  $B = +22^\circ.8 \pm 4^\circ.6$   
corresponding to  $A = 267^\circ.1$ ,  $D = +26^\circ.6$ .

Although the new solution does not differ very much from the standard elements adopted by ADAMS, and the corresponding difference in the corrections for the solar motion never exceeds 1.3 km/sec, the new solution has been used for the computation of the residual velocities discussed below.

<sup>1)</sup> Victoria Publ. V, 195 (1931).

<sup>2)</sup> B.A.N. No. 189, 193 (1930).

FIGURE 1



Statistics of the observed differences between the velocities of the consecutive components in stars with multiple K lines. Divisions *a*, *b*, *c*, *d* refer to observations with the 114" camera, division *e* to those with the 32" camera. In division *a* both components have intensities in ADAMS' scale  $\leq 6$ , in division *b* one of the components has intensity  $\geq 7$ , in division *c* both intensities are  $\geq 7$ . Division *d* is the sum of *a*, *b*, *c*. In division *e* all intensities are plotted. The hatched parts refer to differences between components not fully resolved according to ADAMS.

### 3. The limits of resolution of the absorption lines.

For the interpretation of the observed velocity distribution it is necessary to know the resolving power of the observational equipment. ADAMS, in his first paper, states that the limit of resolution of lines of moderate intensity upon the spectrograms is about 0.10 Å. This corresponds to a velocity difference of about 8 km/sec. A more precise estimate can be obtained from the frequency of the differences between the velocities of consecutive components in stars with multiple lines as given in ADAMS' list. Figure 1 shows the frequency of these differences arranged into three groups according to the intensities of the components. Divisions *a*, *b*, *c* and *d* refer to observations with the 114" camera (dispersion 2.9 Å/mm) and division *e* to those with the 32" camera (dispersion about 10 Å/mm). The distributions *a*, *b* and *c* do not reveal a dependence of the resolving power on the intensities of the components.

Evidently, separation of absorption lines with velocities differing by less than 9 km/sec occurs only in very rare cases. Accordingly we shall assume 9 km/sec to be the lower limit of the resolving power. The chance for lines differing by more than this amount to be separated will increase with the difference itself. In a number of cases ADAMS lists components which could not be fully resolved. Such cases can be recognized because the velocity of the blend of these two components is also given, in parentheses, in ADAMS' list. In the present investigation we have always chosen in these cases the velocities of the individual components and not that of the blend. The differences between these not fully separated components contribute the hatched part of the statistics of Figure 1.

On the basis of the statistics of Figure 1 we shall assume in the case of the observations with the 114" camera, on which the discussion in the following sections will be based, that:

- no absorption lines differing less than 9.0 km/sec are resolved;
- half of the lines differing between 9.0 and 18.0 km/sec are resolved;
- lines differing more than 18.0 km/sec are always resolved.

Assumption b) will be justified in section 5 (page 468) where the statistics of the consecutive differences for the stars in the main group (distances  $< 500$  parsecs) is shown to agree with the predicted statistics in the case of the most plausible form of the velocity distribution (see Table 7).

### 4. Division of the stars according to their distances; exclusion of special regions.

The residual velocities obtained after the elimination of the solar motion still contain the effect of differential galactic rotation. In order to eliminate this as far as is possible before studying the peculiar motions, the rotational effect for half the distance of the stars has been subtracted. Distances for most of the stars were derived from unpublished luminosity classifications by Dr W. W. MORGAN provisionally calibrated by means of proper motions<sup>1)</sup>, and taking into account the interstellar absorption estimated by means of STEBBINS and WHITFORD's colour excesses<sup>2)</sup>. For a few stars distance moduli were taken from the list of Miss RAMSEY<sup>3)</sup>.

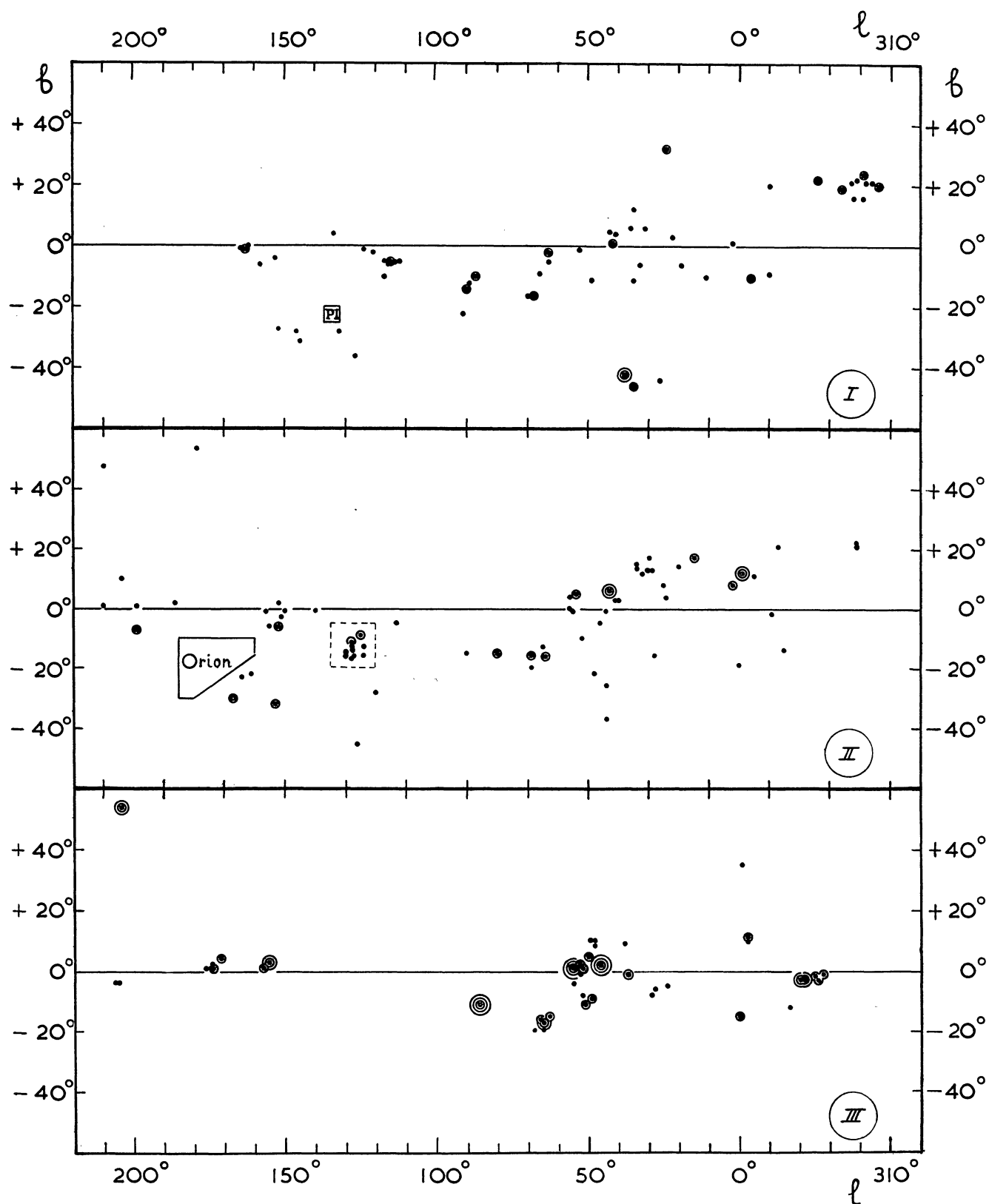
The discussion of the peculiar cloud velocities is based exclusively on the observations with the 114" camera (dispersion 2.9 Å per mm). For 24 of the 231 stars in this category no reliable distances were available. The remaining 207 stars were divided into three

<sup>1)</sup> See the Draft Report of Commission 33 of the I.A.U. (1952).

<sup>2)</sup> *Ap. J.* **91**, 20; *Mt Wilson Cont.* No. 621 (1940).

<sup>3)</sup> *Ap. J.* **111**, 434 (1950).

FIGURE 2



Distribution of the stars of groups I, II and III according to galactic longitude and latitude. The upper division contains the stars of group I (within 220 parsecs), the middle one those of group II (220 to 500 parsecs) and the lower one group III ( $\geq 500$  parsecs). Dots refer to stars with a single component of the K line, dots with one circle to double lines, dots with two circles to triple lines etc. The square with "Pl" in the upper division marks the region of the Pleiades.

groups according to the distance moduli: group I contains the stars with  $m_0 - M \leq 6.7$ , i.e. those within 220 parsecs; group II those with  $m_0 - M = 6.8$  to 8.4 (220 to 500 parsecs); group III those with  $m_0 - M \geq 8.5$  ( $\geq 500$  parsecs). The distribution on the sky of the stars in each of these groups is shown in Figure 2. The numbers of components are indicated by the numbers of circles. The crowded regions of Orion (26 stars) and the Pleiades (9 stars) are not represented. Omitting the Orion region with its exceptionally large number of interstellar components — a feature commented upon by ADAMS at several places — we observe that the frequency of multiple lines is much higher in the distant group III than in groups I and II. In group I, containing 69 stars including the Pleiades, there are 14 stars with double lines and one with triple lines. Group II, with 71 stars, has 12 double lines and 2 triple ones. Group III, with 43 stars, has 16 double, 5 triple and 3 quadruple lines. The distribution of the dots reflects, of course, the distribution of the bright early-type stars.

Within each of the groups I and II there is little or no tendency of the stars with multiple lines to occur in special regions of the sky. ADAMS has remarked that the most complex lines are found in Orion, Sagittarius and Cygnus; for the latter two regions this would seem to be largely an effect of distance. As will be shown below, the more complex structure of the lines for these regions may be due mainly to the larger number of clouds between the sun and these distant stars. This may be also the main cause of the high intensity of the components in these regions. The uniform distribution of the complex lines with respect to the single ones in groups I and II as shown in Figure 2, justifies the assumption that the motions of the interstellar gas clouds outside the direction of Orion may be described, as a first approximation, on the basis of a uniform value of the dispersion of the velocities, independent of galactic longitude.

In a few cases systematic motions of the clouds are observed in limited regions of the sky. ADAMS has drawn attention to such motions in Orion and in the Pleiades. Another rather conspicuous case is in Perseus. The clouds observed in the group of stars about  $l = 128^\circ$ ,  $b = -15^\circ$  in group II all show systematic displacements about  $+4.6$  km/sec. Most of these stars belong to the  $\zeta$  Persei group<sup>1)</sup>. This region is indicated by the dashed square in Figure 2. This outward velocity is a general feature of the cloud motions throughout the region at negative latitudes between longitudes  $110^\circ$  and  $170^\circ$  and was noticed also by MERRILL and SANFORD<sup>2)</sup>. Table 2 shows the distribution of the residual velocities observed in the regions

of the  $\zeta$  Per group, the Pleiades, the Orion region, and the surrounding area at latitudes below  $-20^\circ$ .  $N$  is the number of components,  $\bar{w}$  the mean intensity per component expressed in .01 equivalent Angstrom (see page 464).

TABLE 2

| Limits of residual velocity (km/sec) | region of $\zeta$ Per group |           | Pleiades |           | Orion |           | Remaining stars in region $l = 110^\circ$ to $170^\circ$<br>$b = -20^\circ$ to $-50^\circ$ |           |
|--------------------------------------|-----------------------------|-----------|----------|-----------|-------|-----------|--------------------------------------------------------------------------------------------|-----------|
|                                      | $N$                         | $\bar{w}$ | $N$      | $\bar{w}$ | $N$   | $\bar{w}$ | $N$                                                                                        | $\bar{w}$ |
| -27 to -24                           |                             |           |          |           | 3     | 5.1       |                                                                                            |           |
| -24 -21                              |                             |           |          |           | 4     | 4.5       |                                                                                            |           |
| -21 -18                              |                             |           |          |           | 4     | 3.9       |                                                                                            |           |
| -18 -15                              |                             |           |          |           | 3     | 1.9       | I                                                                                          | I.0       |
| -15 -12                              |                             |           |          |           | 3     | 2.8       |                                                                                            |           |
| -12 -9                               |                             |           |          |           | 4     | 4.6       |                                                                                            |           |
| -9 -6                                |                             |           |          |           | 1     | 0.5       |                                                                                            |           |
| -6 -3                                |                             |           |          |           | 8     | 2.8       |                                                                                            |           |
| -3 0                                 |                             |           |          |           | 2     | 4.8       | I                                                                                          | 5.0       |
| 0 +3                                 |                             |           |          |           | 4     | 7.2       | I                                                                                          | 5.0       |
| +3 +6                                | 7                           | 12.7      | 3        | 1.9       | 13    | 8.4       | 3                                                                                          | 6.7       |
| +6 +9                                | 2                           | 13.1      | 5        | 2.0       | 5     | 10.3      | 5                                                                                          | 5.5       |
| +9 +12                               |                             |           | 1        | 0.5       |       |           |                                                                                            |           |
| +12 +15                              | I                           | 3.7       |          |           |       |           |                                                                                            |           |
| +15 +18                              |                             |           |          |           |       |           |                                                                                            |           |
| +18 +21                              | I                           | I.0       |          |           |       |           |                                                                                            |           |

Excepting the Orion region we find that the majority of the components have velocities between  $+3$  and  $+9$  km/sec. The Orion region shows this concentration of positive velocities superposed on the very wide scatter of velocities occupying the whole range from  $-27$  to  $+9$  km/sec. Inclusion of these regions with pronounced systematic motions in the general statistics of the velocities of groups I and II would present some difficulties. The inclusion would give undue weight to the small but rather crowded regions of the Pleiades and the  $\zeta$  Persei group — an objection that could be met by attaching less weight to their contributions. But a more important difficulty is that the method of analysis, to be applied to this statistics in order to take fully into account the combining effect of the limited resolving power on the spectrograms, is based on the assumption that the motions of the various clouds along a line of sight are distributed at random with respect to the zero velocity left after elimination of the mean solar motion and of the effect of galactic rotation. We have therefore decided to include in the statistics only the numbers of the last column of Table 2.

In the case of the directions of the Pleiades and the  $\zeta$  Persei group the observed distribution of the velocities as well as the mean intensity per component resemble that for the remaining stars of groups I and II when shifted by an amount of  $+6$  km/sec. This means that the state of motion of the clouds in these direc-

<sup>1)</sup> B.A.N. No. 363, 29 (1944).

<sup>2)</sup> *Ap. J.* **87**, 118; *Mt Wilson Cont.* No. 585 (1938).

tions can be described by about the same mean speed,  $\eta = 5$  km/sec, as found for groups I and II, but with respect to an average outward velocity of  $+6$  km/sec.

The Orion region with its entirely abnormal distribution of velocities and mean intensities would deserve a special study, and has also been excluded from the statistics discussed below.

In some cases, listed by ADAMS, components with approximately equal velocities occur in neighbouring stars. If these velocities are high, we probably deal with identical clouds. WHIPPLE, in discussing ADAMS' provisional data for 50 stars, attempted to identify the clouds on the basis of this resemblance of velocities. In the present investigation no such attempt will be made. It is clear that, for the determination of the dispersion of the cloud velocities and the number of clouds per 1000 parsecs along the line of sight, it does not matter whether the same cloud enters more than once in the frequency distribution of velocities derived from all stars. We might select the recognizable cases of identical clouds and count these only once in the statistics,

but, as these are usually the outstanding velocities, this would result in a selection according to the size of the velocities. For the low velocities, as will be shown below, the many nearly identical small values must be due in many cases to the combining effect of the limited resolving power on the spectrograms. Knowledge of the areas of the sky covered by the individual clouds will be of importance for the determination of their sizes; however, this problem is not considered in this paper.

Some data on the three groups after the exclusion announced above are collected in Table 3. The mean distance of the stars of group II is twice that of group I, but the amount of interstellar matter in front of these stars appears to be about the same for the two groups, as the mean values of the equivalent width of the  $K$  line per star as well as the mean colour excesses are nearly equal. For group III these mean values are larger but the mean colour excess is not at all proportional to the distance. The colour excesses are in the scale of *Mt Wilson Cont.* No. 621 with corrections ap-

TABLE 3

|                                                          | I<br>$m_o - M \leq 6.7$ | II<br>$m_o - M = 6.8$ to 8.4 | III<br>$m_o - M \geq 8.5$ |
|----------------------------------------------------------|-------------------------|------------------------------|---------------------------|
| mean distance of stars                                   | 170 ps                  | 310 ps                       | 870 ps                    |
| number of stars                                          | 59                      | 61                           | 43                        |
| number of components                                     | 75                      | 75                           | 80                        |
| mean equivalent width of K line<br>in .01 A; a) per star | 8.5                     | 10.6                         | 25.8                      |
| b) per component                                         | 6.7                     | 8.6                          | 14.0                      |
| mean colour excess                                       | +0.065                  | +0.073                       | +0.108                    |

plied to the normal colours varying from  $-0.04$  for Bo to  $-0.02$  for B7.

A comparison with the data for stars at the same distances in general showed that ADAMS' stars in group I have somewhat higher colour excesses and stronger interstellar absorption lines than the average for stars at their distances, whereas the stars in group II are somewhat below the average. When taken together, the stars of groups I and II can be considered to be representative for stars at their mean distance with respect to colour excess and interstellar line absorption. The mean colour excess for group III is certainly not representative for the stars at that distance in general. Taking all stars in STEBBINS and WHITFORD's catalogue between distance moduli 8.5 and 10.6 and between declinations  $-30^\circ$  and  $+50^\circ$  — a choice which roughly corresponds with group III — we find from 421 stars an average colour excess of  $+0.15$  compared to  $+0.11$  for group III. Moreover, this mean value  $+0.15$  must be expected to be below the average of all stars at that same distance, because the most reddened,

heavily obscured stars will not occur in the Henry Draper catalogue from which the stars were selected.

The equivalent widths used in the discussion below were derived from ADAMS' estimated intensities by means of an average relation between these intensities and equivalent width. This relation was determined with the aid of the K-line intensities published by MERRILL, SANFORD, WILSON and BURWELL<sup>1)</sup>; WIL-

TABLE 4

Relation between ADAMS' intensities and equivalent widths  
(unit .01 A)

| ADAMS | w   | ADAMS | w   | ADAMS | w  | ADAMS | w  | ADAMS | w  |
|-------|-----|-------|-----|-------|----|-------|----|-------|----|
| 1     | 1   | 5     | 7.3 | 9     | 15 | 13    | 23 | 18    | 33 |
| 2     | 2.3 | 6     | 9.2 | 10    | 17 | 14    | 24 | 20    | 36 |
| 3     | 3.7 | 7     | 11  | 11    | 19 | 15    | 26 | 25    | 47 |
| 4     | 5   | 8     | 13  | 12    | 21 | 16    | 29 | 30    | 57 |

<sup>1)</sup> *Ap. J.* **86**, 274; *Mt Wilson Cont.* No. 576 (1937).

LIAMS<sup>1)</sup>; and SPITZER, EPSTEIN and LI HEN<sup>2)</sup>, and is shown in Table 4. As ADAMS' estimates may contain large accidental errors, the equivalent widths thus obtained are rather uncertain and have only statistical value.

5. *Comparison of the observed distribution of velocities with a simple model.*

The observed distribution of the residual K-line velocities without regard to sign for groups I and II together is shown in Tables 5a and 5b and by the smooth curve in Figure 3. The distributions for groups I and II appeared to be very similar; this agrees with our former conclusion that the amount of interstellar material in front of the stars of these two groups is approximately the same. More than half of the components have residual velocities between  $-3$  and  $+3$  km/sec. The preference for this narrow range of velocities is even more pronounced for the components with intensities exceeding 5 in ADAMS' scale, whose contributions to the total frequencies are represented by the dashed curve. In group III the concentration towards the small velocities is less, and the proportion of components with high intensities is much larger. This distribution is shown in Figure 4 and Table 6.

We want to compare these observed distributions with those predicted on the basis of a given distribution of the cloud velocities and making allowance for the blending effect on the spectrograms. Two assumptions will be considered with regard to the true distribution:

A. The velocities have a gaussian distribution with root mean square radial velocity  $\sigma$ .

B. The frequency of the velocities,  $v$ , is proportional to  $e^{-|v|/\eta}$ , where  $\eta$  is the mean radial velocity without regard to sign.

In order to take into account the blending according to the rule defined in section 3, we proceed as follows. For half of the spectrograms we assume that, if there are  $n$  absorption lines (due to  $n$  clouds) and only one of the  $n-1$  differences between consecutive lines exceeds 9 km/sec, this large difference will cut the group of  $n$  lines in two parts and two blends will be produced; if two of the differences between consecutive lines exceed 9 km/sec, there will be three blends, etc. The word blend as used here includes also single lines if they are not combined with neighbouring ones. For the other half of the spectrograms the same procedure is followed with the limit of resolution 18 km/sec instead of 9 km/sec. Next the results of the two groups of spectrograms are added.

<sup>1)</sup> *Ap. J.* 79, 280; *Mt Wilson Cont.* No. 487 (1934).

<sup>2)</sup> *Ann. d'Astroph.* 13, 147 (1950).

For the cases  $n=2$  and  $n=3$  the frequency distribution of velocities of the blends which will result from this procedure has been rigorously computed. It is assumed that the velocity of a blend is the straight average of the individual velocities of the lines included in the blend. We have neglected the fact that the velocity of the blend actually will be a weighted mean of the individual velocities, the weights depending on the intensities of the individual lines. This is important in so far as the dispersion of velocities may be lower for the large clouds causing lines of higher intensity than for the smaller clouds causing lines of low intensity.

Particulars about the computations for  $n=2$  and  $n=3$  are given in the next section. Results based on hypothesis B with  $\eta=5$  km/sec and  $n=3$ , the case which will be shown to fit the observations for groups I+II, are illustrated in Figure 5. The lower diagram refers to the group of spectrograms for which the limit of separation is 9 km/sec. The three dashed curves represent the distributions of velocities produced by blends of three lines, by blends of two lines and by single lines. The drawn curve is the sum of these three. The upper diagram shows the results for the case when the limit of separation is 18 km/sec. The mean of the drawn curves of the two diagrams gives the final frequencies of velocities for  $n=3$ ,  $\eta=5$  km/sec and these, after multiplication by the number of stars, give the calculated velocity distribution which is to be compared with the observed one. As will be seen in the upper diagram, the numbers of velocities due to blends of two lines or to single lines in the total velocity distribution is almost negligible for spectrograms on which the limit of separation is 18 km/sec. The dotted curve represents the frequency distribution that would be observed in the case of a complete resolution of all lines; it is, of course, the same in the two diagrams.

For higher values of  $n$  no rigorous calculations were made. In order to arrive rapidly at a determination of the approximate shape of these distributions, we have confined ourselves to obtaining these in an empirical way. Use was made of large collections of paper slips on each of which was written a number, in such a way that the frequency of these numbers shows a distribution which for one collection corresponds to hypothesis A and for the other to hypothesis B. I am indebted to Dr H. J. LAMMERS of the Laboratory of Anatomy of the University of Leyden for putting the first-mentioned collection at my disposal. By drawing from these collections random samples of  $n$  slips, artificial distributions of numbers were obtained, playing the role of the distributions of velocities of individual clouds in the spectrum of a star. To these samples the rule concerning the combination into blends and the determination of their velocities as described above was applied. The method is admittedly rough, but it

TABLE 5a

Observed and computed numbers of residual velocities for groups I+II (stars within 500 parsecs).  
Hypothesis A (gaussian velocity distribution).

| Limits of velocity (km/sec) | Obs. numbers and mean intensities |           | $n=3$               |       | $n=5$               |       |                     |       | $n=9$               |       |           |                   |
|-----------------------------|-----------------------------------|-----------|---------------------|-------|---------------------|-------|---------------------|-------|---------------------|-------|-----------|-------------------|
|                             |                                   |           | $\sigma=5.5$ km/sec |       | $\sigma=6.0$ km/sec |       | $\sigma=8.2$ km/sec |       | $\sigma=8.2$ km/sec |       |           |                   |
|                             | $N$                               | $\bar{w}$ | $C$                 | $O-C$ | $S$                 | $O-S$ | $S$                 | $O-S$ | $S$                 | $O-S$ | $\bar{m}$ | $\bar{w}/\bar{m}$ |
| 0.0 to 2.9                  | 82                                | 9.0       | 74.3                | +7.7  | 76.0                | +6.0  | 58.8                | +23.2 | 75.6                | +6.4  | 8.8       | 1.0               |
| 3.0 „ 5.9                   | 29                                | 9.5       | 37.9                | -8.9  | 33.2                | -3.2  | 37.2                | -8.2  | 28.8                | +0.2  | 8.6       | 1.1               |
| 6.0 „ 8.9                   | 14                                | 6.9       | 14.1                | -0.1  | 12.8                | +1.2  | 24.0                | -10.0 | 12.6                | +1.4  | 8.0       | 0.9               |
| 9.0 „ 11.9                  | 5                                 | 2.0       | 6.2                 | -1.2  | 5.2                 | -0.2  | 15.6                | -10.6 | 9.6                 | -4.6  | 4.7       | 0.4               |
| 12.0 „ 14.9                 | 5                                 | 1.4       | 2.7                 | +2.3  | 5.2                 | -0.2  | 13.2                | -8.2  | 6.0                 | -1.0  | 1.3       | 1.1               |
| 15.0 „ 17.9                 | 5                                 |           | 0.8                 | +4.2  | 1.2                 | +3.8  | 5.6                 | -0.6  | 3.0                 | +2.0  |           |                   |
| 18.0 „ 20.9                 | 4                                 |           | 0.4                 | +3.6  | 0.8                 | +3.2  | 4.0                 | 0.0   | 5.4                 | -1.4  |           |                   |
| 21.0 „ 23.9                 | 2                                 |           | 0.1                 | +1.9  | 0.0                 | +2.0  | 1.6                 | +0.4  | 2.4                 | -0.4  |           |                   |
| $\geq 24.0$                 | 4                                 |           | 0.1                 | +3.9  | 0.0                 | +4.0  | 0.8                 | +3.2  | 3.6                 | +0.4  |           |                   |

TABLE 5b

Observed and computed numbers of residual velocities for groups I+II. Hypothesis B ( $e^{-|v|/\eta}$ ).

| Limits of velocity<br>(km/sec) | Obs. numbers<br>and mean<br>intensities |           | $n=3$             |       |           |                   | $n=5$             |         |         |         |                   |         |         |         |                 |                   | Poisson<br>$\bar{n}=3$ |  |
|--------------------------------|-----------------------------------------|-----------|-------------------|-------|-----------|-------------------|-------------------|---------|---------|---------|-------------------|---------|---------|---------|-----------------|-------------------|------------------------|--|
|                                |                                         |           | $\eta=5.0$ km/sec |       |           |                   | $\eta=4.3$ km/sec |         |         |         | $\eta=6.0$ km/sec |         |         |         | $\eta=5$ km/sec |                   | $\eta=5$ km/sec        |  |
|                                | $N$                                     | $\bar{w}$ | $C$               | $O-C$ | $\bar{m}$ | $\bar{w}/\bar{m}$ | $\bar{S}$         | $O-S_1$ | $O-S_2$ | $O-S_3$ | $\bar{S}$         | $O-S_1$ | $O-S_2$ | $O-S_3$ | $\bar{m}$       | $\bar{w}/\bar{m}$ | $S$                    |  |
| 0.0 to 2.9                     | 82                                      | 9.9       | 73.5              | +8.5  | 2.7       | 3.3               | 94                | -13     | -17     | -7      | 74                | +13     | +3      | +9      | 4.8             | 1.9               | 69.5                   |  |
| 3.0 „ 5.9                      | 29                                      | 9.5       | 34.3              | -5.3  | 2.6       | 3.6               | 23                | +9      | +10     | 0       | 36                | -9      | -4      | -8      | 4.6             | 2.1               | 33.5                   |  |
| 6.0 „ 8.9                      | 14                                      | 6.9       | 13.8              | +0.2  | 2.3       | 3.0               | 4                 | +8      | +10     | +12     | 12                | -1      | +4      | +3      | 3.6             | 1.9               | 14.2                   |  |
| 9.0 „ 11.9                     | 5                                       | 2.0       | 8.5               | -3.5  | 1.8       | 1.1               | 3                 | +4      | +1      | +2      | 6                 | -2      | 0       | 0       | 1.5             | 1.3               | 7.2                    |  |
| 12.0 „ 14.9                    | 5                                       | 1.4       | 5.9               | -0.9  | 1.1       | 1.3               | 4                 | 0       | +1      | +1      | 7                 | 0       | -3      | -2      | 1.1             | 1.2               | 5.4                    |  |
| 15.0 „ 17.9                    | 5                                       |           | 4.0               | +1.0  |           |                   | 5                 | +1      | -1      | -1      | 5                 | -1      | 0       | 0       |                 |                   | 4.5                    |  |
| 18.0 „ 20.9                    | 4                                       |           | 2.7               | +1.3  |           |                   | 4                 | 0       | +2      | -1      | 7                 | -4      | -3      | -2      |                 |                   | 3.5                    |  |
| 21.0 „ 23.9                    | 2                                       |           | 1.9               | +0.1  |           |                   | 1                 | 0       | +2      | +2      | 7                 | -3      | -6      | -6      |                 |                   | 2.7                    |  |
| > 24.0                         | 4                                       |           | 2.6               | +1.4  |           |                   | 0                 | +3      | +4      | +4      | 9                 | -8      | -1      | -5      |                 |                   | 3.2                    |  |

TABLE 6

Observed and computed numbers of residual velocities for group III (distances  $\geq 500$  parsecs).

| Limits of velocity (km/sec) | Obs. numbers and mean intensities |           | Hypothesis A               |       |           |                   | Hypothesis B $n=5$ |       |                   |       |           |                   |
|-----------------------------|-----------------------------------|-----------|----------------------------|-------|-----------|-------------------|--------------------|-------|-------------------|-------|-----------|-------------------|
|                             |                                   |           | $n=9$ $\sigma=10.0$ km/sec |       |           |                   | $\eta=6.0$ km/sec  |       | $\eta=8.2$ km/sec |       |           |                   |
|                             | $N$                               | $\bar{w}$ | $S$                        | $O-S$ | $\bar{m}$ | $\bar{w}/\bar{m}$ | $S$                | $O-S$ | $S$               | $O-S$ | $\bar{m}$ | $\bar{w}/\bar{m}$ |
| 0.0 to 2.9                  | 27                                | 21.1      | 23.7                       | +3.3  | 8.5       | 2.5               | 26.4               | +0.6  | 21.3              | +5.7  | 21.1      | 4.9               |
| 3.0 „ 5.9                   | 13                                | 22.3      | 9.3                        | +3.7  | 8.2       | 2.7               | 12.9               | +0.1  | 13.3              | -0.3  | 22.3      | 5.3               |
| 6.0 „ 8.9                   | 8                                 | 12.4      | 7.9                        | +0.1  | 7.3       | 1.7               | 4.2                | +3.8  | 8.0               | 0.0   | 12.4      | 3.6               |
| 9.0 „ 11.9                  | 7                                 | 9.1       | 3.9                        | +3.1  | 5.7       | 1.6               | 2.2                | +4.8  | 4.7               | +2.3  | 9.1       | 3.6               |
| 12.0 „ 14.9                 | 5                                 | 6.0       | 3.8                        | +1.2  | 3.2       | 1.9               | 2.4                | +2.6  | 4.7               | +0.3  | 6.0       | 4.1               |
| 15.0 „ 17.9                 | 3                                 | 4.2       | 2.6                        | +0.4  | 2.4       | 1.7               | 1.9                | +1.1  | 1.8               | +1.2  | 4.2       | 2.8               |
| 18.0 „ 20.9                 | 3                                 | 2.9       | 4.9                        | -1.9  | 1.3       | 2.2               | 2.4                | +0.6  | 2.6               | +0.4  | 2.9       | 2.4               |
| 21.0 „ 23.9                 | 4                                 | 3.9       | 1.5                        | +2.5  | 1.3       | 3.0               | 2.5                | +1.5  | 3.3               | +0.7  | 3.9       | 3.1               |
| $\geq 24.0$                 | 10                                | 3.0       | 3.9                        | +6.1  | 1.0       | 3.0               | 3.2                | +6.8  | 9.5               | +0.5  | 3.0       | 2.8               |

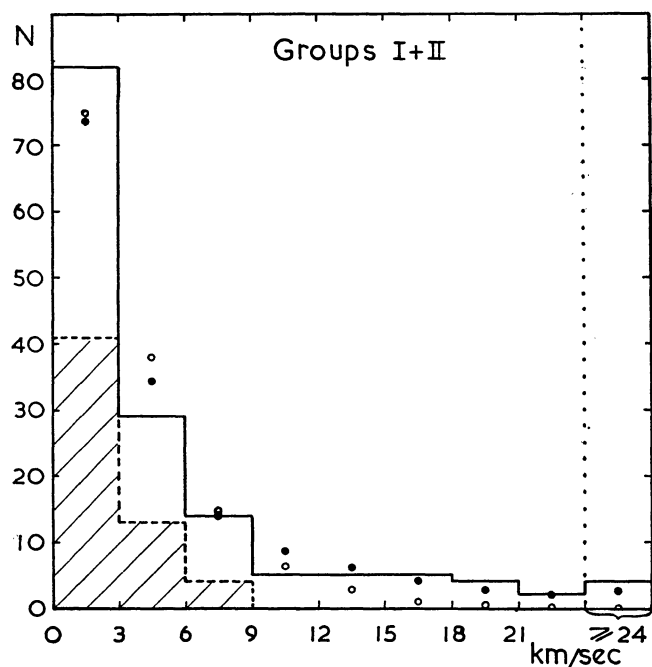
has the advantage of giving in an empirical way also an impression of the random fluctuations that may occur in the numbers of velocities for a given number of stars and for given values of  $n$ ,  $\sigma$  and  $\eta$  and of the limit of resolution.

#### Groups I and II.

The observed distribution of velocities for groups I+II is compared with those derived from the rigorous calculations and from samples in Tables 5a and 5b and in Figure 3.

In the case of hypothesis A, the shape of the theoretical distribution differs more and more from that of the error curve according as  $n$  increases, and the change goes in the direction of better fit to the observed peak at the low velocities. The distribution computed for  $n=3$  and  $\sigma=5.5$  km/sec is shown in the third division of Table 5a and in Figure 3 (open circles). There is a considerable excess in the observed numbers of high velocities. This can be reduced somewhat by an increase of  $\sigma$ , but then the number of small velocities becomes too low.

FIGURE 3



The smooth curve represents the observed numbers of velocities in intervals of 3 km/sec in groups I + II. The dashed curve represents the contribution due to the components with intensities  $\geq 6$  in ADAMS' scale.

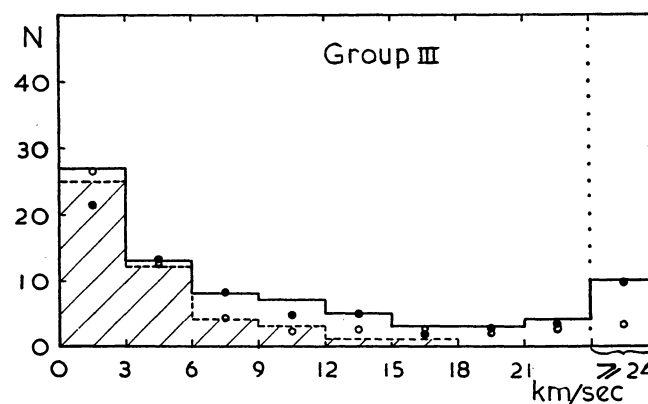
Circles: numbers computed on the basis of hypothesis A with  $n=3$ ,  $\sigma=5.5$  km/sec. Dots: numbers based on hypothesis B with  $n=3$ ,  $\eta=5.0$  km/sec.

The distributions for  $n=5$ ,  $\sigma=6.0$  km/sec and  $\sigma=8.2$  km/sec, derived from samples, are given under  $S$  in the fourth and fifth divisions of Table 5a. For  $n=5$ ,  $\sigma=6.0$  the observed numbers of small velocities are well represented but the calculated numbers of high velocities are too small. The case  $n=5$ ,  $\sigma=8.2$  gives a good representation of these latter, but does not fit the numbers of small velocities. A compromise value  $\sigma=7.0$  is the least unsatisfactory for  $n=5$ .

For  $n=9$ ,  $\sigma=8.2$  km/sec we get the numbers in the fifth division. Although here the residuals  $O-S$  are small and not systematic, there is reason to consider  $n=9$  as perhaps too large. Evidence against larger

values of  $n$  follows from the observed mean intensities  $\bar{w}$  per observed component (expressed in .01 equivalent A) in the second division of Table 5a. As a by-product of the calculations of a velocity distribution we can derive the theoretical mean number of clouds contributing to one blend for each of the intervals of velocity. Denoting this mean number by  $\bar{m}$ , and neglecting the deviation of the curve of growth from linearity in the region of small intensities with which we deal here, we may consider  $\bar{w}/\bar{m}$  as the average K-line intensity due to a single cloud. We expect this average

FIGURE 4



Observed numbers of velocities in group III (smooth curve). Dashed curve: contribution due to components with intensities  $\geq 6$ . Circles: distribution derived from samples for  $n=5$ ,  $\eta=6.0$  km/sec; dots: distribution from samples for  $n=5$ ,  $\eta=8.2$  km/sec; both are based on hypothesis B.

intensity to be below the average for clouds producing the components of high velocity and to exceed the average for components of low velocity. However, for  $n > 9$  the quotients  $\bar{w}/\bar{m}$  run in the other direction, and for that reason  $n=9$  must be considered as an upper limit to  $n$ . We conclude that the best fit on hypothesis A is obtained for  $n$  between 5 and 9, but it is not quite satisfactory.

Definitely better agreement is obtained on the basis of hypothesis B. The values  $n=3$ ,  $\eta=5.0$  km/sec give the distribution under C in the third division of Table 5b, and this case is also shown in Figure 3 (dots). The run of the residuals  $O-C$  as well as that of the quotients  $\bar{w}/\bar{m}$  is satisfactory. A smaller value,  $n=2$ , does not give the desired fit. On the other hand,  $n=5$  works out quite well. In the fourth and fifth divisions of Table 5b are, under  $\bar{S}$ , the numbers obtained as a mean of three samples each based on 100 "stars" and reduced to 120 stars, the number in groups I + II; the differences  $O-S$  for each sample being given next to  $\bar{S}$ . The samples were analysed for  $\eta=4.3$  and 6.0 km/sec and the numbers illustrate the change in the resulting distribution due to a variation of  $\eta$  from 4.3

to 6.0 km/sec, as well as the order of magnitude of the random fluctuations of the numbers of velocities in the various intervals. The mean of these two distributions, corresponding with  $n = 5$ ,  $\eta = 5$  km/sec would fit the observed numbers very well. The quotients  $\bar{w}/\bar{m}$  in the next division are the mean of those corresponding with the two preceding cases for  $n = 5$ . For higher values of  $n$  we again run into improbable values of  $\bar{w}/\bar{m}$ .

Summarizing we may state that, for groups I + II:

- Hypothesis B (velocity distribution  $e^{-|v|/\eta}$ ) represents the observations very well, whereas the gaussian distribution gives only fair agreement.
- The most probable value of  $\eta$  is 5.0 km/sec, the uncertainty being of the order of 1 km/sec.
- The mean number of clouds per star may be from 3 to 6, each of these fitting satisfactorily the observed distributions.

The true number of clouds will not be the same for all stars in groups I + II, as was assumed in the above calculations. Strictly, we should have compared the observed distribution with a proper combination of theoretical distributions for various values of  $n$ . In order to check to what extent this would change the above conclusions we have estimated with the aid of samples the theoretical distribution to be expected in case the numbers of clouds per star are given by Poisson's formula with a mean value  $\bar{n} = 3$ . The computations were done only for hypothesis B with  $\eta = 5$  km/sec, the value arrived at above. The result is shown in the last division of Table 5b. The new distribution differs very little from that for  $n = 3$  given in the third division of the table. The same holds when a combination of distributions with mean value  $\bar{n} = 5$  is compared with the single case  $n = 5$ . The conclusion with regard to the mean number of clouds per star apparently remains valid.

Justification of the assumptions of section 3 (page 461), concerning the resolving power on the spectrograms can be given as follows. From samples based on hypothesis B with  $n = 3$  and  $\eta = 5$  km/sec — the case which fits the observations for groups I + II — we have derived the statistics of the differences between consecutive velocities of the blends in the hypothetical stars. These are compared with the observed statistics. The samples were analysed separately for the limits of resolution 9 and 18 km/sec. Each sample is based on 300 "stars" and reduced to the number of stars, 120, in groups I + II. The second column of Table 7 shows the observed numbers for different values of the differences. In the third and fourth columns are the numbers from samples for the two limits of resolution. The agreement of the mean of these two distributions, in the last column, with the observed numbers, lends

TABLE 7  
Observed and predicted statistics of differences between consecutive velocities of blends for groups I + II;  $n = 3$ ,  $\eta = 5$  km/sec.

| Value of the difference | Observed numbers | Numbers from samples         |                               |                       |
|-------------------------|------------------|------------------------------|-------------------------------|-----------------------|
|                         |                  | Limit of resolution 9 km/sec | Limit of resolution 18 km/sec | Mean of two preceding |
| 0 to 9 km/sec           | 1                | 0                            | 0                             | 0                     |
| 9 to 18 "               | 17               | 35.2                         | 0                             | 17.6                  |
| $\geq 18$ "             | 12               | 11.2                         | 10.8                          | 11.0                  |

support to the assumptions of section 3, particularly to assumption b) according to which one half of the lines with differences in the interval 9 to 18 km/sec are separated.

#### Group III.

Similar comparisons as were made for groups I + II have been made for group III. The total number of components, 80, is only half that observed in groups I + II, and the conclusions to be derived from the comparisons therefore are necessarily somewhat vaguer. Besides it is doubtful whether the basic assumptions with regard to the state of motion of the clouds, and the procedure adopted in combining the individual absorption lines into blends, are still sufficiently applicable to this group. It represents a much larger volume of space and includes more clouds per star.

Comparisons based on hypothesis A show fair agreement for  $n = 9$  and  $\sigma$  between 10.0 and 12.9 km/sec, but the quotients  $\bar{w}/\bar{m}$  run unsatisfactorily. The case  $n = 9$ ,  $\sigma = 10.0$  km/sec is shown in Table 6. Hypothesis B leads again to more satisfactory results. The cases  $n = 5$ ,  $\eta = 6.0$  and  $n = 5$ ,  $\eta = 8.2$  km/sec are shown in Table 6, and in Figure 4. They are based on samples of 300 "stars". It appears that  $n$  may be between 4 and 7, and  $\eta$  about 8.2 km/sec.

The results,  $n =$  between 3 and 6 for groups I + II, and  $n =$  between 4 and 7 for group III may be compared with the mean K-line intensities per star given in Table 3. With  $n = 3$  for groups I + II and  $n = 7$  for group III we find an average K-line absorption of about .035 equivalent Angström per cloud. The number of clouds intersected by the line of sight probably is between 8 and 12 per 1000 parsecs.

The average velocity,  $\eta = 8.2$  km/sec for the region within 900 parsecs agrees roughly with determinations based on discussions of the change of equivalent width of interstellar lines with distance and galactic latitude. SPITZER <sup>1)</sup> found justification of an assumed root mean square velocity of 9 km/sec from a study of the Na D lines, but his discussion is based on the as-

<sup>1)</sup> *Ap. J.* **108**, 276 (1948).

sumption of a rectangular profile for the velocity distribution. MELNIKOV<sup>1)</sup>, in an earlier investigation, found 7 km/sec for the root mean square velocity from the D and the K lines, assuming a maxwellian velocity distribution.

The value  $\eta = 5.0$  km/sec for distances within 300 parsecs agrees with provisional observations of the 21-cm interstellar hydrogen emission line, the origin of which probably is also in the vicinity of the sun<sup>2)</sup>.

In the preceding analysis we have only superficially taken advantage of the information which data on line intensities might furnish with respect to the determination of the numbers of clouds per star. The available data on line intensities do not justify a more careful treatment. Of the 120 stars on which the discussion of groups I + II is based, 45 occur in lists of accurate photometry of the K line. The majority of these were published by SPITZER, EPSTEIN and LI HEN in the paper referred to above. However, these authors have selected their material according to criteria which make their measurements less useful for our purpose. Most of the stars common to groups I + II and the Princeton list were chosen for measurement because the estimated K-line intensity is 5 in ADAMS' scale. These data do not, therefore, allow the determination of representative values of the mean intensity  $\bar{w}$  which might replace those used in our discussion. Extension of the measurements to the stars with lower intensities would be very valuable.

The mean speed of the clouds in the direction of the line of sight,  $\eta = 5.0$  km/sec for the region up to distances of about 300 parsecs, and  $\eta = 8.2$  km/sec for the region up to 900 parsecs, is small compared to the limits of resolution found in section 3. As a consequence the majority of the observed components are actually blends of several absorption lines. For groups I + II, the total number of components, 150, is 42% of the number of clouds, if we assume  $n = 3$ . Thus, even for these relatively nearby stars, when observed with the most powerful spectroscopic equipment at present available, there would seem to be far from complete resolution of the interstellar K lines.

A final remark should be made in connection with an observed asymmetry in the distribution of the velocities of the K-line components. An excess of negative residual velocities was noticed which may not be explicable entirely as due to random selection. From all stars listed by ADAMS, including those observed with the 32" camera, we find the following numbers of high velocities:

<sup>1)</sup> A. J. U.S.S.R. 24, 73 (1947).

<sup>2)</sup> C. A. MULLER and J. H. OORT, *Nature* 168, No. 4270, 357 (1951).

|                               |               |
|-------------------------------|---------------|
| $\leq -24$ km/sec,            | 17 components |
| $-24$ km/sec to $-15$ km/sec, | 34 „          |
| $+15$ km/sec to $+24$ km/sec, | 13 „          |
| $\geq +24$ km/sec,            | 12 „          |

Although the evidence is not conclusive, we should bear in mind that perhaps part of the observed negative velocities may not be of interstellar origin but have some connection with the star itself, like for instance the components with negative velocities of the K line observed by MERRILL in the star HD 190073<sup>1)</sup>.

#### 6. The theoretical frequency distributions of the velocities for $n = 2$ and $n = 3$ .

We denote by

$v$ , the residual radial velocity of a cloud,  
 $\varphi(v) dv$ , the probability that  $v$  lies between  $v$  and  $v + dv$ ,  
 so that

$$\int_{-\infty}^{+\infty} \varphi(v) dv = 1,$$

$n$ , the number of clouds between the sun and the star.

The two hypotheses with regard to the form of  $\varphi(v)$  considered in the preceding section are:

$$A. \quad \varphi(v) dv = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{v^2}{2\sigma^2}} dv, \quad (1)$$

$$\text{or} \quad \frac{1}{V \sqrt{2\pi}} e^{-\frac{v^2}{2}} dv,$$

if  $v$  is expressed in  $\sigma$  as a unit,  $\sigma$  being the root mean square residual velocity;

$$B. \quad \varphi(v) dv = \frac{1}{2\eta} e^{-\frac{|v|}{\eta}} dv, \quad (2)$$

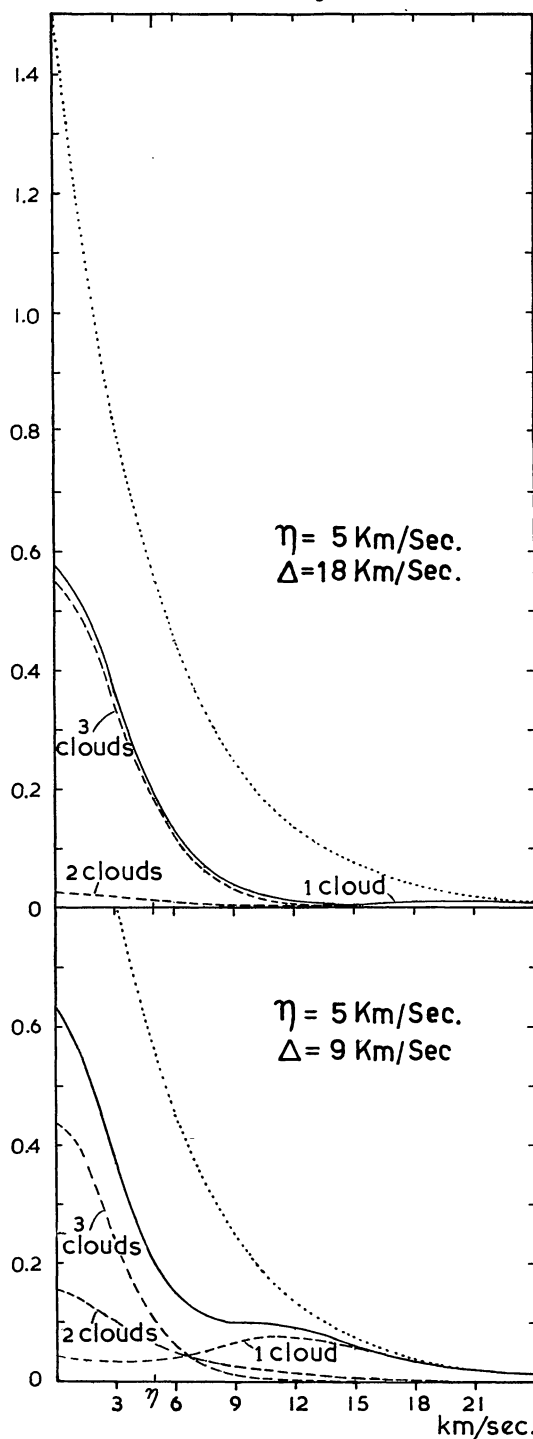
$$\text{or} \quad \frac{1}{2} e^{-|v|} dv,$$

if  $v$  is expressed in  $\eta$  as a unit,  $\eta$  being the mean speed.

We further denote by  $\Delta$  the limit of the resolving power on the spectrograms. Adopting the rules defined in sections 3 and 5, we assume that on half of the spectrograms all absorption lines can be separated which differ more than  $\Delta$  km/sec, but that lines are combined into a blend if they differ less than  $\Delta$  km/sec. For the other half of the spectrograms the same rule is adopted with the limit of resolution twice as large. The velocity of a blend is supposed to be the mean of the velocities of the lines included.

<sup>1)</sup> *Ap. J.* 113, 55 (1950).

FIGURE 5



Computed frequency distributions of velocities due to one cloud ( $P(v')$ ), to blends of two clouds ( $Q(v'')$ ) and to blends of three clouds ( $R(v''')$ ), in the case of 3 clouds per star. The drawn curves represent  $P + Q + R$ . The dotted line shows the frequency distribution of the individual cloud velocities given by  $\frac{3}{2} e^{-|v|/\eta}$  with  $\eta = 5$  km/sec. Upper diagram: limit of resolution is 18 km/sec; lower diagram: limit of resolution is 9 km/sec.

The frequency distribution of the velocities of blends and single lines is the sum of the distributions of the velocities due to one, two or three clouds. Each of these will be computed separately. The mean of the total distributions obtained for the two values of  $\Delta$  is the basis for the comparison with the observed distribution, discussed in the preceding section.

We shall express the velocities in  $\sigma$  or in  $\eta$  as a unit for the two hypotheses A and B, respectively, and accordingly introduce

$$p = \frac{\Delta}{\sigma} \quad \text{and} \quad q = \frac{\Delta}{\eta}.$$

We shall further denote by  $v_1, v_2, v_3$  the velocities of the clouds in the order of their distances from the sun, and by

$v'$ , velocities measured on the spectrograms and due to one cloud,

$v''$ , velocities of blends due to two clouds,

$v'''$ , velocities of blends due to three clouds.

The case  $n = 2$ .

Let  $F(v')$   $dv'$  denote the probability that a velocity  $v'$  be between  $v'$  and  $v' + dv'$ . This velocity may be either  $v_1$  or  $v_2$  and their probabilities are the same. We have:

$$F(v') = 2\varphi(v') \left( 1 - \int_{v' - \Delta}^{v' + \Delta} \varphi(v_1) dv_1 \right). \quad (3)$$

Let  $G(v'')$   $dv''$  denote the probability that a velocity  $v''$  be between  $v''$  and  $v'' + dv''$ . Suppose the velocity of the first cloud is between  $v_1$  and  $v_1 + dv_1$ . In order that the velocity of the blend be between  $v''$  and  $v'' + dv''$ , the velocity of the second cloud should be between  $2v'' - v_1$  and  $2v'' - v_1 + 2dv''$ . The probability that this situation will occur is  $2\varphi(v_1)\varphi(2v'' - v_1)dv''dv_1$ . Inte-

grating from  $v_1 = v'' - \frac{\Delta}{2}$  to  $v_1 = v'' + \frac{\Delta}{2}$  we get:

$$G(v'') = 2 \int_{v'' - \frac{\Delta}{2}}^{v'' + \frac{\Delta}{2}} \varphi(v_1) \varphi(2v'' - v_1) dv_1. \quad (4)$$

These expressions become:  
for hypothesis A:

$$F(v') = \sqrt{\frac{2}{\pi}} e^{-\frac{v'^2}{2}} \left[ 1 - \Omega\left(\frac{v' + p}{2}\right) + \Omega\left(\frac{v' - p}{2}\right) \right] \\ G(v'') = \frac{1}{\sqrt{\pi}} e^{-v''^2} \Theta\left(\frac{p}{2}\right), \quad (5)$$

where

$$\Theta(t) = \frac{2}{\sqrt{\pi}} \int_0^t e^{-x^2} dx, \quad \Omega(t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^t e^{-x^2} dx;$$

for hypothesis B:

$$F(v') = \frac{1}{2} e^{-q} \left( 1 + e^{-2|v'|} \right) \quad \text{for } |v'| \leq q$$

$$\frac{1}{2} e^{-|v'|} \left( 2 - e^{-|v'|+q} + e^{-|v'|-q} \right) \quad \text{for } |v'| \geq q, \quad (6)$$

$$G(v'') = \frac{1}{2} \left[ e^{-2|v''|} \left( 1 + 2|v''| \right) - e^{-q} \right] \quad \text{for } |v''| \leq \frac{1}{2} q$$

$$\frac{1}{2} q e^{-2|v''|} \quad \text{for } |v''| \geq \frac{1}{2} q.$$

In the case of hypothesis B we find the following simple expressions for the total contributions of the two velocity distributions:

$$\int_{-\infty}^{+\infty} F(v') dv' = (2 + q) e^{-q}$$

$$\int_{-\infty}^{+\infty} G(v'') dv'' = -\frac{1}{2} (2 + q) e^{-q} + 1. \quad (7)$$

The sum of these is the predicted average number of measured velocities per star:

$$1 + \frac{1}{2} (2 + q) e^{-q}. \quad (8)$$

Equations (7) satisfy the relation:

$$\int_{-\infty}^{+\infty} F(v') dv' + 2 \int_{-\infty}^{+\infty} G(v'') dv'' = 2.$$

The case  $n = 3$ .

Let  $P(v') dv'$  denote the probability that a velocity  $v'$  be between  $v'$  and  $v' + dv'$ . This velocity may be either  $v_1$ ,  $v_2$  or  $v_3$ . Each of these has the same probability and the total probability is

$$P(v') = 3 \varphi(v') \left( 1 - \int_{v'-\Delta}^{v'+\Delta} \varphi(v_1) dv_1 \right)^2. \quad (9)$$

Let  $Q(v'') dv''$  denote the probability that a velocity  $v''$  be between  $v''$  and  $v'' + dv''$ . The blend with this velocity may be due to the pairs of lines with velocities  $v_1, v_2$  or  $v_1, v_3$  or  $v_2, v_3$ , each of these being equally probable. The total probability is

$$Q(v'') = 3 G(v'') \left( 1 - \int_{v''-\frac{3}{2}\Delta}^{v''+\frac{3}{2}\Delta} \varphi(v_1) dv_1 \right) \quad (10)$$

$$+ 6 \int_{v''-\Delta}^{v''-\frac{3}{2}\Delta} \varphi(v_1) dv_1 \int_{2v''-v_1-\Delta}^{2v''-v_1+\Delta} \varphi(v_2) \varphi(2v''-v_2) dv_2$$

$$+ 6 \int_{v''+\frac{3}{2}\Delta}^{v''+\Delta} \varphi(v_1) dv_1 \int_{v_1-\Delta}^{v_1+\Delta} \varphi(v_2) \varphi(2v''-v_2) dv_2.$$

The first term of the right-hand member represents those cases where the velocity of one cloud, for instance  $v_1$ , differs more than  $\frac{3}{2}\Delta$  from  $v''$ , so that the remaining two velocities,  $v_2$  and  $v_3 = 2v'' - v_2$ , which form the blend, may assume any value within the range  $v'' - \frac{\Delta}{2}$  to  $v'' + \frac{\Delta}{2}$ . The second and third terms represent those

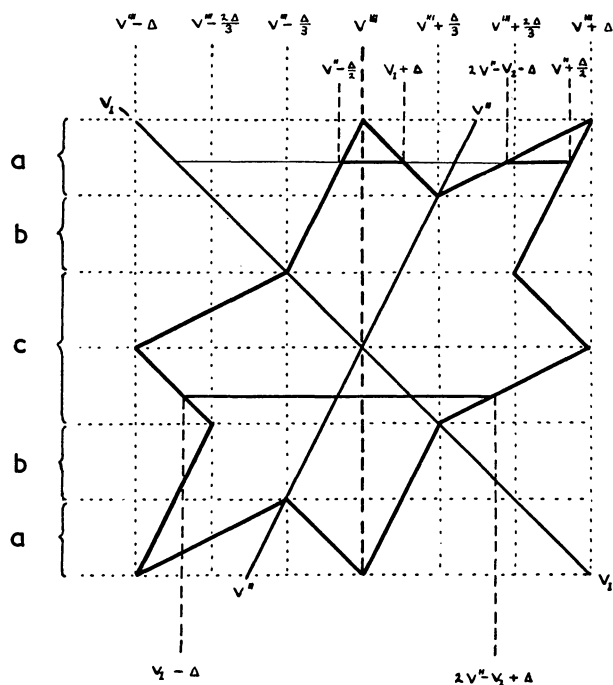
cases where  $v_1$  differs less than  $\frac{3}{2}\Delta$ , but more than  $\Delta$ , from  $v''$ . The remaining velocities which form the blend may then assume any value within the range  $v_1 + \Delta$  to  $2v'' - v_1 - \Delta$  if  $v_1 < v''$ , and within the range  $2v'' - v_1 + \Delta$  to  $v_1 - \Delta$  if  $v_1 > v''$ .

Let  $R(v''') dv'''$  denote the probability that a velocity  $v'''$  be between  $v'''$  and  $v''' + dv'''$ . Each value of  $v'''$  can be considered to be the weighted mean of the velocity  $v_1$  of the first cloud and the mean velocity  $v''$  of the second and third clouds:

$$3v''' = v_1 + 2v''.$$

The probability for  $v''$  to be in the interval  $v'', v'' + dv''$  is different from that given by formula (4), because in the present case  $v_2$  and  $v_3 = 2v'' - v_2$  do not always assume the values from  $v'' - \frac{\Delta}{2}$  to  $v'' + \frac{\Delta}{2}$ . The range of values of  $v_2$  and  $v_3$  depends on the value of  $v_1$ . The possible combination of values of  $v_1, v_2$  and  $v_3$  which contribute to the formation of a mean velocity  $v'''$  are illustrated by Figure 6. The velocities are

FIGURE 6



counted in horizontal direction. Running along the line marked  $v_1$ ,  $v_1$  assumes all values between  $v''' - \Delta$  and  $v''' + \Delta$ . For a given value of  $v_1$ , the range of values allowed for  $v_2$  and  $v_3$  is represented by the section of a horizontal line passing through  $v_1$ , that lies inside the heavily outlined star-shaped area. The probability

$$g(v'') dv'' = 2 dv'' \int \varphi(v_2) \varphi(2v'' - v_2) dv_2$$

that  $v''$  will be in the interval  $v'', v'' + dv''$  is found by integration of  $v_2$  over this range of values. The value of  $v''$  is at the intersection of the horizontal line and the line marked  $v''$ . Two cases are illustrated in the diagram.

The frequency distribution of velocities  $v'''$  is given by

$$R(v''') = \frac{3}{2} \int_{v''' - \Delta}^{v''' + \Delta} \varphi(v_1) g\left(\frac{3v''' - v_1}{2}\right) dv_1. \quad (11)$$

Three intervals of values of  $v_1$  can be distinguished as indicated in the diagram:

$$a) \quad \frac{2}{3} \Delta \leq |v_1 - v'''| \leq \Delta$$

with the ranges for  $v_2$  defined by

$$|v'' - v_1| - \Delta \leq |v_2 - v''| \leq \frac{1}{2} \Delta,$$

$$b) \quad \frac{1}{3} \Delta \leq |v_1 - v'''| \leq \frac{2}{3} \Delta$$

$$\text{with} \quad |v_2 - v''| \leq \frac{1}{2} \Delta,$$

$$c) \quad |v_1 - v'''| \leq \frac{1}{3} \Delta$$

$$\text{with} \quad |v_2 - v''| \leq \Delta - |v'' - v_1|.$$

In the case of hypothesis B, different expressions are obtained for  $g(v'')$  depending on the intervals of  $v_1$  and  $v''$ . For the interval b) of  $v_1$ , these expressions are the same as those for  $G(v'')$  given by (6).

The functions  $P$ ,  $Q$  and  $R$  become for hypothesis A:

$$\left. \begin{aligned} P(v') &= \frac{3}{\sqrt{2\pi}} e^{-\frac{v'^2}{2}} \left[ 1 - \Omega\left(\frac{v' + p}{\sqrt{2}}\right) + \Omega\left(\frac{v' - p}{\sqrt{2}}\right) \right]^2. \\ Q(v'') &= \frac{3}{\sqrt{\pi}} e^{-\frac{v''^2}{2}} \left[ \Theta\left(\frac{p}{2}\right) \left\{ 1 - \Omega\left(\frac{v'' + \frac{3}{2}p}{\sqrt{2}}\right) + \Omega\left(\frac{v'' - \frac{3}{2}p}{\sqrt{2}}\right) \right\} + \right. \\ &\quad \left. + \frac{\frac{3}{2}p - v''}{\sqrt{2}} \int_{\frac{p - v''}{\sqrt{2}}}^{\frac{3}{2}p - v''} \Theta(t\sqrt{2} + v'' - p) e^{-t^2} dt + \frac{\frac{3}{2}p + v''}{\sqrt{2}} \int_{\frac{p + v''}{\sqrt{2}}}^{\frac{3}{2}p + v''} \Theta(t\sqrt{2} + v'' - p) e^{-t^2} dt \right]. \\ R(v''') &= \sqrt{\frac{3}{2\pi}} e^{-\frac{3v'''^2}{2}} \left[ \Theta\left(\frac{\sqrt{3}p}{2}\right) \Theta\left(\frac{p}{2}\right) - \Theta\left(\frac{p}{2\sqrt{2}}\right) \Theta\left(\frac{p}{2}\right) + \right. \\ &\quad \left. + \frac{\frac{p}{2\sqrt{3}}}{\sqrt{2}} \int_0^{\frac{p}{2\sqrt{3}}} e^{-t^2} \Theta(p - t\sqrt{3}) dt - \frac{\frac{\sqrt{3}p}{2}}{\sqrt{2}} \int_{\frac{p}{\sqrt{3}}}^{\frac{\sqrt{3}p}{2}} e^{-t^2} \Theta(t\sqrt{3} - p) dt \right]. \end{aligned} \right\} \quad (12)$$

For hypothesis B:

$$\left. \begin{aligned}
 P(v') &= \frac{3}{8} e^{-|v'| - 2q} \left( e^{|v'|} + e^{-|v'|} \right)^2 && \text{for } |v'| \leq q \\
 &= \frac{3}{8} e^{-|v'|} \left( 2 - e^{-|v'| + q} + e^{-|v'| - q} \right)^2 && \text{for } |v'| \geq q \\
 Q(v'') &= \frac{3}{2} \left( 1 + e^{-2|v''|} \right) \left( e^{-|v''| - q} - \frac{1}{3} e^{|v''| - \frac{5}{2}q} - \frac{2}{3} e^{-2|v''| - q} \right) && \text{for } |v''| \leq \frac{1}{2}q \\
 &= \frac{3}{2} \left( 1 + e^{-2|v''|} \right) \left( e^{-|v''| - q} - e^{-|v''| - \frac{3}{2}q} \right) && \text{for } \frac{1}{2}q \leq |v''| \leq q \\
 &= \frac{3}{2} \left[ 2(|v''| - q) e^{-2|v''|} - e^{-|v''| - \frac{3}{2}q} + e^{-3|v''|} \left( e^q + e^{-q} - e^{-\frac{3}{2}q} \right) \right] && \text{for } q \leq |v''| \leq \frac{3}{2}q \\
 &= \frac{3}{2} \left[ q e^{-2|v''|} + e^{-3|v''|} \left( e^q + e^{-q} - e^{\frac{3}{2}q} - e^{-\frac{3}{2}q} \right) \right] && \text{for } \frac{3}{2}q \leq |v''|. \\
 R(v''') &= \frac{9}{32} \left[ (6v'''^2 + 6v''' + 2) e^{-3v'''} - 3 e^{-v'''} \left( e^{\frac{4}{3}q} - e^{-2q} \right) - 3 e^{v''' - \frac{4}{3}q} + e^{3v''' - 2q} \right] && \text{for } |v'''| \leq \frac{1}{3}q \\
 &= \frac{9}{32} \left[ \left( 4qv''' + 2q - \frac{2}{3}q^2 \right) e^{-3v'''} - 3 e^{-v'''} \left( e^{\frac{4}{3}q} - e^{-2q} \right) \right] && \text{for } \frac{1}{3}q \leq |v'''| \leq \frac{2}{3}q \\
 &= \frac{9}{32} \left[ \left( -6v'''^2 - 6v''' + 12qv''' + 6q - \frac{10}{3}q^2 - 3 \right) e^{-3v'''} + 3 e^{-v''' - 2q} \right] && \text{for } \frac{2}{3}q \leq |v'''| \leq q \\
 &= \frac{3}{4} q^2 e^{-3v'''} && \text{for } q \leq |v'''|.
 \end{aligned} \right\} \quad (13)$$

In the case of hypothesis B we find the following simple expressions for the total contributions of the three velocity distributions:

$$\left. \begin{aligned}
 \int_{-\infty}^{+\infty} P(v') dv' &= \frac{5}{2} e^{-q} + e^{-2q} - \frac{1}{2} e^{-3q} \\
 \int_{-\infty}^{+\infty} Q(v'') dv'' &= \frac{5}{2} e^{-q} - \frac{7}{2} e^{-2q} + e^{-3q} \\
 \int_{-\infty}^{+\infty} R(v''') dv''' &= 1 - \frac{5}{2} e^{-q} + 2e^{-2q} - \frac{1}{2} e^{-3q}.
 \end{aligned} \right\} \quad (14)$$

The sum of these is the predicted average number of measured velocities per star:

$$1 + \frac{5}{2} e^{-q} - \frac{1}{2} e^{-2q}. \quad (15)$$

Equations (14) satisfy the relation

$$\int_{-\infty}^{+\infty} P(v') dv' + 2 \int_{-\infty}^{+\infty} Q(v'') dv'' + 3 \int_{-\infty}^{+\infty} R(v''') dv''' = 3.$$

The frequency distributions  $P$ ,  $Q$  and  $R$  for  $n=3$ ,  $\eta=5$  km/sec and  $\Delta=9$  and  $18$  km/sec ( $q=1.8$  and  $3.6$ , respectively) are shown in Figure 5.

I want to thank Mr A. B. MULLER for his assistance in the first stages of this investigation.

## CONTENTS

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