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COMMUNICATIONS FROM THE OBSERVATORY AT LEIDEN

METHODS FOR COMPUTING THE ORIGINAL ORBITS OF COMETS

BY E. H. BILO AND H. C. VAN DE HULST

The approximative methods for obtaining the original semi-major axis of a comet orbit are reviewed with special reference to the magnitude of the errors made. It is found that formulae linear in the masses of the disturbing planets and based on perturbations computed for a parabolic comet orbit have sufficient practical accuracy, provided that the reduction to the barycentre is correctly made and that the effects of all major planets are included. Where possible, the analytical estimates are illustrated by actual numerical examples.

1. *Introduction*

When a long-period comet penetrates into the region where the attraction by the planets becomes appreciable, its orbital elements are changed by planetary perturbations. So the elements of the osculating orbit, defined at a certain moment during the period in which the comet was observed, differ from the elements of the original orbit, described while the comet was at a large distance from the Sun. The calculation of the original orbits is of special interest for comets that move in nearly parabolic orbits. The major axes of these orbits indicate the extent of the large cloud of comets surrounding the solar system, which is the reservoir from which the nearly-parabolic comets appear to come (OORT 1950).

The element that is of primary importance for this problem is the reciprocal semi-major axis, $1/a$. The eccentricity, e, which is related to it by $1 - e^2 = p/a$, is subject to a nearly equivalent change because both q, the perihelium distance, and $p = q(1 + e)$, the "parameter" of the orbit, change relatively little. In the present article we shall consider only the calculation of the original value of $1/a$.

The problem thus defined has a long standing. Much work has been done by ELIS STRÖMGREN and his collaborators. A list of comets for which reliable original orbits were computed was given by SINDING (1948). It has since been extended by DIRIKIS (1956), who lists 26 comets. More recent work will be referred to below.

A wide variety of methods has been employed, or suggested. They range from fully numerical methods, which are straightforward but laborious, to fully analytical methods, which are too complicated for practical use. The most common practice has been to use Encke's method for a certain time interval of backward integration, say 15 years, and then to make the reduction to barycentric elements, whereas an upper limit of the perturbations over all earlier years is estimated. Practice varies on the point of inclusion of more perturbing planets than Jupiter and Saturn.

The present paper was written in a first draft in 1953 when one of us (E.H.B.) started his computations on the original orbits of some comets. The aim, at that time, was to present a method that was simpler than Encke's and yet could be shown to result in an error of $1/a$ not larger than one unit in the fifth decimal. An accuracy of this order was deemed sufficient both in view of the usual accuracy of the definitive osculating orbits and in view of the statistical discussion in OORT's 1950 paper. It may not be quite sufficient compared to the accuracy that has been reached for the best observed comets in recent years. It was concluded at that stage of the work that, within the stated accuracy,

(1) it is permitted to compute the perturbations

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using the positions that the comet would have in an unperturbed parabolic orbit,

(2) the integration may usually be limited to 10 years, and

(3) the perturbations by Uranus and Neptune may not generally be omitted.

These conclusions are verified by a more precise discussion in the present paper. In the discussion of errors we needed certain numerical data from the work of other authors. This necessitated a comparison of the different methods, although a full survey has not been aimed at. Reference may be made to STRÖMGREN's classical paper (1914) and to a recent survey by DIRIKIS (1956).

In judging the merits of a method it is useful to separate two sets of considerations, which are completely mixed in all papers we have seen on this subject. The first set refers to the ease and rapidity of the calculations, by which two mathematically equivalent methods may not be equally suitable in practice. The results obtained by such methods may differ only by interpolation and rounding errors. Under this heading fall, for instance, the choice between Encke's and Cowell's method, or between rectangular and spherical coordinates, or between the use of time or true anomaly as integration variables.

The second set of considerations refer to the accuracy reached. The errors caused by all approximations and omissions should be estimated. Under this heading fall, for instance, the questions whether to use more perturbing planets than Jupiter and Saturn, how far to extend the integration time, and whether it is permissible to compute the perturbing forces from the position of the comet in an unperturbed orbit.

We concentrate in the present paper on the second set of questions. For clarity it seems advisable to say as little as possible about the first set of questions, which are irrelevant to the final results. Accordingly, we use vector notation throughout this paper. For, although it is convenient to think of a vector $\mathbf{r}$ as a short notation for the three coordinates (x, y, z), a vector formula is equally valid in any system of rectangular or spherical coordinates. The choice may be made on the basis of convenience when the numerical work starts.

2. *Notations*

Units and coordinates. Time is measured in mean solar days; distance in astronomical units; unit of mass = Sun's mass; gravitational constant,

$$k^2 = 0.000\,295\,912\,21.$$

Vectors are in Clarendon type; their rectangular components are given in parentheses. The origin of coordinates is the Sun.

Osculating orbit of comet. The usual notation for the elements has been followed:

v = true anomaly,
$\mathbf{P}$ (P_x, P_y, P_z) unit vector to perihelion.
$\mathbf{Q}$ (Q_x, Q_y, Q_z) unit vector to position of $v = 90°$.
$\mathbf{r}$ (x, y, z) radius vector in true (perturbed) orbit; the length of this vector is r.
$\mathbf{V}$ (V_x, V_y, V_z) velocity in true (perturbed) orbit with respect to the sun; the length of this vector is V.

Parabolic orbit. This is defined by the same values of the elements q, Ω, i, ω, T as the osculating orbit. The true anomaly is found from the well-known relation

$$M(v) \equiv \frac{\sqrt{2}}{k} \tan \tfrac{1}{2} v + \frac{\sqrt{2}}{3k} (\tan \tfrac{1}{2} v)^3 = \frac{t - T}{q^{3/2}}. \qquad (1)$$

Perturbations

m_i (i = 1 to 9) masses of the planets.
$\mathbf{r}_i$ (x_i, y_i, z_i) coordinates of perturbing planet.
$\mathbf{r}_i - \mathbf{r}$ $(x_i - x, y_i - y, z_i - z)$ vector from comet to perturbing planet; the length of this vector is ρ_i.
$k^2\mathbf{u}$ (k^2u_x, k^2u_y, k^2u_z) acceleration due to the perturbations,

$$\mathbf{u} = \sum_i m_i \left\{ \frac{\mathbf{r}_i - \mathbf{r}}{\rho_i^3} - \frac{\mathbf{r}_i}{r_i^3} \right\}. \qquad (2)$$

3. *The fundamental equation*

The sum of the kinetic and potential energy of a comet with mass μ moving in an undisturbed orbit around the sun is

$$-\frac{\mu k^2}{2a}.$$

Here a is the semi-major axis of the orbit the comet would describe, if from the moment considered it would move solely under the attraction of the Sun. The work done per unit time by the perturbative force acting on the comet is

$$\mu k^2 \mathbf{V} \mathbf{u}.$$

Equating this to the time derivative of the energy we obtain the rigorous result:

$$\frac{d(1/a)}{dt} = -2\,\mathbf{V}\mathbf{u}. \qquad (3)$$

This fundamental equation may also be derived by means of the equation of motion as follows:

$$\frac{d(1/a)}{dt} = \frac{d}{dt}\left(\frac{2}{r} - \frac{V^2}{k^2}\right) = -\frac{1}{r^3}\frac{d}{dt}(\mathbf{r}\,.\,\mathbf{r}) - \frac{1}{k^2}\frac{d}{dt}(\mathbf{V}\,.\,\mathbf{V}) = -\frac{2}{r^3}\mathbf{r}\mathbf{V} - \frac{2}{k^2}\mathbf{V}\left(\frac{-k^2\mathbf{r}}{r^3} + k^2\mathbf{u}\right) = -2\,\mathbf{V}\mathbf{u}, \qquad (4)$$

which may be written fully:

$$\frac{d(1/a)}{dt} = -2\mathbf{V}\sum_i m_i\left(\frac{\mathbf{r}_i-\mathbf{r}}{\rho_i^3}-\frac{\mathbf{r}_i}{r_i^3}\right). \tag{5}$$

We now compute $\mathbf{V}$ and $\mathbf{u}$ on the assumption that the comet moves in a fixed parabolic orbit. The approximation introduced by this assumption is discussed in section 6. The position and velocity are given by

$$\mathbf{r} = q(1-\tan^2\tfrac{1}{2}v)\,\mathbf{P} + 2q\tan\tfrac{1}{2}v\,\mathbf{Q}, \tag{6}$$

$$\mathbf{V} = \frac{d\mathbf{r}}{dt} = \frac{2k}{\sqrt{p}}\cos\tfrac{1}{2}v\,(-\mathbf{P}\sin\tfrac{1}{2}v + \mathbf{Q}\cos\tfrac{1}{2}v), \tag{7}$$

where $p = 2q$. It follows that

$$V = \frac{2k}{\sqrt{p}}\cos\tfrac{1}{2}v \tag{8}$$

and the fundamental equation becomes

$$-\frac{d(1/a)}{dt} = \frac{2k}{\sqrt{p}}\left\{\mathbf{Q}(1+\cos v) - \mathbf{P}\sin v\right\}\,.\,\mathbf{u}. \tag{9}$$

Upon multiplication by $dt/dv = r^2/k\sqrt{p}$ equation (9) becomes

$$-\frac{d(1/a)}{dv} = \frac{r^2}{q}\left\{\mathbf{Q}(1+\cos v) - \mathbf{P}\sin v\right\}\,.\,\mathbf{u}, \tag{10}$$

which is the form used by MAKOVER (1955), BARTENEVA (1955), and GALIBINA (1958).

4. *The barycentric orbit*

The barycentre (index c) is the centre of gravity of the Sun and the nine planets[1]). Its position with respect to the Sun is

$$\mathbf{r}_c = \frac{\sum_i m_i\mathbf{r}_i}{1+m}, \tag{11}$$

where $m = \sum_i m_i$. Its velocity with respect to the Sun, $\mathbf{V}_c$, is formed similarly from $\mathbf{V}_i$; its acceleration with respect to the Sun is (exactly):

$$\frac{d^2\mathbf{r}_c}{dt^2} = \frac{d\mathbf{V}_c}{dt} = -k^2\sum_i\frac{m_i\mathbf{r}_i}{r_i^3}. \tag{12}$$

Let at a certain epoch $\mathbf{r}$ and $\mathbf{V}$ denote the position and velocity of the comet relative to the Sun. The barycentric osculating orbit is the orbit the comet would pursue if from this epoch it would move solely under attraction of the mass $1+m$ placed at the barycentre. Let a' denote the semi-major axis of this orbit, then

$$\frac{1}{a(t)} = \frac{2}{r} - \frac{V^2}{k^2} \tag{13}$$

and

$$\frac{1}{a'(t)} = \frac{2}{|\mathbf{r}-\mathbf{r}_c|} - \frac{(\mathbf{V}-\mathbf{V}_c)^2}{k^2(1+m)}. \tag{14}$$

The difference

$$c(t) = \frac{1}{a'(t)} - \frac{1}{a(t)} \tag{15}$$

is called the reduction to the barycentre for this epoch. Differentiation of (14), in the same manner as (4) may be derived from (13), rigorously gives

$$\frac{d}{dt}\left(\frac{1}{a'}\right) = -\frac{2(\mathbf{V}-\mathbf{V}_c)\,\mathbf{q}}{(1+m)}, \tag{16}$$

where

$$\mathbf{q} = \frac{\mathbf{r}-\mathbf{r}_c}{|\mathbf{r}-\mathbf{r}_c|^3}(1+m) - \frac{\mathbf{r}}{r^3} + \sum_i m_i\frac{\mathbf{r}_i-\mathbf{r}}{\rho_i^3}\,; \tag{17}$$

$k^2\mathbf{q}$ may be called the perturbative acceleration in the barycentric system. Further (15) gives

$$\frac{dc(t)}{dt} = 2\,\mathbf{V}\,\mathbf{u} - \frac{2(\mathbf{V}-\mathbf{V}_c)\,\mathbf{q}}{1+m}. \tag{18}$$

It is permissible to use first-order perturbations, i.e. to linearize these equations in terms of the small quantities m_i, m, $|\mathbf{r}_c|/r$ and $|\mathbf{V}_c|/V$. In this approximation the denominator of (11) may be omitted. We then obtain instead of (15):

$$c(t) = \frac{2\,\mathbf{r}\,\mathbf{r}_c}{r^3} + \frac{1}{k^2}(2\,\mathbf{V}\,\mathbf{V}_c + V^2m), \tag{19}$$

instead of (16):

$$\frac{d}{dt}\left(\frac{1}{a'}\right) = -2\mathbf{V}\left[\left(m+\frac{3\,\mathbf{r}\,\mathbf{r}_c}{r^2}\right)\frac{\mathbf{r}}{r^3} - \frac{\mathbf{r}_c}{r^3} + \sum_i m_i\frac{\mathbf{r}_i-\mathbf{r}}{\rho_i^3}\right] \tag{20}$$

and instead of (18):

$$\frac{dc(t)}{dt} = -2\,\mathbf{V}\left(m+\frac{3\,\mathbf{r}\,\mathbf{r}_c}{r^2}\right)\frac{\mathbf{r}}{r^3} - \frac{\mathbf{r}_c}{r^3} - \frac{1}{k^2}\frac{d\mathbf{V}_c}{dt}. \tag{21}$$

A few alternative forms may be mentioned. Equation (20) can be transformed into

$$\frac{d}{dt}\left(\frac{1}{a'}\right) = -2\,\mathbf{V}\sum_i m_i\left[(\mathbf{r}_i-\mathbf{r})\left(\frac{1}{\rho_i^3}-\frac{1}{r^3}\right) + \frac{3(\mathbf{r}\,\mathbf{r}_i)\,\mathbf{r}}{r^5}\right], \tag{22}$$

which form has been used e.g. by STRÖMGREN (1914)

[1]) This is the classical definition of the barycentre. It is somewhat unfortunate that the recent issue "Planetary Coordinates 1960-1980" understands by the term barycentre a quite different thing, namely a point approximately coinciding with the centre of gravity of the Sun and the four inner planets only (cf. eq. 28 of this paper).

and by MAKOVER (1955), whereas (5) may also be written in the form

$$\frac{d}{dt}\left(\frac{1}{a}\right) = -2\,\mathbf{V}\left\{\sum_i \frac{m_i(\mathbf{r}_i - \mathbf{r})}{\rho_i^3} + \frac{1}{k^2}\frac{d\,\mathbf{V}_c}{dt}\right\}. \quad (23)$$

Finally, for a nearly parabolic orbit (19) may be approximated by

$$c(t) = \frac{2}{r}\left(\frac{\mathbf{r}\,\mathbf{r}_c}{r^2} + \frac{2\,\mathbf{V}\,\mathbf{V}_c}{V^2} + m\right) \quad (24)$$

but the other formulae are not correspondingly simplified.

5. *The overall method*

The original semi-major axis is the value of a' at $t = -\infty$. The semi-major axis of the "definitive orbit" is the value of a at the osculation date t_2. Hence, the strict equation is

$$\left(\frac{1}{a'}\right)_{\text{orig}} = \left(\frac{1}{a}\right)_{t_2} + \int_{t_2}^{t_1} d\left(\frac{1}{a}\right) + c(t_1) + \int_{t_1}^{-\infty} d\left(\frac{1}{a'}\right). \quad (25)$$

We shall further confine ourselves to first-order perturbations. The heliocentric perturbations (second term) may then be computed from (5) or from (23), the barycentric perturbations (fourth term) may be computed from (20) or (22) and the reduction to the barycentre (third term) from (19).

For a nearly-parabolic orbit each term may be approximated by using the positions and velocities of a parabolic orbit (6), (7). The second term then transforms to (9) and the fourth term to a similar expression with $\mathbf{q}$ instead of $\mathbf{u}$. Further, the third term may be written in the form (24), where it is irrelevant to the first order whether the true position and velocity or the parabolic approximations are used.

The integration interval should be chosen practically, i.e., in such a manner that the integrand does not change too rapidly. It may be varied as required, for instance, taken smaller during a close approach to a planet and relatively large when the comet is very distant. However, there is no point in selecting awkward intervals in time corresponding to prescribed intervals in v. Nor is MAKOVER's claim justified that a smaller total number of steps is then required. In particular the last step, from $|v| = 170°$ to $|v| = 180°$ ($t = \pm\infty$), cannot be taken at once, for $\mathbf{q}$ keeps its oscillating character with the periods of the planets. There is no correct way but an analytical estimate as extensively discussed by STRÖMGREN (1914) and others.

The equations in the preceding section show that the conversion date t_1 may be chosen freely. The result is mathematically independent of this choice and only reasons of convenience may cause preference for a certain choice. The normal practice is that, for convenience, we wish to neglect the fourth term altogether. It then becomes vital to choose $t_2 - t_1$ sufficiently large.

In a first-order theory the conversion time t_1 may be chosen separately for each of the planets because not only the perturbation equations (5) and (22) but also the "reduction to the barycentre" (19) may be broken into contributions from each planet:

$$c(t) = \sum_i c_i(t) = \sum_i m_i \left\{\frac{2\,\mathbf{r}\,\mathbf{r}_i}{r^3} + \frac{2\,\mathbf{V}\,\mathbf{V}_i}{k^2} + \frac{V^2}{k^2}\right\}. \quad (26)$$

It is convenient to apply $c_i(t)$ for the four inner planets right at the start of the calculation, i.e., at the moment t_2, because their contributions are small. The forms of eqs. (13) and (14) suggest that this should be done by defining the semi-major axis a'' by

$$\frac{1}{a''} = \frac{2}{\bar{r}} - \frac{\bar{V}^2}{S_4 k^2}, \quad (27)$$

where (in accordance with the notations used in "Planetary Co-ordinates 1960—1980"):

$$\left.\begin{aligned} \bar{\mathbf{r}} &= \mathbf{r} - \mathbf{r}_b \text{ (length } \bar{r}\text{)}, \\ \bar{\mathbf{V}} &= \mathbf{V} - \mathbf{V}_b \text{ (length } \bar{V}\text{)}, \\ S_4 &= 1 + \sum_{i=1}^{4} m_i, \\ \mathbf{r}_b &= \frac{1}{S_4}\sum_{i=1}^{4} m_i\,\mathbf{r}_i. \end{aligned}\right\} \quad (28)$$

The proof that the difference of (27) and (13) consists to the first order of the terms $i = 1$ to 4 of (26) is precisely parallel to the derivation of (26) from the difference of (14) and (13). Hence we may call a'' the semi-major axis reduced to the barycentre of the Sun and the four inner planets.

It may be noted that $\mathbf{r}_b$ as defined by (28) is not precisely identical to the $\mathbf{r}_b$ tabulated in "Planetary Co-ordinates 1960—1980", in which Mercury was omitted. The difference is $< 7 \times 10^{-8}$.

If the computation of the original orbit is started with $1/a''$ by (27), the strict first-order procedure for the further computation is to apply the barycentric perturbations (22) for the inner planets and the heliocentric perturbations (4) for the outer planets. These terms together form $d(1/a'')/dt$. The reduction to the true barycentre should be made, at another time t_1, by adding the last five terms of (26), which may also be lumped together in a form similar to (19):

$$\frac{1}{a'(t)} - \frac{1}{a''(t)} = \sum_{i=5}^{9} c_i(t) = \frac{2\mathbf{r}\mathbf{r}_w}{r^3} + \frac{1}{k^2}\left(2\,\mathbf{V}\mathbf{V}_w + V^2\sum_{i=5}^{9} m_i\right). \quad (29)$$

Here $\mathbf{r}_w$ is identical to the vector tabulated in *Astronomical Papers for the American Ephemeris and Nautical Almanac XIII*, part 4 (and there written x, y, z). This is the radius vector of the heliocentric position of the centre of mass of the Sun (increased by the mass of the four inner planets) and the five outer planets. We have accurately for the true barycentre

$$\mathbf{r}_c = \frac{S_4}{1+m}\mathbf{r}_b + \mathbf{r}_w \qquad (30)$$

but to the first order simply

$$\mathbf{r}_c = \mathbf{r}_b + \mathbf{r}_w \,. \qquad (31)$$

The error in (31) is $< 7 \times 10^{-9}$.

It may be remarked that the factors $1+m$, or S_4, in front of k^2 are essential in (14) and (27), where the terms are of order zero in the masses but that the addition or omission of such a factor in (19) or (29) does not make any difference in the first order.

6. *Discussion of errors*

Any practical method of computation contains omissions or approximations, which introduce errors. The magnitude of these errors may be estimated from the strict equations derived above. We shall not try to estimate the error of $1/a$ in the "definitive orbit". The mean error usually is of the order of 10^{-5}, or 10^{-6} for exceptionally well observed comets. Moreover a systematic error may arise from the omission of the perturbations by any planets in calculating the definitive orbit. For the sake of argument, we assume that the given orbit at the time t_2 represents the true heliocentric osculating orbit. We try to estimate only the errors made from there on.

A. *Confinement to first-order perturbations*

Omission of some terms higher than the first order in the planetary masses occurs as soon as the decision is taken to compute the perturbations of one element $(1/a)$ only. It then appears logical not to care about any other terms of the second or higher order. The error thus made cannot strictly be estimated. However, the "total change"

$$\Delta\left(\frac{1}{a}\right) = \frac{1}{a'(-\infty)} - \frac{1}{a(t_2)} \qquad (32)$$

statistically is $(+55 \pm 25$ m.e.$) \times 10^{-5}$ (SINDING 1948). The second-order terms should not normally exceed 10^{-3} times this amount, i.e. (3 to 8) $\times 10^{-7}$. We also computed the exact second-order terms in $c(t)$. The difference between $c(t)$ computed to the second order from (13) to (15) and to the first order from (19) is:

$$(15)-(19) = -\frac{1}{k^2}(V_b^2 + 2m\mathbf{V}\mathbf{V}_b + m^2V^2) + \frac{3(\mathbf{r}\mathbf{r}_b)^2}{r^5} - \frac{r_b^2}{r^3} \,. \qquad (33)$$

A numerical check was made for comet 1914 III at the time 1906 May 29.5 ($t - T = 8.25$ years) and for comet 1930 IV at the time 1920 Jul. 14.5 ($t - T = 10.20$ years). This check gave the values:

	comet 1914 III	comet 1930 IV
(15)	−.000 002 89	+.000 040 79
(19)	−.000 002 71	+.000 041 09
(15)−(19)	−.000 000 18	−.000 000 30
(33)	−.000 000 18	−.000 000 32

which shows that the first-order theory is practically sufficient. The reduction refers to the four outer planets only.

B. *Omission of any planets in the reduction to the barycentre*

In order to estimate orders of magnitude we replace V by the value in a parabolic orbit: $V^2/k^2 = 2/r$ and r_i and V_i by the values in a circular orbit: $r_i = a_i$, $V_i^2/k^2 = 1/a_i$. We then have

$$c_i(t) = m_i\left\{\frac{2a_i}{r^2}\cos\alpha_i + \frac{2\sqrt{2}}{\sqrt{r\,a_i}}\cos\beta_i + \frac{2}{r}\right\}, \qquad (34)$$

where α_i and β_i are the angles between the vectors multiplied in (26). Depending on the ratio of a to r the first or the second term is more important than the last one, so that we cannot make both oscillating terms vanish.

For $t \to \pm\infty$, $r \to \infty$, all terms go to 0[1]), but the middle term vanishes very slowly: for $r = 10\,000$ ($t - T = 74\,000$ years) the coefficient of $\cos\beta_i$ for Jupiter is still 118×10^{-7}, so that it is not practical to avoid the reduction to the barycentre by carrying the heliocentric perturbations sufficiently far. Table 1 shows the values of the terms of (34) in units of the 7th decimal for $r = 10$, a distance which is reached about 5 years before and after perihelium. It is seen that within a desired accuracy of 10^{-5} Jupiter, Saturn, Uranus and Neptune should all be retained and that within an accuracy of 10^{-6} only Mercury and Mars might be omitted.

At a smaller distance, for instance $r=1$, the terms of (34) are larger, the multipliers being 100, $\sqrt{10}$ and 10, for the consecutive terms. This brings the

[1]) The remark made by FAYET (1900) and repeated by GENNARO (1937, p. 263) and DIRIKIS (1956, p. 9) that the heliocentric osculating elements oscillate between very wide limits for very large $|t-T|$ is not to the point. For it is based on the fact that for large $|t-T|$ eventually V becomes smaller than V_c, which occurs in a parabolic orbit only if $r >$ 50 parsec.

contribution of each term of (34) for the combined inner planets to the order of 100×10^{-7}. These reductions are automatically taken into account if the computation of the original orbit is started by (27). Numerical estimates $r = 1$, $V = 0.025$, $r_b = 5 \times 10^{-6}$ (maximum), $V_b = 1.0 \times 10^{-7}$ (maximum), $S_4 = 1.0000060$, give indeed 380×10^{-7} as the maximum difference between (27) and (13). The conclusion is that this difference should not be neglected and that it is not permitted to use (13) and simply to ignore the inner planets.

TABLE 1

Magnitude of terms in reduction to the barycentre at $r = 10$ A.U. (units of 7th decimal).

	coeff. of $\cos\,\alpha_i$	coeff. of $\cos\,\beta_i$	$\frac{2m_i}{r}$
Mercury	0	3	0
Venus	0	25	5
Earth	1	27	6
Mars	0	2	1
Jupiter	993	3740	1910
Saturn	545	826	571
Uranus	168	89	87
Neptune	312	84	104
Pluto	22	4	6

C. *Omission of the barycentric perturbations*

The formulae for the heliocentric perturbations are more convenient for numerical work than those for the barycentric perturbations. For, (23) is simpler than (20) (terms of the various planets combined) and (5) is simpler than (22) (terms of each planet separate). It follows that it is practical to make the reduction to the barycentre at such a time t_1 that it is permissible to neglect the barycentric perturbations in the interval t_1 to $-\infty$ entirely. This is indeed common practice. The error made by this omission may be estimated in two ways:

(a) *Numerical computation.* STRÖMGREN (1914) has calculated by means of eq. (22) the barycentric perturbations of 8 comets by Jupiter and Saturn in the approximate period $t - T = -12$ to -6 years. GALIBINA (1958) has calculated by essentially the same formula the original values of $1/a'$ for 3 comets (one common with STRÖMGREN) and the future values for 20 comets. In practice her integrations stop at $r = 30$, $|t - T| = 13$ years, because the products (integrand $\times$ interval) become 0 in the sixth decimal. At our request Mrs I. VAN HOUTEN-GROENEVELD determined by interpolation in GALIBINA's tables the barycentric perturbations by the four planets separately, over the interval $|t - T| = 6$ to 12 years. These values are given in Table 2. The full integral to $|t - T| = \infty$ was also recomputed from GALIBINA's data. The changes arising from the perturbations in the interval 12 years to ∞ were small and may not be very certain as GALIBINA's intervals in that period are very large. However, by including them, the results could be checked against GALIBINA's own. The differences were 3×10^{-6} at the most. They may be interpreted as rounding errors. Only for comet 1889 I a difference of 17×10^{-6} arose; this comet was omitted from Table 2.

Table 3 gives the averages and the dispersions of the values for individual comets from STRÖMGREN's as well as GALIBINA's material. Please note that the numbers behind the $\pm$ signs are not mean errors of the averages, but are the root-mean-square deviations from the average displayed by the individual comet orbits.

(b) *Analytical estimate.* In the present context it is convenient to remark that, in consequence of the footnote on p. 123, the "original" value of $1/a$ may be considered to be well-defined and to be identical to the original value of $1/a'$. Hence we do not have to determine the total change $\Delta(1/a)$ from (32), or from the last three terms of (25), but can directly integrate (23) from t_2 to $-\infty$.

TABLE 2

Barycentric perturbations for $|t - T| = 6$ to 12 years on the basis of GALIBINA's computations (units of 6th decimal)

Comet		Jup.	Sat.	Ur.	Nept.	All four
1864 III	fut	−2	−1	0	−9	−12
1889 II	fut	+6	−4	−2	+4	+4
1892 I	fut	−1	+4	+2	−9	−4
1892 II	fut	−3	−2	−2	−3	−10
1897 I	fut	0	+1	−1	−2	−2
1898 VII	orig	−3	−4	−2	−1	−10
1898 VII	fut	+3	0	+3	−10	−4
1899 I	fut	0	0	0	0	0
1904 I	fut	+2	+3	+3	−9	−1
1907 I	fut	−2	−3	−3	+1	−7
1908 III	fut	+3	0	+1	−12	−8
1914 III	orig	−6	0	0	−6	−12
1914 III	fut	−2	−2	−2	−5	−11
1914 V	fut	−3	−3	−2	0	−8
1915 II	fut	0	−3	0	−6	−9
1919 V	fut	+1	+2	0	−7	−4
1925 I	fut	+5	+4	0	−9	0
1930 IV	orig	−2	−1	−4	+1	−6
1930 IV	fut	+4	−4	0	+1	+1
1932 VI	fut	−3	−3	−3	−4	−13
1936 I	fut	+6	+5	−4	−4	+3
1937 IV	fut	+1	−3	−2	+3	−1

TABLE 3
Average barycentric perturbations (units of 6th decimal)

Author	Interval $\|t-T\|$	Number	Jup.	Sat.	Ur.	Nept.	All four
STRÖMGREN	6 — 12 yr	8	− 1 ± 2	− 1 ± 6			
GALIBINA	6 — 12 yr	22	0 ± 3	− 1 ± 3	− 1 ± 2	− 3 ± 5	− 5 ± 5
„	6 — ∞ yr	22	0 ± 4	0 ± 3	− 1 ± 2	− 4 ± 6	− 5 ± 7
Theory (eq. 40)	6 — 12 yr	—	0	0	− 0.6	− 2.5	− 3.1
„	6 — ∞ yr	—	0	0	− 0.6	− 2.7	− 3.3

Even if we take the planet orbits circular, the exact expressions thus obtained contain four irreducible parameters: two for the orientation of the comet orbit, one for q and one for the position the planet has at the time of perihelium passage. STRÖMGREN (1914) and others have derived series expansions which may replace this exact expression for large values of r but which have not proved very practical.

A less ambitious plan is to compute only the average value assumed by the perturbations that are incurred by comet orbits with the same q in all possible orientations. This problem was first set by SINDING (1948), but as his solution appears incorrect[1]), we shall newly derive the result.

We assume: one planet i moves in a circular orbit with radius a_i, and the comet moves in a parabola with perihelium distance q. Let $<>$ denote the average values of the enclosed expression, taken over all orientations. Random orientation implies that $<\mathbf{r}\mathbf{r}_i> = <\mathbf{V}\mathbf{r}_i> = 0$. However

$$\mathbf{r}\mathbf{V} = rV\sin\tfrac{1}{2}v = \pm k\sqrt{2(r-q)}. \qquad (35)$$

Here and in (37) the lower sign holds before and the upper sign after perihelium passage. Let for a moment $\mathbf{r}$ be fixed in space and let $\mathbf{r}_i$ assume all positions on a sphere with radius a_i. Direct integration then gives

$$\left\langle\frac{\mathbf{r}-\mathbf{r}_i}{\rho_i^3}\right\rangle = \begin{cases} 0 \text{ for } r < a_i \\ \mathbf{r}/r^3 \text{ for } r > a_i \end{cases} \qquad (36)$$

The integration giving this result is identical to that yielding the attraction by a spherical shell. We now can at once average the perturbation equations but the resulting formulae are simpler if we first multiply the time derivatives by

$$\frac{dt}{dr} = \frac{\pm r}{k\sqrt{2(r-q)}}. \qquad (37)$$

Let
$$\left.\begin{aligned} \Delta(1/a) &= \frac{1}{a(-\infty)} - \frac{1}{a(t)}, \\ \Delta(1/a') &= \frac{1}{a'(-\infty)} - \frac{1}{a'(t)}. \end{aligned}\right\} \qquad (38)$$

We then find from (5) and (22) with the aid of (35) to (37) the simple but rigorous results:

$$\left.\begin{aligned} &\text{for } r < a_i: \left\langle\frac{d(1/a)}{dr}\right\rangle = 0 \quad, \left\langle\frac{d(1/a')}{dr}\right\rangle = -\frac{2m_i}{r^2} \\ &\text{for } r > a_i: \left\langle\frac{d(1/a)}{dr}\right\rangle = +\frac{2m_i}{r^2}, \left\langle\frac{d(1/a')}{dr}\right\rangle = 0 \end{aligned}\right\} \qquad (39)$$

and upon integration:

$$\left.\begin{aligned} &\text{for } r < a_i: <\Delta(1/a)> = \frac{2m_i}{a_i}, <\Delta(1/a')> = \frac{2m_i}{a_i} - \frac{2m_i}{r} \\ &\text{for } r > a_i: <\Delta(1/a)> = \frac{2m_i}{r}, <\Delta(1/a')> = 0, \end{aligned}\right\} \qquad (40)$$

so that in any case

$$<c(t)> = <\Delta(1/a) - \Delta(1/a')> = \frac{2m_i}{r}, \qquad (41)$$

which follows also directly from (26).

Values of $\Delta(1/a')$ computed from (40) with the average values $r = 17$ for $|t-T| = 6$ years, $r = 28$ for $|t-T| = 12$ years are given in Table 3. They are in good agreement with the numerical averages in the same table. In conclusion it may be said that the systematic errors made by omitting the barycentric perturbations beyond $|t-T| = 6$ years are quite small. However, for a reasonable certainty that the error for any individual comet falls below 1×10^{-5} it is better to extend the integration several years further.

[1]) SINDING assumes without reason that the difference of two oscillating zero-order terms in m in his expression for $\Delta(1/a)$ has no systematic part of the first order in m. In the limit of $r/a \to \infty$ SINDING's result is a factor 3/2 higher than that derived here.

D. *Calculation of perturbations for a fixed parabolic orbit*

It is very convenient to take the positions and velocities used in the perturbation formulae from a fixed parabolic orbit (eqs. 6-9) instead of the true orbit. The error caused by this approximation has been estimated in two ways.

(a) The original orbit of comet 1892 II was computed by Encke's method as well as by eq. 9. The resulting second term in eq. (25) from 1892 May 5 to 1886 April 27 was +.0011268 by Encke's method and +.0011282 by the parabolic approximation, if in both methods only perturbations by Jupiter and Saturn are considered. The original value of $1/a'$ for comet 1932 VI was computed by VAN BIESBROECK (1937). From his results we derived that the second term from 1932 Oct. 23 to 1927 Dec. 6 is +.0006505 by Encke's method and perturbations by Jupiter and Saturn only. The parabolic approximation gives +.0006502 for the same time interval using the same two planets. For comet 1898 VII the result was identical in the sixth decimal by the exact method (STRÖMGREN 1914) and by the parabolic approximation (GALIBINA 1958). The differences of 1, 0 and 0, in the sixth decimal in these three examples are small enough to be permitted.

(b) The error in the perturbations of $1/a$ can be written as

$$\delta\left(-\frac{d(1/a)}{dt}\right) = \delta\left(\frac{2}{k^2}\mathbf{u}\,\mathbf{V}\right) = \frac{2}{k^2}\mathbf{V}\,\delta\mathbf{u} + \frac{2}{k^2}\mathbf{u}\,\delta\mathbf{V}. \quad (42)$$

Here $\delta\mathbf{u}$ depends on $\delta\mathbf{r}$ so it is necessary to compute the position and velocity deviations of the comet moving in the actual orbit from those of a body moving in the unperturbed parabolic orbit. A sharp error estimate cannot easily be made but an upper limit to the errors is probably found if we replace the actual orbit by a fixed hyperbolic orbit with

$$q = 2,\ e = 1.002,\ 10^5\frac{1}{a} = -100.$$

A direct but somewhat tedious computation gives the following expressions, correct to the first power of $e - 1$, for the differences between the hyperbolic orbit and the parabolic orbit with the same q. The comparison is made at the same time after perihelion passage, i.e. not at the same value of the true anomaly v.

$$s = \tan \tfrac{1}{2} v$$

$$r\,\delta v = \frac{-2\,qs}{1+s^2}\left(-\tfrac{1}{4} + \tfrac{1}{4}s^2 + \tfrac{1}{5}s^4\right)(e-1) \quad (43)$$

$$\delta r = \frac{qs^2}{1+s^2}\left(1 + \tfrac{1}{2}s^2 + \tfrac{1}{10}s^4\right)(e-1) \quad (44)$$

$$|\delta\mathbf{r}| = \left\{(r\delta v)^2 + (\delta r)^2\right\}^{\frac{1}{2}} \quad (45)$$

$$\delta(V^2) = \frac{k^2}{q(1+s^2)}\left\{1 + \frac{s^2(-1+s^2+\frac{4}{5}s^4)}{(1+s^2)^2}\right\}(e-1) \quad (46)$$

By order of magnitude, (42) may be replaced by

$$-\frac{d(1/a)}{dt}\cdot\left(\frac{\delta u}{u} + \frac{\delta V}{V}\right). \quad (47)$$

The maximum of $\delta u/u$ is found if $\delta\mathbf{r}$ is along the line joining the comet and the planet. Then:

$$\frac{\delta u}{u} = \frac{2\,\delta r}{|\mathbf{r} - \mathbf{r}_{\rm Jup}|}. \quad (48)$$

Further, $$\frac{\delta V}{V} = \frac{\delta(V^2)}{2V^2} = \frac{r\,\delta(V^2)}{4k^2}. \quad (49)$$

Employing these expressions with the numerical values adopted above, we obtain for the Jupiter perturbations the values in Table 4.

The values in the last two lines form probable upper limits to the fractional changes in $-\frac{d(1/a)}{dt}$ due to $\delta\mathbf{u}$ and to $\delta\mathbf{V}$ separately. It is seen that their sum barely exceeds 1 per cent. The empirical average of the total change found from the numerical integration is 55×10^{-5}. Hence, we may conclude that the error in the result due to the replacement of the actual orbit by the parabolic orbit should almost

TABLE 4

Estimated maximum errors made by using a parabolic orbit

s	1	1.732	2.287	3.334
$\|t - T\|$	0.85 year	2.21 year	4.00 year	9.99 year
r	4.00	8.00	12.46	24.23
δr	.0033	.0129	.0287	.0917
$\|\mathbf{r} - \mathbf{r}_{\rm Jup}\|$ (minimum)	1.2	2.8	7.3	19.0
$\frac{\delta u}{u}$ (maximum)	.0055	.0093	.0079	.0097
$\frac{\delta V}{V}$	.0006	.0014	.0023	.0046