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CONTINUED FRACTIONS AND LINEAR RECURRENCES

H W LENSTRA, JR AND J O SHALLIT

Dedicated to the memory of D H Lehmer

ABSTRACT We prove that the numerators and denominators of the convergents to a real irrational number θ satisfy a linear recurrence with constant coefficients if and only if θ is a quadratic irrational. The proof uses the Hadamard Quotient Theorem of A van der Poorten.

Let θ be an irrational real number with simple continued fraction expansion $[a_0, a_1, a_2, \dots]$. Define the numerators and denominators of the *convergents* to θ as follows:

$$(1) \quad p_{-2} = 0; \quad p_{-1} = 1; \quad p_n = a_n p_{n-1} + p_{n-2} \quad \text{for } n \geq 0;$$
$$(2) \quad q_{-2} = 1; \quad q_{-1} = 0; \quad q_n = a_n q_{n-1} + q_{n-2} \quad \text{for } n \geq 0.$$

By the classical theory of continued fractions (see, for example, [2, Chapter X]), we have

$$\frac{p_n}{q_n} = [a_0, a_1, \dots, a_n].$$

In this note, we consider the question of when the sequences $(p_n)_{n \geq 0}$ and $(q_n)_{n \geq 0}$ can satisfy a linear recurrence with constant coefficients. If, for example, $\theta = \sqrt{3}$, then $\theta = [1, 1, 2, 1, 2, 1, 2, \dots]$, and it is easy to verify that $q_{n+4} = 4q_{n+2} - q_n$ for all $n \geq 0$. Our main result shows that this exemplifies the situation in general.

Theorem 1. *Let θ be an irrational real number. Let its simple continued fraction expansion be $\theta = [a_0, a_1, \dots]$, and let (p_n) and (q_n) be the sequence of numerators and denominators of the convergents to θ , as defined above. Then the following four conditions are equivalent:*

- (a) $(p_n)_{n \geq 0}$ satisfies a linear recurrence with constant complex coefficients;
- (b) $(q_n)_{n \geq 0}$ satisfies a linear recurrence with constant complex coefficients;
- (c) $(a_n)_{n \geq 0}$ is an ultimately periodic sequence;
- (d) θ is a quadratic irrational.

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Our proof is simple, but uses a deep result of van der Poorten known as the Hadamard Quotient Theorem. We do not know how to give a short proof of the implication (b) \implies (c) from first principles.

Proof. The equivalence (c) \iff (d) is classical. We will prove the equivalence (b) \iff (c); the equivalence (a) \iff (c) will follow in a similar fashion.

(c) \implies (b): It is easy to see (cf. Frame [1]) that

$$(3) \quad \begin{bmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{bmatrix} = \begin{bmatrix} a_0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} a_n & 1 \\ 1 & 0 \end{bmatrix}.$$

Now if the sequence $(a_n)_{n \geq 0}$ is ultimately periodic, then there exists an integer $r \geq 0$, and r integers b_0, b_1, \dots, b_{r-1} , and an integer $s \geq 1$ and s positive integers c_0, c_1, \dots, c_{s-1} such that

$$\theta = [b_0, b_1, \dots, b_{r-1}, c_0, c_1, \dots, c_{s-1}, c_0, c_1, \dots, c_{s-1}, \dots].$$

Now for each integer i modulo s , define

$$M_i = \prod_{0 \leq j < s} \begin{bmatrix} c_{i+j} & 1 \\ 1 & 0 \end{bmatrix}.$$

Then for all $n \geq r$, we have, by equation (3),

$$(4) \quad \begin{bmatrix} p_{n+s} & p_{n+s-1} \\ q_{n+s} & q_{n+s-1} \end{bmatrix} = \begin{bmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{bmatrix} M_{n-r}.$$

Since for all pairs (i, j) it is possible to find matrices A, B such that $M_i = AB$ and $M_j = BA$, and since $\text{Tr}(AB) = \text{Tr}(BA)$, it readily follows that $t = \text{Tr}(M_i)$ does not depend on i . Hence the characteristic polynomial of each M_i is $X^2 - tX + (-1)^s$. Since every matrix satisfies its own characteristic polynomial, we see that $M_{n-r}^2 - tM_{n-r} + (-1)^s I$ is the zero matrix. Combining this observation with equation (4), we get

$$\begin{bmatrix} p_{n+2s} & p_{n+2s-1} \\ q_{n+2s} & q_{n+2s-1} \end{bmatrix} - t \begin{bmatrix} p_{n+s} & p_{n+s-1} \\ q_{n+s} & q_{n+s-1} \end{bmatrix} + (-1)^s \begin{bmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{bmatrix} = 0.$$

Therefore, $q_{n+2s} - tq_{n+s} + (-1)^s q_n = 0$ for all $n \geq r$, and hence the sequence $(q_n)_{n \geq 0}$ satisfies a linear recurrence with constant integral coefficients.

(b) \implies (c): The proof proceeds in two stages. First we show, by means of a theorem of van der Poorten, that if $(q_n)_{n \geq 0}$ satisfies a linear recurrence, then so does $(a_n)_{n \geq 0}$. Next we show that the a_n are bounded because otherwise the q_n would grow too rapidly. The periodicity of $(a_n)_{n \geq 0}$ then follows immediately.

Let us recall a familiar definition: if the sequence of complex numbers $(u_n)_{n \geq 0}$ satisfies a linear recurrence with constant complex coefficients

$$u_n = \sum_{1 \leq i \leq d} e_i u_{n-i}$$

for all n sufficiently large, and d is chosen to be as small as possible, then $X^d - \sum_{1 \leq i \leq d} e_i X^{d-i}$ is said to be the *minimal polynomial* for the linear recur-

rence Also recall that a sequence of complex numbers $(u_n)_{n \geq 0}$ satisfies a linear recurrence with constant coefficients if and only if the formal series $\sum_{n \geq 0} u_n X^n$ represents a rational function of X

Define the two formal series $F = \sum_{n \geq 0} (q_{n+2} - q_n) X^n$ and $G = \sum_{n \geq 0} q_{n+1} X^n$ Clearly F and G represent rational functions We now use the following theorem of van der Poorten [4, 5, 6]

Theorem 2 (Hadamard Quotient Theorem) *Let $F = \sum_{i \geq 0} f_i X^i$ and $G = \sum_{i \geq 0} g_i X^i$ be formal series representing rational functions in $\mathbb{C}(X)$ Suppose that the f_i and g_i are complex numbers such that $g_i \neq 0$ and f_i/g_i is an integer for all $i \geq 0$ Then $\sum_{i \geq 0} (f_i/g_i) X^i$ also represents a rational function*

Since $q_{n+2} = a_{n+2}q_{n+1} + q_n$ for all $n \geq 0$, it follows from this theorem that $\sum_{n \geq 0} a_{n+2} X^n$ represents a rational function, and hence the sequence of partial quotients $(a_n)_{n \geq 0}$ also satisfies a linear recurrence with constant coefficients

We now require the following lemma

Lemma 3. *Suppose that $(y_n)_{n \geq 0}$ and $(z_n)_{n \geq 0}$ are sequences of complex numbers, each satisfying a linear recurrence, with the property that the minimal polynomial of $(z_n)_{n \geq 0}$ divides the minimal polynomial of $(y_n)_{n \geq 0}$ Let d denote the degree of the minimal polynomial of $(y_n)_{n \geq 0}$ Then there exist constants $c > 0$ and n_0 such that for all $n \geq n_0$ we have*

$$\max(|y_{n-d+1}|, |y_{n-d+2}|, \dots, |y_n|) > c|z_n|$$

Proof Put $Y = \sum_{n \geq 0} y_n X^n = f/g$ with $\gcd(f, g) = 1$ and $\deg g = d$, and $Z = \sum_{n \geq 0} z_n X^n = h/g$, here $f, g, h \in \mathbb{C}[X]$ Since $\gcd(f, g) = 1$, we can find a polynomial $k = \sum_{0 \leq i < d} k_i X^i$ of degree $< d$ such that $kf \equiv h \pmod{g}$ Then $Z = kY + m$, for a polynomial m , and $z_n = \sum_{0 \leq i < d} k_i y_{n-i}$ for $n > n_0 = \deg m$ It follows that

$$|z_n| \leq \left(\sum_{0 \leq i < d} |k_i| \right) \max(|y_{n-d+1}|, |y_{n-d+2}|, \dots, |y_n|),$$

and the lemma follows, with $c = (1 + \sum_{0 \leq i < d} |k_i|)^{-1}$ \square

Since $(a_n)_{n \geq 0}$ satisfies a linear recurrence, we may express a_n as a generalized power sum

$$a_n = \sum_{1 \leq i \leq d} A_i(n) \alpha_i^n,$$

for all n sufficiently large Here the α_i are distinct nonzero complex numbers (the "characteristic roots") and the $A_i(n)$ are polynomials in n

Now take $y_n = a_n$ and $z_n = n^l \alpha^n$, where $\alpha = \alpha_i$ and $l = \deg A_i$ for some i Then the hypothesis of Lemma 3 holds, and we conclude that at least one of $a_{n-d+1}, a_{n-d+2}, \dots, a_n$ is greater than $cn^l |\alpha|^n$, for all n sufficiently large Then, using equation (2), we have

$$q_{dm} \geq \prod_{1 \leq j \leq dm} a_j > c^d \cdot c^m \cdot d^{lm} \cdot (m!)^l \cdot (|\alpha|^d)^{m(m+1)/2}$$

for some positive constant c' and all $m \geq 1$. But $(q_n)_{n \geq 0}$ satisfies a linear recurrence, and therefore $\log q_{dm} = O(dm)$. It follows that $|\alpha_\iota| \leq 1$ for all ι , and further that $\deg A_\iota = 0$ for those ι with $|\alpha_\iota| = 1$. Hence the sequence $(a_n)_{n \geq 0}$ is bounded. But a simple argument using the pigeonhole principle (see, for example, [3, Part VIII, Problem 158]) shows that any bounded integer sequence satisfying a linear recurrence is ultimately periodic. This completes the proof. \square

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