Superconductivity in the mean-field anyon gas

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The linear electromagnetic response of an anyon gas at zero temperature is obtained from the mean-field (Hartree) Hamiltonian. Zero resistivity and the Meissner effect follow from the integer quantum Hall effect in the fictitious (statistical) magnetic field of the flux tubes bound to the anyons—provided that the electric field induced by the motion of the flux tubes is taken into account.

The introduction of particles with fractional statistics (*anyons*[1]) into condensed-matter physics has led to a number of interesting applications. Among these, of particular importance is the suggestion that high-temperature superconductivity might originate from these (quasi-) particles.[2] Indeed, it appears now to be well established that a two-dimensional ideal gas of noninteracting anyons has a superconducting ground state.[3–11]

Recently, Leggett has addressed the more limited, but highly interesting, question whether the anyon gas is superconducting *when solved in the mean-field approximation.*[12] In that approximation the anyon gas is replaced by a gas of fermions, subject to a perpendicular magnetic field $\mathbf{B}^f(\mathbf{r}) = (h/pe)n(\mathbf{r})\hat{\mathbf{z}}$ proportional to the particle density $n(\mathbf{r})$ (in the x-y plane). The strength of this fictitious (or "statistical") magnetic field adjusts itself in such a way to variations in the density, that p Landau levels are kept fully occupied. (The value $p = 2$ is expected to be relevant for high-temperature superconductivity.[2]) The single-particle eigenstates of the mean-field Hamiltonian are extended along equipotentials of the electrostatic potential. Leggett uses the insensitivity of these eigenstates to variations in the boundary conditions (in a Corbino-disk geometry), to argue that the mean-field anyon gas is an *insulator*—rather than a superconductor.

In this paper we reexamine the mean-field theory of the anyon gas. We show that the mean-field Hamiltonian contains, in addition to the fictitious magnetic field $\mathbf{B}^f$ mentioned above, also a fictitious *electric* field $\mathbf{E}^f(\mathbf{r}) = (h/pe^2)\hat{\mathbf{z}} \times \mathbf{j}(\mathbf{r})$ proportional to the current density $\mathbf{j}(\mathbf{r})$. This electric field arises because in the original Hamiltonian the anyons are composed of fermions bound to a flux tube, of strength h/pe and of infinitesimal cross section. In the mean-field approximation, the flux tubes are smeared out, and one obtains the fictitious magnetic field $\mathbf{B}^f$ proportional to n. However, the flux tubes remain bound to the particles. When a flux tube at $\mathbf{r}_a$ moves with the velocity $\mathbf{v}_a$, it induces an electric field $\mathbf{E}(\mathbf{r}) = -\mathbf{v}_a \times \mathbf{b}(\mathbf{r} - \mathbf{r}_a)$, where $\mathbf{b}$ is the magnetic field of the flux tube. As we will show below, the fictitious electric field transforms the mean-field anyon gas from an insulator into a superconductor. We consider the case of an ideal (impurity-free) anyon gas in detail, but we will argue that the superconductivity persists in the presence of disorder (by using results from the integer quantum Hall effect).

The anyon Hamiltonian in the fermion-gauge representation is given by

$$\mathcal{H} = \sum_i \frac{1}{2m} (\mathbf{p}_i - e\mathbf{A}_i)^2 , \tag{1}$$

$$\mathbf{A}_i = \sum_{j\neq i} \mathbf{a}(\mathbf{r}_{ij}),\ \mathbf{a}(\mathbf{r}) = \frac{h}{pe}\frac{\hat{\mathbf{z}} \times \mathbf{r}}{2\pi r^2}. \tag{2}$$

The Hartree-Fock equations are obtained by approximating the ground state Ψ of the many-body Hamiltonian (1) by a Slater determinant of single-particle wave functions $\{\psi_i\}$ and minimizing the energy $E_{\rm HF} =< \Psi_{\rm HF}|\mathcal{H}|\Psi_{\rm HF} >$. Disregarding the exchange terms, we obtain in a straightforward manner the single-particle mean-field (Hartree) Hamiltonian for anyons,

$$\mathcal{H}^H = \frac{1}{2m} \left(\mathbf{p} - e\mathbf{A}^f\right)^2 + e\Phi^f. \tag{3}$$

The fictitious electromagnetic potentials Φ^f and $\mathbf{A}^f$ are related to the particle density n and charge current density $\mathbf{j}$ by

$$\mathbf{A}^f(\mathbf{r},t) = \int d\mathbf{r}'\, \mathbf{a}(\mathbf{r}-\mathbf{r}')\, n(\mathbf{r}',t), \tag{4}$$

$$e\Phi^f(\mathbf{r},t) = \int d\mathbf{r}'\, \mathbf{a}(\mathbf{r}-\mathbf{r}') \cdot \mathbf{j}(\mathbf{r}',t). \tag{5}$$

The corresponding fictitious electric and magnetic fields take the form

$$\mathbf{B}^f(\mathbf{r},t) = \nabla \times \mathbf{A}^f = \frac{h}{pe} n(\mathbf{r},t)\hat{\mathbf{z}}, \tag{6}$$

$$\mathbf{E}^f(\mathbf{r},t) = -\partial_t \mathbf{A}^f - \nabla\Phi^f = \frac{h}{pe^2}\hat{\mathbf{z}} \times \mathbf{j}(\mathbf{r},t). \tag{7}$$

The wave functions $\{\psi_i\}$ are determined by solving the Schrödinger equation $\mathcal{H}^H\psi_i = E_i\psi_i$ together with Eqs. (4) and (5) self-consistently.

The term $e\Phi^f$ in Eq. (3) accounts for the fictitious electric field induced by the moving flux tubes. Such a term is required by Galilean invariance, but was omitted in a previous mean-field theory of the anyon gas.[11] Other approaches to the problem,[3–10] being Galilean invariant, include this term implicitly. Since the extra term $e\Phi^f$ is proportional to the current density [see Eq. (5)], it may presumably be disregarded in calculations of the density response.[11] However, it plays a crucial role in the current response (i.e., in the conductivity), as we now show.

When an external electromagnetic field $(\Phi^{\rm ex}, \mathbf{A}^{\rm ex})$ is switched on, the anyon density and current distributions are modified. Therefore, the perturbation Hamiltonian $\Delta\mathcal{H}$ contains terms due to the variations $\delta\Phi^f$ and $\delta\mathbf{A}^f$ in the internal fictitious fields from their ground-state values $\Phi^{\rm gr}$ and $\mathbf{A}^{\rm gr}$. To first order in the perturbation, one obtains

$$\mathcal{H}^H = \mathcal{H}_0 + \Delta\mathcal{H}, \tag{8}$$

$$\mathcal{H}_0 = \frac{1}{2m}\left(\mathbf{p} - e\mathbf{A}^{\rm gr}\right)^2 + e\Phi^{\rm gr}, \tag{9}$$

$$\Delta\mathcal{H} = -\frac{e}{m}\left(\mathbf{p} - e\mathbf{A}^{\rm gr}\right)\cdot\left(\mathbf{A}^{\rm ex} + \delta\mathbf{A}^f\right) + e\left(\Phi^{\rm ex} + \delta\Phi^f\right). \tag{10}$$

A straightforward application of the Kubo formalism,[13] yields in the long-wavelength limit $(\mathbf{k} = \mathbf{0})$ the current density response

$$\delta\mathbf{j}(\omega) = \boldsymbol{\sigma}_0(\omega)\cdot\left[\mathbf{E}^{\rm ex}(\omega) + \delta\mathbf{E}^f(\omega)\right], \tag{11}$$

where $\mathbf{E}^{\rm ex} = {\rm i}\omega\mathbf{A}^{\rm ex} - \nabla\Phi^{\rm ex}$. The conductivity tensor $\boldsymbol{\sigma}_0$ is associated with the Hamiltonian $\mathcal{H}_0$ with unperturbed potentials $\Phi^{\rm gr}$ and $\mathbf{A}^{\rm gr}$. To obtain the true conductivity tensor $\boldsymbol{\sigma}$, one still needs to eliminate $\delta\mathbf{E}^f$ from Eq. (11) by applying the self-consistency relation (7), which we write in the form

$$\delta\mathbf{E}^f = -\frac{h}{pe^2}\boldsymbol{\epsilon}\cdot\delta\mathbf{j}. \tag{12}$$

Here, $\boldsymbol{\epsilon}$ is the antisymmetric tensor of rank two ($\epsilon_{xx} = \epsilon_{yy} = 0, \epsilon_{xy} = -\epsilon_{yx} = 1$). The solution to Eqs. (11) and (12) is

$$\delta\mathbf{j}(\omega) = \boldsymbol{\sigma}(\omega)\cdot\mathbf{E}^{\rm ex}(\omega), \tag{13}$$

$$\boldsymbol{\sigma}(\omega) = \left(\boldsymbol{\rho}_0(\omega) + \frac{h}{pe^2}\boldsymbol{\epsilon}\right)^{-1}, \quad \boldsymbol{\rho}_0 \equiv \boldsymbol{\sigma}_0^{-1}. \tag{14}$$

Now we use that $\mathcal{H}_0$ describes a fermion gas with p fully filled Landau levels. This implies that $\boldsymbol{\rho}_0$ equals the integer quantum Hall effect resistivity tensor at filling factor p, i.e.,

$$(\rho_0)_{xx} = (\rho_0)_{yy} = -{\rm i}\omega\frac{m}{e^2 n_0}, \quad (\rho_0)_{xy} = -(\rho_0)_{yx} = -\frac{h}{pe^2}, \tag{15}$$

where n_0 is the bulk density. Upon substitution of the expression for $\boldsymbol{\rho}_0$ into Eq. (14), one obtains

$$\sigma_{\alpha\beta} = -\frac{1}{{\rm i}\omega}\frac{e^2 n_0}{m}\delta_{\alpha\beta}. \tag{16}$$

The anyon gas is thus a perfect conductor in the limit $\omega \to 0$. We now take the curl of Eq. (13) and substitute Eq. (16) to arrive at the London equation

$$\nabla\times\delta\mathbf{j} = -\frac{e^2 n_0}{m}\mathbf{B}^{\rm ex}. \tag{17}$$

So far we have considered the electromagnetic response for $\mathbf{k} = \mathbf{0}$ in the limit $\omega \to 0$. The existence of the Meissner effect depends on whether Eq. (17) is valid for $\omega = 0$, in the subsequent limit $\mathbf{k} \to \mathbf{0}$.[14] As we will now show, this is indeed the case.

To demonstrate that Eq. (17) holds regardless of the order of the limits $\mathbf{k} \to \mathbf{0}$ and $\omega \to 0$, we consider the general form of the linear response for density and current density in the Fourier space,

$$\delta\mathcal{J}_\mu(\mathbf{k},\omega) = K_{\mu\nu}(\mathbf{k},\omega)\mathcal{A}_\nu(\mathbf{k},\omega), \quad \mathcal{J} = (en, \mathbf{j}), \quad \mathcal{A} = (\Phi, -\mathbf{A}). \tag{18}$$

The response function $K_{\mu\nu}$ is defined in the space-time representation as

$$K_{\mu\nu}(\mathbf{r},t;\mathbf{r}',t') = ({\rm i}\hbar)^{-1}\langle\Psi|\left[\mathcal{J}_\mu(\mathbf{r},t),\mathcal{J}_\nu(\mathbf{r}',t')\right]|\Psi\rangle\theta(t-t') + \frac{n_0}{m}(1-\delta_{\mu 0})\delta_{\mu\nu}. \tag{19}$$

Because of the continuity equations $\partial_\mu K_{\mu\nu} = \partial'_\nu K_{\mu\nu} = 0$, the elements of the tensor $K_{\mu\nu}$ can all be expressed in terms of four coefficients $\chi_0, \xi_0, \eta_0, \zeta_0$, according to

$$K_{00}(\mathbf{k},\omega) = -k^2\chi_0(\mathbf{k},\omega), \tag{20}$$

$$K_{0j}(\mathbf{k},\omega) = -\omega\chi_0(\mathbf{k},\omega)k_j + {\rm i}\xi_0(\mathbf{k},\omega)\epsilon_{jn}k_n, \tag{21}$$

$$K_{i0}(\mathbf{k},\omega) = -\omega\chi_0(\mathbf{k},\omega)k_i + {\rm i}\eta_0(\mathbf{k},\omega)\epsilon_{im}k_m, \tag{22}$$

$$K_{ij}(\mathbf{k},\omega) = -\frac{\omega^2}{k^2}\chi_0(\mathbf{k},\omega)k_ik_j + \frac{{\rm i}\omega}{k^2}\eta_0(\mathbf{k},\omega)\epsilon_{im}k_mk_j + \frac{{\rm i}\omega}{k^2}\xi_0(\mathbf{k},\omega)k_i\epsilon_{jn}k_n + \frac{1}{k^2}\zeta_0(\mathbf{k},\omega)\epsilon_{im}k_m\epsilon_{jn}k_n. \tag{23}$$

Here, roman indices denote the Cartesian components x, y. The physical meaning of these coefficients becomes more apparent if we rewrite the response equation (18) in terms of the density δn and the vorticity $\delta\Omega \equiv \hat{\mathbf{z}}\cdot({\rm i}\mathbf{k}\times\delta\mathbf{j})$. (The component of $\delta\mathbf{j}$ along $\mathbf{k}$ is not an independent variable as it is constrained by the continuity equation ${\rm i}\mathbf{k}\cdot\delta\mathbf{j} = {\rm i}\omega e\delta n$.) The resulting equations are[15]

$$e\delta n(\mathbf{k},\omega) = -\chi_0(\mathbf{k},\omega)k^2\Phi(\mathbf{k},\omega) + \xi_0(\mathbf{k},\omega)B(\mathbf{k},\omega), \tag{24}$$

$$\delta\Omega(\mathbf{k},\omega) = \eta_0(\mathbf{k},\omega)k^2\Phi(\mathbf{k},\omega) - \zeta_0(\mathbf{k},\omega)B(\mathbf{k},\omega). \tag{25}$$

The potential Φ and the magnetic field B contain an external part and a fictitious part, given according to Eqs. (5) and (6) by

$$\Phi = \Phi^{\rm ex} + \delta\Phi^f = \Phi^{\rm ex} - \frac{h}{pe^2k^2}\delta\Omega, \quad (26)$$

$$B = B^{\rm ex} + \delta B^f = B^{\rm ex} + \frac{h}{pe}\delta n. \quad (27)$$

In Eq. (28) we used the result $\mathbf{a}(\mathbf{k}) = (h/pek^2)i\mathbf{k} \times \hat{\mathbf{z}}$ for the Fourier transform of the flux tube vector potential. We now solve Eqs. (26)–(29) for the case $\Phi^{\rm ex} = 0$, and find

$$\delta\Omega(\mathbf{k},\omega) = -\zeta(\mathbf{k},\omega)B^{\rm ex}(\mathbf{k},\omega), \quad (28)$$

$$\zeta \equiv \{[1-(h/pe^2)\xi_0][1+(h/pe^2)\eta_0] + (h/pe^2)^2\chi_0\zeta_0\}^{-1}\zeta_0. \quad (29)$$

The Meissner effect is obtained if $\zeta(\mathbf{k},0) > 0$ in the limit $\mathbf{k} \to \mathbf{0}$. We now calculate this limit and show that it is the same as $\lim_{\omega\to 0}\zeta(0,\omega)$, thereby proving the analyticity of the response function.

By direct evaluation of Eq. (19), for the case of an ideal (impurity-free) two-dimensional electron gas at the filling factor p, one obtains the following relations:

$$\chi_0(0,\omega) = \chi_0(0,0) + O(\omega^2), \chi_0(\mathbf{k},0) = \chi_0(0,0) + O(k^2), \quad (30)$$

$$\xi_0(0,\omega) = \frac{pe^2}{h} + O(\omega^2), \xi_0(\mathbf{k},0) = \frac{pe^2}{h} + O(k^2), \quad (31)$$

$$\eta_0(0,\omega) = -\frac{pe^2}{h} + O(\omega^2), \eta_0(\mathbf{k},0) = -\frac{pe^2}{h} + O(k^2), \quad (32)$$

$$\zeta_0(0,\omega) = O(\omega^2), \zeta_0(\mathbf{k},0) = O(k^2). \quad (33)$$

In Eq. (30), the constant $\chi_0(0,0)$ is given by

$$\chi_0(0,0) = \frac{mp^2e^2}{n_0h^2}. \quad (34)$$

Substitution of these relations into Eq. (29) yields

$$\lim_{\mathbf{k}\to\mathbf{0}}\zeta(\mathbf{k},0) = \lim_{\omega\to 0}\zeta(0,\omega) = \Lambda^{-2}, \quad (35)$$

$$\Lambda \equiv \frac{h}{pe^2}[\chi_0(0,0)]^{1/2} = \left(\frac{m}{e^2n_0}\right)^{1/2}. \quad (36)$$

We conclude that $\zeta(\mathbf{k},\omega)$ is analytic at $(\mathbf{k},\omega) = (\mathbf{0},0)$, so that the anyon gas shows the perfect conductivity and the Meissner effect in accordance with the London equation (17) . This completes our demonstration of superconductivity of the anyon gas in the mean-field approximation. We conclude this paper with a discussion of the symmetry of the conductivity tensor and of the influence of impurities.

Since the anyon Hamiltonian (1) is not invariant under time reversal, it is possible in principle to have a nonsymmetric conductivity tensor. In the foregoing analysis we have seen that although $\boldsymbol{\sigma}_0$ is not a symmetric tensor, the true conductivity $\boldsymbol{\sigma}$ is symmetrical. We believe that the symmetry of $\boldsymbol{\sigma}$, derived here at $T = 0$ in the mean-field approximation, holds also at higher temperatures and particularly in the normal state. A heuristic way to see this is to replace $\boldsymbol{\rho}_0$ by the classical resistivity

$$(\rho_0)_{xx} = (\rho_0)_{yy} = \frac{m}{e^2n_0\tau}, \quad (\rho_0)_{xy} = -(\rho_0)_{yx} = -\frac{h}{pe^2}, \quad (37)$$

where τ is a relaxation time. Substitution of this expression into Eq. (14) leads to the Drude conductivity tensor, i.e., a symmetrical $\boldsymbol{\sigma}$. The question of the symmetry of $\boldsymbol{\sigma}$ in the normal state is relevant for the recent experimental search by Gijs *et al.*[16] for a spontaneous Hall effect in zero magnetic field. In this experiment a symmetrical conductivity tensor was found within the experimental resolution. In our description of the anyon gas the Hall electric field originating from the fictitious magnetic field is fully compensated by the fictitious electric field induced by the moving flux tubes. An explanation in different terms has recently been put forward by Wiegmann.[8]

The demonstration of superconductivity given above can be generalized to include a uniform distribution of impurities. The key step in this generalization is to show that Eqs. (32)–(35) remain valid. This can be shown if the gap in the density of states for the integer quantum Hall effect Hamiltonian $\mathcal{H}_0$ is not closed by the impurities. The presence of the excitation gap then implies that the response coefficients $\chi_0, \xi_0, \eta_0, \zeta_0$ are analytical at $(\mathbf{k},\omega) = (\mathbf{0},0)$, and hence Eqs. (32) and (35) result. (Whether a *mobility* gap is sufficient for the analyticity is not clear to us.) Equations (31) and (32) are enforced by the quantum Hall effect, since $\zeta_0(\mathbf{0},0) = -\eta_0(\mathbf{0},0) = [\sigma_0(0)]_{xy} = pe^2/h$ regardless of the presence of impurities. Note that the impurities will modify the penetration depth $\Lambda = (h/pe^2)[\chi_0(\mathbf{0},0)]^{1/2}$, through their effect on the susceptibility $K_{00} = -k^2\chi_0$. We hope to return to this effect in a future publication.

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