1934BAN.....7..227K

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COMMUNICATION FROM THE OBSERVATORY AT LEIDEN.

The frequency of physical pairs of separations between 1/20 and 5 parsecs,

by *G. P. Kuiper*.

1. The study of these very wide binary stars has been made as a part of a general survey of the double stars described in *Leiden Annals*, Vol. XIV, 5 (which will soon be published), and is of particular interest for the derivation of the frequency curve of the mutual distances (*l.c.* Problem 2).

Physical pairs with these large mutual distances are, at least among the stars of large parallax, generally not distinguishable by their proper motions alone. When we consider a parallax, $p = 0''\cdot100$, transverse distances of $^1/_{20}$ and 5 parsecs are seen under $0^\circ\cdot3$ and 30° respectively, so that it is necessary to compare the space velocities, in order to establish parallel and equal motion. But by doing so, a considerable amount of uncertainty is introduced, mainly by the parallaxes, which are used to convert the well known angular proper motions into linear velocities. However, especially for the fainter stars ($m > 7$), the radial velocities may also have considerable probable errors. When we confine ourselves to stars nearer than 15 parsecs the uncertainty in the velocities is about 10 times smaller than the dispersion in the velocities of the different stars themselves. Therefore a direct comparison of the calculated space velocities among a few hundred stars must leave a large percentage of the genuine pairs undetected and reveal many spurious combinations. Without a statistical treatment rendering an account of these accidental combinations we cannot obtain results of much significance.

Recently two authors have given lists of approximately parallel moving pairs or star groups. In *Veröff. d. Universitätssternwarte zu Berlin-Babelsberg*, Vol. III., Heft 3, p. 21 (1923), J. HAAS gives a list of such stars, but expresses as his opinion (p. 20): "It should not be said at all that we deal here with stars having the same movement or even with star clusters." In his recent discussion: *Verzeichnis der Sterne innerhalb 15 parsec*, *A.N.* 239, 97–114; 1930, however, HAAS calculates spatial distances in two pairs of stars, "correlated by agreeing velocities," evidently now assuming some real connection between these stars. A further list is given by W. LUYTEN in a paper: On the possible existence of physically connected groups of stars (*H.O.C.* 298, 1926). LUYTEN calculated for 500 stars nearer than 25 parsecs the space velocities and picked out those pairs or groups of stars having nearly the same X, Y, Z components of velocity; this was done by studying plots of the stars on the XY plane as well as on the YZ plane of velocity. LUYTEN's list contains 52 stars (46 Tau ($p = 0''\cdot034$) and the Sun excepted) as belonging to pairs or plural systems, this being 10% of the total number of stars investigated. LUYTEN writes: "Considering the small number of stars investigated, the accordance of the three velocity components within each group is closer than might be expected if the velocities were distributed at random. It is not claimed that each of the groups tabulated here constitutes a moving cluster of physically related objects. In not accepting this explanation, however, we are faced with an alarming number of coincidences, far greater than we should be led to expect from pure chance". No calculations, however, are given on which this last conclusion should be based. From the following lines it will appear that the number of apparently parallel moving pairs is just equal to the number of chance combinations to be expected.

2. In the following investigation an extensive use has been made of the above mentioned "Verzeichnis der Sterne innerhalb 15 parsec" by J. HAAS, which contains the galactic components of the space velocities. This list contains 180 [1]) stars of which 33 had

[1]) Double or multiple stars counted as one.

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1934BAN.....7..227K

in my parallax catalogue [1]) (at the time I executed this work) a probable error in the parallax exceeding $p/8$ or else had according to my parallax value distances larger than 15 parsecs; these 33 stars were rejected. In 7 cases out of the remaining 147 the parallax used by HAAS differed by more than $p/8$ from my value; 2 of these ($11^h44^m + 15°8'$ and $23^h59^m - 37°51'$) differed only because HAAS had rejected older, very bad parallax determinations, which had been included by SCHLESINGER in his Catalogue. I have taken HAAS' unaltered value in these 2 cases. For the remaining 5 stars and 7 others[2]), according to HAAS nearer than 10 parsecs, and differing in p more than 0·07 p from my catalogue values, the galactic components of the space velocity were calculated anew. Thus the parallaxes employed up to distances of 10 parsecs always differed by less than 7 percent from my catalogue values, and those between 10 and 15 parsecs by less than $12\frac{1}{2}$ percent; in no case the probable error of p exceeded $p/8$.

The following principle was used for the statistics. Consider the velocity space in which every star from the list is represented by a point. Two stars having equal and parallel space motions will in the velocity space be represented by two points very close together. A number of physical pairs will make the relative frequency of very small mutual distances abnormally large. In a part of the velocity space where the density d of points is constant we imagine a sphere around an arbitrary image-point and consider all other image-points lying in this sphere. The frequency of the distances of these points to the centre of the sphere will be $4\pi\,\rho^2\delta\rho.d$, therefore proportional to ρ^2. The presence of physical pairs will produce an excess near $\rho = 0$. In the actual computations I have taken 20 km/sec. for the radius of the sphere; and computed all the distances from the centre of the points within this sphere. This was done for each star as a centre in the following way: A large diagram was made, representing the $\dot{\xi}\dot{\eta}$ coordinates (notation of HAAS), i.e. the two velocity components parallel to the plane of the Milky Way. Near the plotted points the $\dot{\zeta}$ component was written down with its sign. For each star as a centre the three coordinate differences were read off with the aid of a reseau, bearing also a circle with a radius of 20 km/sec, thereby limiting the area to be investigated. Each mutual distance occurred twice, but as afterwards all distances were to be collected in one table, it was sufficient to take each combination only once, thus halving the work.

If there were no physical pairs, the frequency (N) of the distances (ρ) would be proportional to ρ^2. The mean error of a counted number N is $\sqrt{N}$; therefore, if we take $\sqrt{N}$ as ordinate instead of N, and ρ as abscissa, we not only obtain for the relation a straight line instead of a parabola, which is a great advantage when an excess at $\rho = 0$ has to be studied (see later), but we also make the mean error of the points determining this straight line a *constant* in the diagram, independent of N, another large advantage.

The position is therefore: when no physical pairs are present in a diagram with $\sqrt{N}$ as ordinate, the observed frequency curve of mutual distance is a straight line (on which the mean error of the points is constant) intersecting the horizontal (ρ) axis at a geometrically determined point; physical pairs will be distributed near this point of intersection and their distribution curve entirely determined by the mean errors of the parallaxes and radial velocities. The fact that the straight line ends in a geometrically fixed point and that, further, the mean errors of the observed points are numerically small at this end, makes a sharp determination of the number of physical pairs possible.

One point has so far been neglected: the finite extension of the velocity picture of the stars. For a star lying near the boundary the exterior part of the sphere will not be filled to the same extent as the remaining part [1]); the only consequence will be that the $\sqrt{N}$ curve is no longer a straight line, but deviates for larger relative distances a little downwards, as a consequence of the convex shape of the equidensity surfaces.

In nearly all cases (the exceptions are stars having very small angular proper motions), the accuracy of the transverse velocity is determined by the probable error of the parallax. As a rule therefore the probable error of the transverse velocity is

$$\frac{\text{p.e. of } p}{p} \ . \ V \text{ transv.};$$

to attain a certain homogeneity in the p.e. of the different velocity components of a star, it seems desirable to divide the velocity space into two parts, an inner sphere of radius R km/sec, Sun as a centre, and the remaining part. The quantity $\frac{\text{p.e. of } p}{p}$ is, in our case, always $< \frac{1}{8}$; for the stars with $p > {}''{\cdot}100$

1) This catalogue is based on SCHLESINGER's Catalogue of 1924 supplemented by all the new determinations. These new values, and also their probable errors, are reduced to SCHLESINGER's system.

2) A number of the differences in p are apparently a consequence of the fact that HAAS did not (or not in all cases) use the corrections to the measured p values which were derived by SCHLESINGER; in a few cases a modern p value had not been used by HAAS; in a few others the difference was not explained.

1) One may compare this with the mechanism causing the surface tension in a liquid.

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1934BAN.....7..227K

the mean value is about $\frac{1}{20}$, for $p < ''\cdot100$ somewhat larger. If we take $R = 40$, the p.e. of the transverse velocity in the inner sphere never exceeds 5 km/sec, and is generally about 1 or 2 km/sec, i.e. of the same order of magnitude as the p.e. of a radial velocity. The accuracy of the velocities in the sphere may therefore be considered to be approximately homogeneous, whereas that of the remaining stars is about proportional to the velocity. I found it easier to take a cylinder of $R = 40$ km/sec parallel to the ζ axis, instead of the sphere between the inner and outer parts of the velocity space. As large ζ velocities in this cylinder do not occur, the homogeneity of the accuracy is not disturbed in it.

3. *The counts. a.* Stars of the inner cylinder (= Region I); $p \geq ''\cdot100$. Here are 38 stars, after having rejected one star with $\frac{\text{p.e. of } p}{p} = \frac{1}{8}$. The frequency of relative velocities (ρ) derived as described above, is given in Table I, column 2. Combined in groups of 2, 3, 4, and 5 respectively, the square roots of the totals are shown in Figs. *a*2, *a*3, *a*4, and *a*5. The mean errors of the points are always nearly $\frac{1}{2}$ in the diagram, as $\sqrt{N \pm \sqrt{N}} \sim \sqrt{N} \pm \frac{1}{2}$, if $N \geq 2$.

TABLE I.

Counts of relative space velocities (ρ).

ρ	*a*	*b*	*c*
km/sec			
0	0	0	0
1	0	2	0
2	1	3	0
3	0	0	0
4	0	4	0
5	2	12	4
6	1	9	1
7	10	12	1
8	3	23	3
9	6	26	6
10	8	19	10
11	4	30	11
12	7	33	6
13	14	42	11
14	10	34	8
15	12	45	11
16	9	45	16
17	14	59	23
18	14	58	16
19	13	48	15
20	13	59	17

It appears that the points actually approximate to a straight line, and that a small "surface-tension effect" is present. If the abscissae of the points are chosen as has been done in the diagrams, the points of intersection of the line with the ρ axis are in the four cases at

$$\rho = 0\cdot37,\ 0.78,\ 1\cdot25 \text{ and } 1\cdot65$$

respectively. (The limits of counting are $0 - 2\frac{1}{2} - 4\frac{1}{2} - 6\frac{1}{2} \ldots\ldots, 0 - 3\frac{1}{2} - 6\frac{1}{2} - 9\frac{1}{2}$, etc.) It at once appears that no physical pairs are present among the 38 stars; a more precise discussion will follow later on.

b. Region I, all stars within 15 parsecs (except the rejected ones); 69 altogether. The frequency of relative velocities is found in Table 1, column 3. Combined in groups of 2, 3, and 4, the square roots are given in Figs. *b*2, *b*3, and *b*4.

c. Stars outside Region I, but inside a cylinder with R = 100 km/sec (= Region II); all stars within 15 parsecs. The frequency of the relative velocities is given in Table 1, column 4. Combined in groups of 4, 5, and 7, the square roots are given in Figs. *c*4, *c*5, and *c*7.

4. Discussion of the results obtained under *a* to *c*. From a glance at the diagrams it appears that all three groups of stars, *a*, *b*, and *c* show that the number of physical pairs among them must be very small. A determination of the *accuracy* of this result is of interest especially in a study of the frequency curve of the mutual distances (Problem 2, *Leiden Annals*, Vol. XIV, 5). Dealing with small numbers, as we are, the exact procedure, of obtaining both the most probable value of the number of physical pairs and its accuracy, is to derive the frequency distribution of this value on the basis of the calculus of probabilities. This is done in Note A to this paper in which the required formulae are derived and applied to the counts of Table 1. These formulae are entirely general and also valid if the number of physical pairs is large.

In this section we prefer to use a simple approximate formula. Define: n_p as the number of physical[1]) pairs present among the stars studied; N as the total number of observed pairs in a certain interval of relative velocity; n_c as the mean number of accidental [1]) combinations in the same interval; and f as the fraction of the total number of physical pairs to be expected in this interval.

[1]) Physical pairs are thought to be of common origin, like members of a binary system or of a moving cluster. We may expect the relative velocities to be less than 1 km/sec or 1·02 parsecs in 10^6 years, because their ages must be much higher than 10^6 years. Accidental pairs, on the contrary, happen to possess nearly the same space velocity and their number is determined by the laws of chance.

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1934BAN.....7..227K

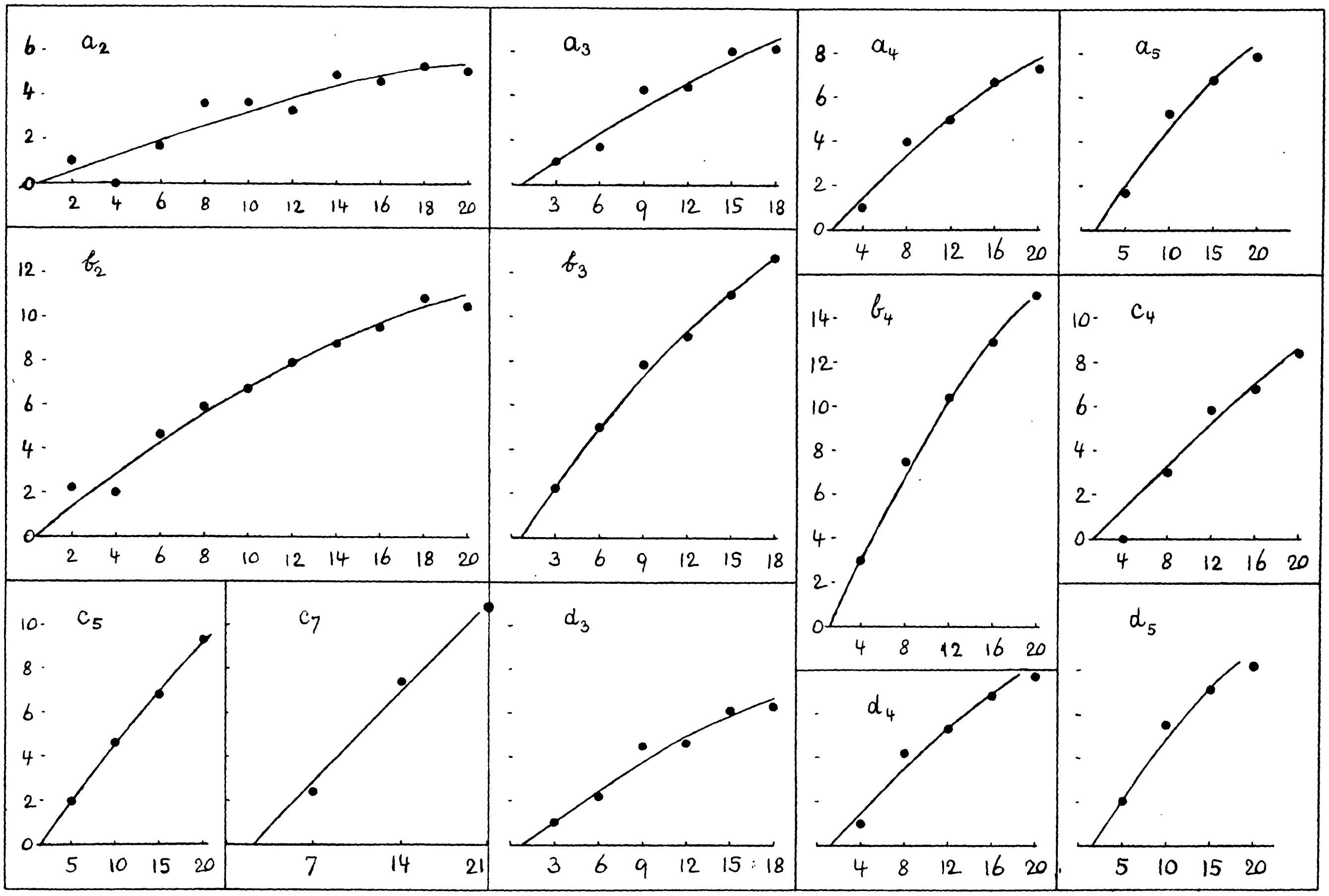

FIGURE 1.

N is counted from Table 1; $\bar{n}_c$ read from the diagrams; f is computed from the mean probable error ε of one component of a space velocity, of which an estimate is given in Note C for all three groups of stars mentioned under a, b, and c. In Note B a table is constructed from which f may be taken as soon as ε is known.

Since from the diagrams it is probable that $f n_p << \bar{n}_c$, we may suppose the deviations of N from its mean value for a certain interval to be given by the dispersion, $\pm \sqrt{\bar{n}_c}$. For n_p we may then use the approximate formula

$$f\,(n_p \pm \text{m.e.}) = N \pm \sqrt{\bar{n}_c} - \bar{n}_c\,. \qquad (1)$$

In Table 2 the values of N, $\bar{n}_c$ and f are given for those combinations of counts that seem to promise the most accurate determination of n_p for the three groups of stars, a, b, c. In the last column, the corresponding values of n_p and their mean errors are given, as computed by formula (1) [1]).

[1]) The exact procedure is much more complicated, see Note A.

TABLE 2.

	N	$\bar{n}_c$	f	n_p + m.e.
a_2	1	0·4	0·59	+ 1·0 ± 1·0
a_3	1	1·0	0·87	0·0 ± 1·2
a_4	1	1·7	0·976	— 0·7 ± 1·3
a_5	3	3·6	0·997	— 0·6 ± 1·9
b_2	5	1·7	0·242	+ 13·6 ± 5·4
b_3	5	5	0·488	0·0 ± 4·5
b_4	9	9	0·717	0·0 ± 4·2
c_7	6	7·7	0·686	— 2·5 ± 4·0

We notice that $\sqrt{\bar{n}_c}$ increases about in proportion to the length of the intervals used in the counts, whereas f increases slowly for very small and very large intervals and quickly in between. Remembering this it is seen from formula (1) and Table 7 that the result must be that n_p is most accurately determined if f is about 0·8.

This is confirmed by inspection of Table 2. For

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1934BAN.....7..227K

instance, the result obtained from b_2 has no value on account of the small value of f.

As the result of Table 2 we may take for the number of physical pairs belonging to groups a, b, and c: $0 \pm 1{\cdot}2$, 0 ± 4, and -2 ± 4 respectively; or, in percentages of the numbers of stars used: $0 \pm 3{\cdot}2$, 0 ± 6 and -3 ± 6 respectively.

5. It has been stated in section 1 that the knowledge of the number of wide pairs is of importance in constructing the frequency curve of the separations of the binaries in a volume of space. A provisional determination of this curve will be published later.

Since this frequency curve probably depends on the absolute magnitude of the stars considered, we derive the curve for stars brighter than $6^{M}{\cdot}5$ (absolute; $p = 0''{\cdot}1$), for which our present knowledge on duplicity is most complete. Furthermore, since stars with very faint companions are not completely known, it is appropriate to set an upper limit to Δm, the difference between the magnitudes of the two components. We take for this limit $\Delta m = 4{\cdot}0$, which corresponds to a mass proportion $\frac{m_2}{m_1}$ of approximately $\frac{2}{5}$.

Accordingly, we adopt $M = 6{\cdot}44$ for the limit of the total brightness of the two members of a physical pair and $\Delta m = 4{\cdot}0$ for their difference in magnitude. Furthermore, it is necessary to make an assumption as to the relation of the magnitudes of the components of these wide physical pairs. For simplicity we suppose the relation to be the same as that of arbitrarily combined pairs of single stars in space. Probably this assumption is extreme and the relation will be closer.

Consider a physical pair of which the brighter component has an absolute magnitude M. Define as N_M the number of stars between $M - \frac{1}{4}$ and $M + \frac{1}{4}$ in the volume of space investigated. Further, we define S_M as the number of stars between M and $M + 4{\cdot}0$ contained in the same volume. According to the above assumption about the relation between the magnitudes of a physical pair, the number of such pairs is proportional to $N_M S_M$: say $c N_M S_M$. Then, the number of physical pairs with $M < 6{\cdot}5$ and $\Delta m < 4{\cdot}0$ equals $c\,(\Sigma N_M S_M + \Delta)$, in which the sum has to include all M values brighter than $6{\cdot}5$; $c\Delta$ is the number of pairs with a total brightness exceeding $6^{M}{\cdot}5$ and with both components fainter than $6^{M}{\cdot}4$. N_M and S_M were taken from VAN RHIJN's luminosity function in Gron. Publ. 38, Fig. 1, for each half magnitude, the last one being $+ 6^{M}{\cdot}2$.

This number may be contrasted with the number actually found, which equals $n_p = \frac{1}{2} c H H_1$; H is the number of stars in the sphere investigated that is contained in our list, and H_1 the fraction of H between the selected limits of spatial velocity. Elimination of c gives the desired "standard" number, with the limits $M < 6{\cdot}5$ and $\Delta m < 4{\cdot}0$:

$$\Big(\sum_{-\infty}^{+6\cdot2} N_M S_M + \Delta\Big)\, n_p \big/ \tfrac{1}{2} H H_1 = F n_p.$$

If we divide this quantity by the number of stars brighter than $6^{M}{\cdot}5$ in the same sphere, being $\sum_{-\infty}^{+6\cdot2} N_M$, we find the "standard" *fraction*.

Results. Group a (see sections 3 and 4). Volume of the sphere 4190 cubic parsecs. Numerator of F: 2520; $H = 79$, $H_1 = 38$, hence $F = 1{\cdot}66$; $\sum_{-\infty}^{+6\cdot2} N_M = 40{\cdot}4$, therefore the "standard" fraction of physical pairs $\frac{1{\cdot}66}{40{\cdot}4}\,(0{\cdot}0 \pm 1{\cdot}2) = 0{\cdot}00 \pm 0{\cdot}05$ (m.e.). Using the result of the more exact derivation in Note A, which makes the quantity always positive, we have for the standard fraction $+ 0{\cdot}05 \pm 0{\cdot}03$ (p.e.) [1]).

For group b the corresponding figures are $0{\cdot}00 \pm 0{\cdot}16$ (m.e.) and $+ 0{\cdot}13 \pm 0{\cdot}08$ (p.e.). For group c they are $- 0{\cdot}09 \pm 0{\cdot}18$ (m.e.) and $+ 0{\cdot}09 \pm 0{\cdot}06^5$ (p.e.).

6. It is interesting to compare the relative merits of the 3 groups of stars for our purpose, in order to see whether something may be gained by another subdivision of the material, and what may be expected from additional observations. The advantages of group a are the small value of ε and the relatively great completeness of the stars in the sphere (the influence of the completeness is quadratic). Disadvantages are the small number of stars and the great density in the velocity space. Group b has a somewhat larger ε, is much less complete, and has a great density in the velocity space. On the other hand, the number of stars is larger; this, however, appears to be insignificant in comparison with the disadvantages. We come to the remarkable conclusion that *the addition of the stars between 10 and 15 parsecs does not improve, but rather lessens the value of the result.*

In group c we have a still larger value of ε, the incompleteness is the same as for group b, the density is much less, the number of stars about the same. The smaller density approximately counterbalances the larger value of ε.

1) We prefer to retain the p.e. in this case, since this is the quantity derived in Note A and since the dispersion is not nearly gaussian.

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1934BAN.....7..227K

On the basis of these conclusions it seems appropriate to discard the stars beyond 10 parsecs entirely and to take all velocities below 100 km/sec together.

Group *d*. (Conclusion); $p \geq 0''\cdot100$; velocity relative to Sun < 100 km/sec; 70 stars in all. The counts of the relative velocities are given in Table 3. Combined in groups of 3,4 and 5 the results are shown in Figs. *d*3, *d*4 and *d*5.

TABLE 3.

ρ	N	ρ	N	ρ	N
km/sec		km/sec		km/sec	
0	0	7	10	14	11
1	0	8	3	15	12
2	1	9	7	16	9
3	0	10	8	17	16
4	0	11	6	18	15
5	3	12	7	19	15
6	2	13	14	20	13

Table 4 corresponds to Table 2 in giving the number of physical pairs derived by means of formula (1) from the quantities N, $\bar{n}_c$ and f. As the result of the n_p values in Table 4 we adopt $0\cdot0 \pm 1\cdot6$ (m.e.) as the best determination. In Note A we find $+ 1\cdot2 \pm 0\cdot9$ (p.e.) for the same stars. These values make the "standard" fraction of physical pairs $0\cdot00 \pm 0\cdot036$ (m.e.) or $+ 0\cdot027 \pm 0\cdot020$ (p.e.) [1]).

TABLE 4.

	N	$\bar{n}_c$	f	n_p ± m.e.
d_3	1	1·0	0·639	0·0 ± 1·6
d_4	1	2·3	0·849	— 2·0 ± 1·8
d_5	4	4·0	0·952	0·0 ± 2·1

7. One point remains to be considered: the separations in the pairs for which this result is valid. Pairs wider than 20 parsecs are not included among those stars, for which $p \geq 0''\cdot100$. If s is the spatial distance between the components of a physical pair expressed in terms of the *diameter* of the sphere investigated (i.e. 20 parsecs in our case), and N is the density of such pairs, V the volume of the sphere, we have for the number of pairs included with both components in the sphere: $NV\left(1 + \frac{s}{2}\right)(1 - s)^2$. If R is the distance between the two components in astronomical units, for $\log R = 6\cdot25$ the coefficient of NV is 0·395. For $\log R = 6\cdot00$, it is 0·646; for $\log R = 5\cdot75$ it is 0·800. The effective upper limit may be considered to be about $\log R = 6\cdot05$, corresponding to $5\frac{1}{2}$ parsecs.

The lower limit is the separation of the widest binaries, since, in this paper, the binaries have been treated as single objects. The widest binary in the sphere of 10 parsecs is α with Proxima Centauri, for which $\log D = 4\cdot01$ (D is the projection of R on the sphere). In this binary, however, $\Delta m < 4\cdot0$ and for the next widest pair, ζ Ret, $\log D = 3\cdot5$ [1]).

It appears that the interval in $\log R$ covered by the present investigation is about 2. Accordingly, the fraction, per unit interval in $\log R$, of all stars brighter than $6^{M}\cdot5$ (absolute, $p = 0''\cdot1$) that is double with $\Delta m < 4\cdot0$ is $0\cdot00 \pm 0\cdot018$ (m.e.) or $+ 0\cdot014 \pm 0\cdot010$ (p.e.) [2]), valid for $4 < \log R < 6$.

For the same reason that made it desirable to take in this investigation 10 parsecs as an upper distance limit rather than 15, it is not very feasible to extend this search to still wider pairs. Future *improvement* in the determination of the number of wide physical pairs will mainly depend on the following items:

1. Increase in the completeness of the known stars up to distances of 15 or 20 parsecs (the effect on the accuracy is quadratic).
2. Increase in the accuracy of the trigonometric parallaxes for these stars by repeated measurement.
3. Determination of the radial velocities of these stars within 1 or 2 km/sec, if possible.

NOTES.

Note A. We suppose that in a certain interval of relative velocity (for instance from 0 to $3\frac{1}{2}$ km/sec) a number N pairs of stars (physical and accidental combinations) have been observed; further, that we know the probable error of a velocity component for the stars considered, so that it immediately follows how large a fraction of the number of physical pairs, present in the material studied, may be expected to be included in the number N; we call this fraction f. The stars studied form an arbitrary sample (of number M) of the stars in general; the number of physical pairs present in this sample will, as a rule, not be equal to the mean of the numbers of all possible samples, which mean, however, we desire

1) Ibid. The preference of one of these values over the other is a matter of taste. The determination gives a frequency curve with the maximum at 0 but no negative values.

1) In the sphere of 15 parsecs the widest known physical pair is 26 *Dra AB, C* for which $\log D = 4\cdot04$. There is a remote possibility that *Co.D.* — 42° 5678 ($9^{h}47^{m}\cdot1$ — 43°01′; 1900) is connected with LUYTEN 268*a*, *H.C.* 283, which is 11550″ distant. If the connection is real, $\log D = 5\cdot10$. The reality, however, is very doubtful and, moreover, the star is not contained in our list of stars with known space velocities.

2) See note under previous column.

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1934BAN.....7..227K

to know. We call this mean number of physical pairs (per M stars): $\bar{n}_p$; the chance to have n_p physical pairs among the M stars investigated is then:

$$C = \frac{(\bar{n})_p^{\,n_p}}{(n)_p\,!} \cdot e^{-\bar{n}_p} \quad {}^{1)}.$$

We consider this also to be the chance that $f.n_p$ physical pairs among the M stars fall into the first interval of relative velocity; we have thereby neglected the fluctuation of f by chance and taken its mean value, derived in Note B.

The number of accidental combinations among the N pairs found in the first interval is $N - fn_p$; the probability that this number occurs is:

$$C' = \frac{\bar{n}_c^{\,(N - fn_p)}}{(N - fn_p)\,!}\; e^{-\bar{n}_c},$$

where $\bar{n}_c$ is the mean number of accidental pairs, which is accurately known from Fig. 1. The chance to have fn_p physical pairs and $N - fn_p$ accidental ones is the product CC'; and to find the total number N (in the supposition that, in the mean, $\bar{n}_p$ physical pairs are present among M stars): $\varkappa = \Sigma\, CC'$; the summation to be extended over all integral values of $fn_p \leq N$. Now $\varkappa$ gives the relative probability that $\bar{n}_p$ is the actual mean number of physical pairs included among M stars [2]).

The analysis in the preceding Note is due to a conversation with Dr. J. H. Oort.

With the above formulae the frequency curves of $\bar{n}_p$ were derived for all 4 groups of stars; for the groups a and d the details are given below. The required values of N, $\bar{n}_c$ and f are taken from Tables 2 and 4.

Group a; a_2. The sum $\varkappa = \Sigma\, CC'$ consists of two terms, corresponding to $fn_p = 0$ and 1. We find:

$$\varkappa = \left(0.4 + \frac{\bar{n}_p^{\,1\cdot 7}}{(1\cdot 7)!}\right) e^{-(0\cdot 4 + \bar{n}_p)}\,,$$

where $0\cdot 4 = \bar{n}_c$ and $1\cdot 7 = \dfrac{1}{f}$.

Dividing $\varkappa$ by

$$\int_0^\infty \varkappa\,(\bar{n}_p)\, d\bar{n}_p = \left(0\cdot 4 + \frac{\Gamma\,(2\cdot 7)}{\Gamma\,(2\cdot 7)}\right) e^{-0\cdot 4}\,,$$

we find for the normalized frequency curve:

$$\frac{e^{0\cdot 4}}{1\cdot 4}\,\varkappa = (0\cdot 286 + 0\cdot 462\,\bar{n}_p^{\,1\cdot 7})\; e^{-\bar{n}_p}\,.$$

This curve is given in Table 5, column 2.

In the same way the columns for a_3, a_4 and a_5 were derived. The sum $\varkappa$ consists of 2, 2 and 4 terms respectively in these cases.

Table 5.

Frequency curves of $\bar{n}_p$.

$\bar{n}_p$	a_2	a_3	a_4	a_5
	0–2½	0–3½	0–4½	0–5½
1	·286	·500	·630	·412
1	·275	·359	·367	·316
2	·242	·210	·186	·210
3	·163	·109	·088	·127
4	·094	·052	·039	·071
5	·050	·024	·017	·038
6	·025	·011	·007	·019

It is evident from Table 5 that group a_4, i.e. the interval 0–4½ km/sec, gives the sharpest determination of n_p. This, however, is partly accidental as is seen by comparison with the approximate determination in section 4, Table 2; there a negative number is found for group a_4. Accordingly, to avoid a spurious accuracy we shall use a_3.

Since no negative values for $\bar{n}_p$ are possible, the probable or median value is always positive. As the probable error of the median value of $\bar{n}_p$ we may take half the interval that includes from ¼ to ¾ of the area enclosed by the frequency curve. In this way we find $\bar{n}_p = +\,1\cdot 2 \pm 0\cdot 8$.

Group d. The frequency curves of $\bar{n}_p$ as derived from d_3, d_4 and d_5 are given in Table 6. The interval 0–4½ km/sec seems the best suited to determine $\bar{n}_p$, but inspection of Table 4 shows that this is partly due to the fact that $N < \bar{n}_c$. The interval 0–3½ gives $\bar{n}_p = 1\cdot 39 \pm 0\cdot 98$; the interval 0–4½ : $\bar{n}_p = 0\cdot 98 \pm 0\cdot 75$. We adopt $1\cdot 2 \pm 0\cdot 9$ (p.e.).

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[1]) This is easily derived by considering the problem in one dimension: If m points (m very large) are thrown on a line of unit length, and the interval of the line investigated has a length s (s very small), the mean number of points falling in the interval s is $sm = \bar{\mu}$. The chance that no single point drops into the interval is $(1-s)^m$; one point: $ms\;(1-s)^{m-1}$; p points $\binom{m}{p}\; s^p\,(1-s)^{m-p}$, which approaches $\dfrac{\bar{\mu}^p}{p\,!}\; e^{-\bar{\mu}}$ for $m \rightarrow \infty$, and $s \rightarrow 0$ ($\bar{\mu}$ is kept constant).

[2]) In the summation for $\varkappa$ the quantity fn_p takes all integral values from 0 to N; therefore, the computation of C' presents no difficulties. However, in general f is an arbitrary real fraction and hence n_p a fractional number; this is not illogical since n_p depends on the choice of M. From the derivation of $\varkappa$ it is clear that we want in this case an interpolatory value of C, which is found if we have such a value for $n_p\,!$, the only discontinuous function in C. We shall use for this: $n! = \Gamma\,(n + 1)$.

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1934BAN.....7..227K

TABLE 6.

n_p	0–3½	0–4½	0–5½
0	·500	·697	·451
½	·377	·497	·382
1	·316	·359	·318
2	·211	·180	·205
3	·125	·085	·122
4	·067	·039	·068
5	·033	·017	·037

Note B. Consider a number of wide physical pairs, which can be recognized only by their space velocities. In Note 1, p. 229, it is explained why the relative velocities in such systems probably are negligible. The relative *observed* velocities, however, will follow a certain frequency curve, which is entirely determined by the probable error ε of one component of an individual space velocity. (In this Note ε is considered to be the same in all three components). Considering ε to be known, the problem of this Note is to find this frequency curve, which is needed for the computation of f. (See Note A) [1]).

Consider two observed velocities (taken relative to the *real* motion of the pair), viz. c_1 and c_2 $(c_2 > c_1)$ [2]). The relative observed velocity is

$$r = \sqrt{c_1^2 + c_2^2 - 2c_1\, c_2 \cos \vartheta},$$

if ϑ is the angle between c_1 and c_2. We first determine the frequency curve $f_1\,(r)$ of r if c_1 and c_2 are fixed, and ϑ is allowed to vary. The chance that ϑ lies between ϑ and $\vartheta + d\vartheta$ is $\frac{1}{2} \sin \vartheta\, d\vartheta$; expressing ϑ in r, c_1 and c_2, we find:

$$f_1\,(r)\; dr = \frac{r\, dr}{2 c_1\, c_2},$$

valid for $c_2 - c_1 < r < c_2 + c_1$. ($f_1\,(r) = 0$ outside these limits).

The next step is to allow c_2 to vary from 0 to ∞ [2]). The frequency of c_2 being between c_2 and $c_2 + d\,c_2$ is $\frac{4h^3}{\sqrt{\pi}}\, c_2^2\, e^{-h^2 c_2^2}\; dc_2$. Multiplying $f_1\,(r)$ by this frequency and integrating with respect to c_2 [3]), we find the frequency curve $f_2\,(r)$ of the possible relative velocities with one fixed velocity, c_2. Repeating the multiplication and integration for c_1, we find the general frequency curve $F\,(r)$ for r:

$$F\,(r)\; dr = h^3\, r^2 \sqrt{\frac{2}{\pi}}\; e^{-\frac{h^2 r^2}{2}}\; dr.$$

Expressing the modulus h in terms of the probable error ε by the relation $h\,\varepsilon = 0{\cdot}47694$, we find for the frequency curve of the relative observed velocities:

$$F\,(V_r)\, d\,V_r = 0{\cdot}0864 \left(\frac{V_r}{\varepsilon}\right)^2 e^{-0{\cdot}114\left(\frac{V_r}{\varepsilon}\right)^2} d\left(\frac{V_r}{\varepsilon}\right) = $$

$$= f\left(\frac{V_r}{\varepsilon}\right)\, d\left(\frac{V_r}{\varepsilon}\right).$$

We want the quantity

$$f = \int_0^a F\,(V_r)\; d\,V_r = \int_0^{\frac{a}{\varepsilon}} f\left(\frac{V_r}{\varepsilon}\right)\, d\,\frac{V_r}{\varepsilon},$$

in which a represents the interval of relative observed velocity studied. From the last integral it appears that f may be tabulated as a function of $\frac{a}{\varepsilon}$ [1]), as has been done in the following Table 7:

TABLE 7.

$\frac{a}{\varepsilon}$	f	$\frac{a}{\varepsilon}$	f	$\frac{a}{\varepsilon}$	f
0·5	·004	3·5	·575	6·5	·978
1·0	·027	4·0	·698	7·0	·989
1·5	·085	4·5	·798	7·5	·995
2·0	·178	5·0	·873	8·0	·998
2·5	·301	5·5	·925	8·5	·999
3·0	·438	6·0	·958	9·0	1·000

Note C. The purpose of this Note is: 1. To obtain an approximate value of the mean p.e. of a radial velocity used in our list of space velocities. 2. To estimate the mean p.e. of one component of a space velocity (ε).

1. The radial velocities of the stars in our list between 0^h and 4^h R. A., obtained by different observers, were compared after the systematic corrections given in Lick Publications, 16, p. XXXI had been applied. The probable errors as derived from the internal agreement between the measures by one observatory were compared with those derived by intercomparing the mean values obtained at different institutions. In this way it was found that the former

1) The formulae in the textbooks on the Kinetic Theory of Gases give the *mean* relative observed velocity, expressed in the value of ε. However, we need the whole frequency curve.

2) Therefore, c_1 and c_2 are the errors of observation.

3) This has to be done in two steps: from c_1 to ∞, so that $c_1 > c_2$; and from 0 to c_2, in this case interchanging the roles of c_1 and c_2.

1) This was anticipated, of course.

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1934BAN.....7..227K

p.e. had to be multiplied by about 1·2 in order to represent the true accuracy [1]).

For the mean corrected p.e. of the radial velocity for 15 stars with $p > 0''\cdot100$, $\pm$ 0·56 km/sec was found. For 12 stars with $p < 0''\cdot100$ (fainter, on the average), it was $\pm$ 1·1 km/sec; for the mean of all these 27 stars, $\pm$ 0·8 km/sec. These figures have been used for all the stars in our list, except that the first p.e. has been changed to $\pm$ 0·6 km/sec.

2. We shall estimate the mean p.e. ε of one component of a space velocity in two ways:

a. Consider the X component (X_1) of the velocity of a star relative to the Sun. X_1 partly originates from the radial, partly from the transverse velocity; as both are arbitrarily distributed with respect to the X axis (taking all the stars in our list together), half the error of the radial velocity and half the error of the transverse velocity contribute to the error in X_1 [2]). For the first contribution we have (group *a*; section 3): $\frac{1}{2} \times (\pm 0\cdot6$ km/sec$) = \pm 0\cdot3$ km/sec. The mean space velocity in Region I (section 3) is about 30 km/sec relative to the Sun; the mean transverse velocity, therefore, $30 . \pi/4 = 24$ km/sec. The mean $\frac{\text{p.e. of } p}{p}$ for group *a* is 0·053; hence the mean p.e. of a transverse velocity $\pm 0\cdot053 \times 24 = \pm 1\cdot2$ km/sec. The second contribution of errors on the X axis, therefore, amounts to $\pm$ 0·6 km/sec. If we neglect the criticism of Note 2 and the influence of the non-gaussian form of the two error curves, the mean p.e. of the X component is about $\pm$ 0·7 km/sec.

b. Somewhat better is the following computation. Take for a small area of the sky the Z axis along the line of sight; then $\bar{\varepsilon}_z = 0\cdot6$ km/sec. For the parallax stars considered, ε_τ of the transverse velocity will be mainly determined by ε_p of the parallax, the relation being $\varepsilon_\tau = 4\cdot74 \frac{\mu}{p^2} \varepsilon_p$ (μ = proper motion, p = parallax, both in seconds; ε_p = p.e. of the parallax, in seconds). We may assume that the errors in the parallaxes used are fairly well represented by a gaussian error curve, with a p.e. $= \bar{\varepsilon}_p$. But we can hardly expect that the errors in the transverse velocities are gaussian, with a p.e. $= \bar{\varepsilon}_\tau$, found from $4\cdot74\overline{\left(\frac{\mu}{p^2}\right)}\cdot\bar{\varepsilon}_p$. However, it probably is an approximation close enough for our purpose. We find: $\bar{\varepsilon}_p = 0''\cdot0072$; $\overline{\left(\frac{\mu}{p^2}\right)} = \frac{32\cdot2}{1''}$; $\bar{\varepsilon}_\tau = 1\cdot1$ km/sec.

From this we find for $\bar{\varepsilon}_x$ and $\bar{\varepsilon}_y$: $\frac{\bar{\varepsilon}_\tau}{\sqrt{2}} = 0\cdot8$ km/sec, neglecting a new source that makes the distribution of the errors non-gaussian, viz. the effect of the various position angles of the transverse velocity, which makes the frequency curve of the errors in the X and Y coordinates the superposition of a great number of frequency curves (one for every position angle), all with a different dispersion. For ε we may take the average of ε_x, ε_y and ε_z, because the parallax stars considered are distributed over the whole sky and so the Z axis takes all different directions. Hence $\varepsilon = 1/3\ (0\cdot6 + 0\cdot8 + 0\cdot8) = 0\cdot7$ km/sec, agreeing with the value found under *a*.

For the three other groups (*b*, *c*, *d*) of stars we find, following the second method of computation, the values of $\bar{\varepsilon}_p$, $\bar{\varepsilon}_\tau$ and ε given in the accompanying table.

Group	$\bar{\varepsilon}_p$	$\bar{\varepsilon}_\tau$	ε
	″	km/sec	km/sec
b	0·0072	1·61	1·1
c	0·0066 [1])	3·46	1·9
d	0·0070	1·70	0·93

1) This factor is about the same as that for trigonometric parallaxes.

2) This simple reasoning neglects the fact that for every individual star the radial and the transverse velocity are not independent in their orientation to the X axis.

1) These stars with large proper motion have been measured more frequently for parallax, which reduces $\bar{\varepsilon}_p$.

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