Review

DUMMETT, MICHAEL. ***The elements of intuitionism.*** Oxford: Oxford University Press, 1977, x + 467 pages.

Over the last twenty years, Dummett has written a long series of papers advocating a view on meaning which has become known as "anti-realism". Now, in the book under review, we are given a thorough treatment of a *mathematical* version of antirealism; the sort of version from which extensions to non-mathematical contexts take their departure. The book is, in fact, an expansion and revision of previously circulated lecture notes from the Mathematical Institute, Oxford and the new parts comprise 150 pages of mainly *philosophical* material on the claims of intuitionism.

In the first two, out of seven, chapters some standard concepts of intuitionistic mathematics are introduced, in particular the notion of *constructive proof*, with the reading of mathematical truth as possession of constructive proof and the accompanying logical laws. The discussion of these matters is lucid, elegant and the most exhaustive in the literature. In these chapters, one also finds a treatment of the intuitionistic version of the real number continuum, where a real is a *constructive* Cauchy-sequence of rationals, as well as a description of the Gödel-Gentzen double-negation interpretation of classical formal systems into the corresponding intuitionistic formal system. When discussing the completeness of the intuitionistic continuum, viz., that every constructive Cauchy-sequence of intuitionistic reals has a constructive limit, Dummett rightly stresses the intuitionistic validity of the *Axiom of choice*. Indeed, from a constructive proof of $\forall x \exists y A(x, y)$, one immediately extracts a constructive function b such that $\forall x A(x, b(x))$, and therefore one is entitled to assert the implication $\forall x \exists y A(x, y) \rightarrow \exists f \forall x A(x, f(x))$. The traditional "constructivist" qualms over the *Axiom of choice* only arise on positions intermediate between the full-blooded platonism of the cumulative hierarchy and constructivism. Examples of such positions are those of Russell and Poincaré.

The third chapter—in the opinion of the reviewer, the best in the book—is devoted to a meticulous examination of the intuitionistic properties of the quantifier-combinations $\forall f \exists n$ and $\forall f \exists g$. The treatment is focussed on various notions of *continuity*—very roughly, the n which exists given the f depends only on a finite initial segment of the f—and *Bar-induction*—an induction principle linking up the meaning of the prefix with a certain type of inductive definition. The famous Brouwer "proof" of Bar-induction is given a novel and profound discussion. Using a counter-example of Kleene's, Dummett is able to pinpoint exactly where the Brouwerian argument goes astray; he shows that, as so often, the crux of the matter is intuitionistic invalidity of the distributive law $\forall x(A \vee Bx) \rightarrow A \vee \forall x Bx$. This chapter, apart from the novel treatment of Brouwer's "proof", also contains a beautiful exposition of a host of original research-material by, e.g., Kleene, Kreisel, Howard, Troelstra, etc, which makes many otherwise rather inaccessible results readily available in a uniform notation.

From there, Dummett goes on to treat the modern formal counterparts to Brouwer's "fully analysed" proofs, i.e., cut-free derivations in sequent calculi and normal derivations in natural deduction systems. The cut-elimination and normalisation theorems are proved by standard methods. This chapter, although the exposition is most competent, is strangely left hanging in the air: it is not made sufficiently clear *why* the theorems proved are of great interest. Apart from some well-known applications, Dummett makes no further use of the normal forms he has so laboriously derived, nor does he attempt to explain the "philosophical significance" of normalisation. This is a criticism which may be voiced against almost all proof-theoretical practicians. Kreisel, in particular, has tried to rectify the situation, but so far without conclusive success; we are still in want of the proper *concepts* with which to formulate the theorems that are established by the proof-theoretical transformation techniques. On the whole, this chapter could well have been replaced by some other left-out part of the meta-mathematics of intuitionism, as its content is easily available in many places.

Next comes a rapid shift into the *semantics* of intuitionistic formal systems. The author starts out by introducing the Beth- and Kripke-models via the topological interpretations and valuation systems. Pedagogically, this seems to the reviewer to be a mistake. In order to get an understanding of the semantics in question, a retreat to, say, the exposition by Fitting seems to be called for, because of the rather tortuous way the models are introduced in the book. The classical *completeness* of the Beth-tableaux is established along the usual pattern and then follows one of the high-spots of the book: a complete survey of the Gödel-Kreisel results on the *incompleteness* of first order intuitionistic predicate calculus. This survey will, like chapter III, save the reader much time and trouble in mastering an otherwise rather inaccessible area. One of the more interesting recent contributions to the subject is the Nijmegen-school results on completeness for deviant Beth-models (By allowing absurdity to be true at some nodes, De Swart and Veldman discovered a way to circumvent the Gödel-Kreisel result). Dummett devotes the rest of chapter V to various versions of these deviant models and discusses their (doubtful) intuitive rationale. The technicalities involved are quite complicated and perhaps do not belong in *The "elements" of intuitionism*.

Chapter VI is one of the more useful chapters: it contains surveys of current formal systems such as Kleene's FIM and Kreisel-Troelstra's CS with statements, though not proofs, of central meta-results. There is a fine guide to the jungle of realisability interpretations including the Kleene-slash. The chapter ends with a thoughtful analysis of Brouwer's theory of the creative subject and the difficulties which arise when one tries to incorporate statements about *tense* in mathematics.

The final chapter is a lengthy philosophical treatment of the meaning theory underlying intuitionism, very much like Dummett's article in *Logic colloquium* '73 (Eds. Rose & Sheperdson, North-Holland, 1975). He also treats the notions of canonical argument and demonstrations; the former are the normative constructions used to define the meanings of the logical constants, whereas the latter are what is needed for the *assertion* of a statement. There arises the need to convince oneself that each demonstration can be brought to a canonical argument. On the formal level this is quite similar to the normalisation procedure for arbitrary derivations. Dummett then relates these notions to the Beth-trees in a novel way and ends the book with a survey of choice sequences.

The philosophical position of Dummett on intuitionistic logic was subjected to a sympathetic scrutiny by Prawitz, "Meaning and proofs" (*Theoria* **43**, (1977), pp. 1—xx), and I will here only raise some points which have not been covered either by Prawitz or Dummett. As Prawitz rightly stresses, the demand of harmony between the grounds for asserting, and the consequences following from, an utterance is closely related to a Gentzen-inversion principle and the removal of maxima from natural deduction derivations.

At the Kiel conference in 1974, Marcel Crabbé caused something of a commotion among proof theorists when he produced a counter-example to normalisation for what seems a reasonable formalisation of the set-theoretic *Aussonderung*-axiom. We introduce restricted comprehension-terms and rules for them: from the two premisses $t \in a$ and $F(t)$ infer $t \in \{x \in a : F(x)\}$ and from this conclusion either premiss may be inferred. An inversion principle is immediate. However, form the relativised Russell-class $R_a =_{def} \{x \in a : x \notin x\}$. In the most straightforward way one proves that $R_a \notin a$, which derivation will contain a maximum. A few removals of maxima bring us back to the original derivation again and *harmony is blocked*. Against the background of the Prawitz-Dummett arguments, it is interesting to note the following:

+ the system is *very* weak (*Aussonderung* + existence of a set)
+ the logic used is intuitionistic (even *minimal*)
+ full type theory (an incomparably stronger system) admits normalisation and harmony.

What then is there in this conception of a set that precludes harmony? The most likely cause seems to the reviewer to be the explicit suppression of an implicitly understood type structure: an object is always given as an object of a certain kind or type. This is true also of set-theory but the formalisms currently used do not reflect this fact, whereas the formalism of an even cumulative type theory does. The matter is in need of a detailed investigation and it is a pity that neither

Prawitz nor Dummett addresses himself to the issue.

Another point worth mentioning deals with the meaning of implication. We usually say that a proof of an implication is a construction which transforms constructions proving the antecedent into constructions proving the consequent. On page 399, Dummett flirts with the Kreisel-idea of having another construction included as well which should prove that the first indeed does what it is supposed to do. To the reviewer this comes close to demanding that one should *prove that a proof is a proof*, which one cannot. A proof has to be *understood* as a proof of its conclusion—it cannot be *proved* to be a proof of that conclusion. In a non-mathematical version of anti-realism, the analogous demand would require us to have sense-impressions verifying other sense-impressions verifying certain elementary sentences, etc. The reviewer for one would not like such a theory.

Furthermore it should be remarked that it is not clear that the argument based on the proof-theoretical meaning theory, with its presupposition of normalisation and harmony, actually justifies *intuitionistic* analysis. The reviewer is willing to accept that systems adequate for a formalisation of Bishop's *Constructive mathematics*, e.g., Martin-Löf's type theory, can be justified using the Dummett argument. He cannot see, however, how to proceed to the *intuitionistic* notion of *choice sequence* within the Dummett framework.

A final, non-related point: Dummett is quite sceptical about the possible interest of *eclecticism*, i.e., the position which seeks to develop interesting meaning-theories for both platonism and constructivism. If the former can be made to work, the latter would be uninteresting. Kreisel, who is branded as an eclectic by Dummett, points out in conversation that philosophically eclecticism is at least as interesting as either of the two alternatives: it gives a way to link the external mathematical universe of platonism with the idealistic universe of intuitionism, "which gives rich possibilities for speculative metaphysics and should delight a professional philosopher".

Almost as interesting to discuss as the contents of the book is perhaps (the possible reasons for) the exclusion of some topics from the book. The most regrettable is Dummett's intentional omission of Gödel's *Dialectica* interpretation. Of all the subjects within intuitionistic meta-mathematics this is the one where there remains a great need for an elegant survey. Dummett had a great opportunity here and would that he had taken it in the style of chapter III! The book would have been improved by a shortening of the semantics chapter, and the possible exclusion of chapter IV, and the addition of a chapter on functional interpretations.

Another remarkable omission lies in the fact that Dummett nowhere in close to 500 pages on intuitionism refers to the works of Per Martin-Löf. He twice in the text refers to the "intended interpretation" and mentions the Kreisel-Goodman theory of constructions. A reference to the type theory of Martin-Löf would have been well-motivated here. Also in the normalisation chapter, Martin-Löf's old conjecture that identity between proofs can be explicated as congruence of normal forms ought to have been mentioned. Finally there is the point alluded to above that the type theory of Martin-Löf may well be the natural mathematical home for Dummett's meaning theoretical views, in spite of the differences between their respective positions on meaning theory.

Dummett states in the preface that he wanted to write a book which would help the beginner to attack more specialized research papers within the field by people like Kleene, Troelstra, Kreisel, etc. In this task he succeeds completely throughout most of the book. In spite of the reservations about chapters IV and V, the reviewer's overall impression is one of admiration. The book remains a tremendous achievment and will for many years to come be *the* book on intuitionism.

Göran Sundholm Worcester & Magdalen Colleges, Oxford