Relativistic moment equations in media with strong velocity gradients

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Abstract. The one-dimensional stationary PSTF moment equations for radiative transfer in relativistically differentially moving media are examined. It is found that the equations can have two different types of critical points. In regions of strong velocity gradients, solutions might have pathological behavior around such critical points. The moment method may therefore yield unphysical solutions for the radiation field in such flows.

Key words: hydrodynamics – relativity – radiative transfer – shock waves

1. Introduction

In this paper I investigate an aspect of the mathematical structure of the equations governing the dynamics of relativistically moving radiative plasmas. This can be of relevance to problems of accretion onto black holes and neutron stars, problems concerning the origin and powering of relativistic jets and problems related to gamma–ray bursts. In relativistically moving plasmas ($v \gtrsim 0.3c$), radiative transfer can have an important dynamical effect over length–scales of the order of a photon mean free path. It is difficult to take such transport effects into account, because the radiative transfer equation is a Boltzmann equation for photons. As soon as the optical depth is of order unity, approximative treatments (e.g. diffusion and Fokker-Planck) break down and the full transfer equation should be solved. Moreover, for relativistically differentially moving plasmas this Boltzmann equation is very complex (Thomas 1930, Lindquist 1966, Mihalas 1980) due to relativistic aberration effects. Solving these equations analytically is almost always impossible, and even numerically it is quite costly to do so in sufficient detail to be useful in a dynamical calculation.

There is an alternative to solving the full Boltzmann equation. In dynamical calculations one is usually only interested in the radiative energy- and momentum density, and its pressure. Of course these follow from detailed transfer calculations, but they are costly. It is better to expand the transfer equation into moments, and solve these moment equations directly. The first three moments are precisely the energy density, momentum density and the pressure (for frequency integrated moments), so that the transfer calculation is performed directly on the quantities of interest. Unfortunately the equations for these three quantities do not form a closed set, which necessitates the inclusion of higher moments and their equations. The transfer problem remains just as complex; it is only rephrased in another basis which better suits the purpose of dynamical calculations. Of course, an infinite set of equations cannot be solved numerically, so one must truncate this infinite set by adopting a closure assumption for the higher moments. This way one may tune the accuracy of the calculation by choosing the number of moments taken into account.

The idea of using a moment expansion of the Boltzmann equation is due to Maxwell, and was later cast into a consistent theory by Grad (1949). Its application to radiative transfer originated from the work of Eddington, Milne and many others in the early 20th century, but it took a long time before consistent moment equations for relativistic radiative transfer were derived (Lindquist 1966, Anderson & Spiegel 1972, Mihalas 1980, Thorne 1981, Udey & Israel 1982, Schweizer 1982).

Thorne's paper presents a complete moment formalism for relativistic flow, without any approximations. It is called the PSTF formalism, and it is a good starting point for any study involving moments of radiation. For spherically symmetric problems this formalism has been used by several authors (e.g. Thorne et al. 1981, Turolla & Nobili 1988, Nobili et al. 1991, 1993). It has proven to be a powerful tool for tackling problems of relativistic radiative transfer and relativistic radiation hydrodynamics for 1D flows. Despite its success, however, the results from a study using the moment method should always be regarded with some amount of suspicion, especially when used in the regime of marginal optical thickness.

In this paper I investigate the mathematical structure of the moment equations, and focus on their behavior in regions of vigorous differential fluid motion where strong aberration effects can be expected. I specifically concentrate on relativistic radiation pressure dominated shocks. It is shown that, especially around critical points, aberration effects can cause the moment method to go severely astray.

This paper is organized as follows. I start with a review of the moment equations, Sects. 2 and 3. The matrix form of the equations is presented in Sects. 4 and 5. The boundary conditions are discussed in Sects. 6, and then the behavior of solutions around the critical points is studied in Sects. 7 and 8.

2. The relativistic transfer equation

Radiation in a thin to mildly optically thick environment is a non-equilibrium system and must therefore be described by a photon distribution function $f(p^\sigma)$. Here p^σ is the four-momentum of the photon. The distribution function is proportional to the photon occupation number $N(p^\sigma)$:

$$f(p^\sigma) = \frac{2}{h^3} N(p^\sigma) \tag{1}$$

This distribution function is only defined on the light cone $p^\sigma p_\sigma = 0$ and is therefore not a genuine function of all the four components of p^σ. I will retain this misleading notation of $f(p^\sigma)$ in certain cases in order not to sacrifice covariance where this is desired. However, in order to be able to do physics, one needs to reduce this redundant dimensionality. This can only be done by choosing a local spatial hyper surface upon which the light cone can be projected, thus leading to a function of three variables $f(\mathbf{p})$. To do so one has to choose a local tetrad[1] $e^{\hat\alpha}{}_\mu$ so that the photon four-momentum p^μ can be cast into a local tetrad form

$$p^{\hat\alpha} = e^{\hat\alpha}{}_\mu p^\mu \tag{2}$$

This tetrad defines the 'local observer frame' in which the radiative quantities will be expressed. A well-defined spatial projection of $p^{\hat\alpha}$ is $p^{\hat a}$ and $f(p^{\hat a})$ is now a genuine distribution function in momentum space. It is physically appealing to choose the tetrad to be 'comoving' with the fluid, i.e. to take $e^\mu{}_{\hat 0} = u^\mu$, where u^μ is the four-velocity of the fluid. From now on let us choose a plane–parallel geometry as the simplest example. The following analysis can be extended to any geometry without problems. In a plane-parallel situation the comoving tetrad is

$$e^\mu{}_{\hat 0} = \begin{pmatrix} \gamma \\ \gamma\beta \\ 0 \\ 0 \end{pmatrix} \qquad e^\mu{}_{\hat 1} = \begin{pmatrix} \gamma\beta \\ \gamma \\ 0 \\ 0 \end{pmatrix} \tag{3}$$

$$e^\mu{}_{\hat 2} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad e^\mu{}_{\hat 3} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{4}$$

The dynamics of the photon distribution function is governed by the photon Boltzmann equation, which in a relativistic setting reads

$$p^\mu \left[\frac{\partial}{\partial x^\mu} - \omega^{\hat a}{}_{\mu\nu} p^\nu \frac{\partial}{\partial p^{\hat a}} \right] f(p^\sigma) = C[f] \tag{5}$$

The $\omega^{\hat a}{}_{\mu\nu} \equiv \omega^{\hat a}{}_{\mu\hat b} e^{\hat b}{}_\nu$ is the spin connection, defined in Appendix A. The source term $C[f(p^\sigma)]$ on the right hand side takes into account all the interactions of the photons with matter.

In order to obtain a more tractable form of the transfer equation I define μ and ν by

$$\mu = p^{\hat 1}/p^{\hat 0} \tag{6}$$

$$\nu = p^{\hat 0} \tag{7}$$

[1] See Appendix A for definitions.

and derivatives in the tetrad frame (i.e. comoving)

$$\partial_{\hat 0} \equiv \gamma \left(\frac{1}{c} \frac{\partial}{\partial t} + \beta \frac{\partial}{\partial x} \right) \tag{8}$$

$$\partial_{\hat 1} \equiv \gamma \left(\beta \frac{1}{c} \frac{\partial}{\partial t} + \frac{\partial}{\partial x} \right) \tag{9}$$

and introduce the somewhat more familiar quantity of radiative intensity in the tetrad frame:

$$I(\mu, \nu) \equiv \frac{h^4 \nu^3}{c^2} f(\mu, \nu) \tag{10}$$

By explicitly substituting the components of the spin connection $\omega^{\hat a}{}_{\mu\nu}$ into the Boltzmann equation Eq. (5), using full plane-parallel symmetry and considerable algebra one finds the transfer equation of Mihalas (1980) for plane parallel flow,

$$\begin{aligned} &\partial_{\hat 0} I + \mu \partial_{\hat 1} I \\ &+ \frac{\partial}{\partial \mu} \left[\gamma^2(\mu^2 - 1)(\partial_{\hat 0}\beta + \mu\partial_{\hat 1}\beta) I \right] \\ &- \frac{\partial}{\partial \nu} \left[\gamma^2 \mu\nu \, (\partial_{\hat 0}\beta + \mu\partial_{\hat 1}\beta) I \right] \\ &+ \gamma^2 \left[2\mu \, \partial_{\hat 0}\beta + (\mu^2 + 1)\partial_{\hat 1}\beta \right] I \\ &= j - \alpha I \end{aligned} \tag{11}$$

Here the source terms are explicitly written out as emission and absorption terms, as is conventional in this notation.

3. Moment expansion

The radiative intensity $I(\mu, \nu)$ can be expanded in Legendre polynomials $P_n(\mu)$ defined by

$$P_n(\mu) = \frac{1}{2^n} \sum_{m=0}^{[n/2]} (-1)^m \binom{n}{m} \binom{2(n-m)}{n} \mu^{n-2m} \tag{12}$$

which satisfy the recursion relation

$$\mu P_k(\mu) = \frac{k P_{k-1}(\mu) + (k+1) P_{k+1}(\mu)}{2k+1} \tag{13}$$

the orthogonality relation

$$\int_{-1}^{+1} P_k(\mu) P_l(\mu) d\mu = \delta_{kl} \frac{2}{2k+1} \tag{14}$$

and the differential equation

$$(\mu^2 - 1) \frac{dP_k(\mu)}{d\mu} = k\mu P_k(\mu) - k P_{k-1}(\mu) \tag{15}$$

We define the moments of $I(\mu, \nu)$ as

$$w_\nu^k \equiv \frac{2^k (k!)^2}{2 \, (2k)!} \int_{-1}^{+1} I(\mu, \nu) P_k(\mu) d\mu \tag{16}$$

and the intensity $I(\mu, \nu)$ can then be found back by

$$I(\mu) = \sum_{n=0}^{\infty} \frac{(2n+1)!}{2^n (n!)^2} w_\nu^n P_n(\mu) \tag{17}$$

Such an expansion is effectively an expansion of the angular dependency of $I(\mu, \nu)$ in spherical harmonics. Since we have full symmetry around the x^1 axis the Y_{nm} modes are all zero except for the Y_{n0}, so that one only needs the Legendre polynomials for the expansion.

By multiplying the transfer equation (11) by

$$\frac{2^k (k!)^2}{2 \, (2k)!} P_k(\mu) \tag{18}$$

and integrating, using Eqs. (13,15,16), one obtains the transfer equations for the moments (Thorne 1981)

$$
\begin{aligned}
\partial_{\hat{0}} w_\nu^k &+ \partial_{\hat{1}} (w_\nu^{k+1} + a_k w_\nu^{k-1}) \\
&+ \gamma^2 [b_k \partial_{\hat{1}} \beta \cdot w_\nu^{k-2} + c_k \partial_{\hat{0}} \beta \cdot w_\nu^{k-1} \\
&\quad + d_k \partial_{\hat{1}} \beta \cdot w_\nu^k + e_k \partial_{\hat{0}} \beta \cdot w_\nu^{k+1} \\
&\quad + f_k \partial_{\hat{1}} \beta \cdot w_\nu^{k+2}] \\
&- \partial_\nu [\gamma^2 \nu \{ g_k \partial_{\hat{1}} \beta \cdot w_\nu^{k-2} + h_k \partial_{\hat{0}} \beta \cdot w_\nu^{k-1} \\
&\quad + i_k \partial_{\hat{1}} \beta \cdot w_\nu^k + \partial_{\hat{0}} \beta \cdot w_\nu^{k+1} \\
&\quad + \partial_{\hat{1}} \beta \cdot w_\nu^{k+2} \}] = s_\nu^k
\end{aligned}
\tag{19}
$$

where $a_k, b_k, c_k, d_k, e_k, f_k, g_k, h_k, i_k$ are constants defined as

$$a_k = \frac{k^2}{(2k-1)(2k+1)} \tag{20}$$

$$b_k = \frac{(k-1)^2 k^2 (k+2)}{(2k-3)(2k-1)^2(2k+1)} \tag{21}$$

$$c_k = \frac{k^2(k+3)}{(2k-1)(2k+1)} \tag{22}$$

$$d_k = \frac{7k^2 + 7k - 4}{(2k-1)(2k+3)} \tag{23}$$

$$e_k = (2 - k) \tag{24}$$

$$f_k = (1 - k) \tag{25}$$

$$g_k = \frac{k^2(k-1)^2}{(2k-3)(2k-1)^2(2k+1)} \tag{26}$$

$$h_k = \frac{k^2}{(2k-1)(2k+1)} \tag{27}$$

$$i_k = \frac{2k^2 + 2k - 1}{(2k-1)(2k+3)} \tag{28}$$

This set of equations is complete only when an infinite number of moments $w_\nu^0 \ldots w_\nu^\infty$ are taken into account. Of course this is numerically prohibitive, so that one needs to cut off this infinite series of moments by a suitable closure assumption for the highest two moments. For optically thick systems a consistent closure relation would be to put to zero all moments higher than the highest moment one wishes to retain,

$$w^{\bar{k}+1} = 0 \quad w^{\bar{k}+2} = 0 \quad \cdots \tag{29}$$

and then solve for $w^0 \cdots w^{\bar{k}}$. Such a closure has a clear physical meaning: one restricts oneself to a finite set of spherical harmonics in the analysis. This is a mathematically appealing way of limiting the angular resolution of the equations. When taking $\bar{k} = 1$ one finds the Eddington approximation to radiative

transfer, cast in a relativistic setting. For low enough velocity gradients ($dv/dx \lesssim 0.2 \, c/\tau$) it will often produce correct results. However, it fails to reproduce the shear viscosity predicted by Thomas[2], because this is an effect on w^2. It is, however, an instructive system to study because of its simplicity and the fact that it covers all first-order effects of the transfer. When one takes $\bar{k} = 2$ all relativistic second-order effects will be present and the Thomas shear viscosity will enter. However, for regions in which large velocity gradients are present, it can be essential to include more, sometimes even many more moments.

In realistic applications, it is often numerically costly to solve the equations for a large number of moments. It is therefore customary to choose a low cut-off order (mostly $\bar{k} = 1$) and use a non-linear closure for the higher moments. The expression for such a non-linear closure is in general inspired by the type of solutions one expects and is designed to guess the behavior of the higher moments as correctly as possible. However, a good procedure will always involve a test program of this closure assumption (e.g. by testing the results against a calculation using many more moments, or calculations solving the Boltzmann equation directly).

4. Matrix form of the equations

I wish to study the the structure of the moment equations (19) when a minimal closure is imposed, i.e. when taking $w^{\bar{k}+1} = w^{\bar{k}+2} = \cdots = 0$. The discussion will be restricted to the case of frequency integrated moments, defined as

$$w^k \equiv \int_0^\infty w_\nu^k d\nu \tag{30}$$

Also, I will only study systems in which the velocity profile $\beta(x, t)$ is constant in time, $\beta(x, t) = \beta(x)$, since I am particularly interested in stationary solutions to the transfer equation. Using

$$\partial_{\hat{0}} \beta = \gamma \beta \frac{\partial \beta}{\partial x} \tag{31}$$

$$\partial_{\hat{1}} \beta = \gamma \frac{\partial \beta}{\partial x} \tag{32}$$

and the definition of the frequency integrated moments, the moment equations reduce to

$$
\begin{aligned}
\gamma \partial_{ct} w^k &+ \gamma \beta \partial_{ct} (w^{k+1} + a_k w^{k-1}) \\
&+ \gamma \beta \partial_x w^k + \gamma \partial_x (w^{k+1} + a_k w^{k-1}) \\
&\quad + \gamma^3 [b_k \partial_x \beta \cdot w^{k-2} + c_k \beta \partial_x \beta \cdot w^{k-1} \\
&\quad + d_k \partial_x \beta \cdot w^k + e_k \beta \partial_x \beta \cdot w^{k+1} \\
&\quad + f_k \partial_x \beta \cdot w^{k+2}] = s^k
\end{aligned}
\tag{33}
$$

It is convenient to cast these equations into a matrix equation of the form

$$\mathsf{A}(\beta)\partial_{ct}\mathbf{w} + \mathsf{B}(\beta)\partial_x\mathbf{w} + \mathsf{C}(\beta, \partial_x\beta)\mathbf{w} = \mathbf{s} \tag{34}$$

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[2] A possible bulk viscosity is a more subtle matter, since it involves the interplay between the gas pressure and the radiation pressure. It will not be considered here.

The vectors $\mathbf{w}$ and $\mathbf{s}$ are

$$\mathbf{w} = (w^0, w^1, w^2, \cdots, w^{\bar{k}}) \tag{35}$$

$$\mathbf{s} = (s^0, s^1, s^2, \cdots, s^{\bar{k}}) \tag{36}$$

As an example I work out the matrices for $\bar{k} = 1$,

$$A = \gamma \begin{pmatrix} 1 & \beta \\ \frac{1}{3}\beta & 1 \end{pmatrix} \tag{37}$$

$$B = \gamma \begin{pmatrix} \beta & 1 \\ \frac{1}{3} & \beta \end{pmatrix} \tag{38}$$

$$C = \gamma^3 \begin{pmatrix} \frac{4}{3} & 2\beta \\ \frac{4}{3}\beta & 2 \end{pmatrix} \partial_x \beta \tag{39}$$

and for $\bar{k} = 2$,

$$A = \gamma \begin{pmatrix} 1 & \beta & 0 \\ \frac{1}{3}\beta & 1 & \beta \\ 0 & \frac{4}{15}\beta & 1 \end{pmatrix} \tag{40}$$

$$B = \gamma \begin{pmatrix} \beta & 1 & 0 \\ \frac{1}{3} & \beta & 1 \\ 0 & \frac{4}{15} & \beta \end{pmatrix} \tag{41}$$

$$C = \gamma^3 \begin{pmatrix} \frac{4}{3} & 2\beta & 1 \\ \frac{4}{3}\beta & 2 & \beta \\ \frac{16}{45} & \frac{4}{3}\beta & \frac{38}{21} \end{pmatrix} \partial_x \beta \tag{42}$$

In general the matrices A and B can be written in terms of the B in the fluid frame, $B^{(0)}$, as

$$A = \gamma \left(\beta B^{(0)} + 1 \right) \tag{43}$$

$$B = \gamma \left(B^{(0)} + \beta \right) \tag{44}$$

so that the matrices A and B are simultaneously diagonalizable, and their eigenvectors are independent of β. The eigenvalues are also related to the eigenvalues of $B^{(0)}$, denoted by $\lambda_i^{(0)}$,

$$\psi_i = \gamma(\beta \lambda_i^{(0)} + 1) \tag{45}$$

$$\lambda_i = \gamma(\lambda_i^{(0)} + \beta) \tag{46}$$

As an example I give the eigenvectors $\mathbf{e}_i$ for the case $\bar{k} = 1$,

$$\mathbf{e}_1 = \begin{pmatrix} \sqrt{3} \\ -1 \end{pmatrix} \qquad \mathbf{e}_2 = \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix} \tag{47}$$

and the eigenvalues ψ_i of A and λ_i of B,

$$\begin{aligned} \psi_1 &= \gamma\left(-\frac{1}{\sqrt{3}}\beta + 1\right) & \lambda_1 &= \gamma\left(-\frac{1}{\sqrt{3}} + \beta\right) \\ \psi_2 &= \gamma\left(\frac{1}{\sqrt{3}}\beta + 1\right) & \lambda_2 &= \gamma\left(\frac{1}{\sqrt{3}} + \beta\right) \end{aligned} \tag{48}$$

For $\bar{k} = 2$ one has

$$\mathbf{e}_1 = \begin{pmatrix} \frac{15}{4} \\ -\frac{3}{4}\sqrt{15} \\ 1 \end{pmatrix} \qquad \mathbf{e}_2 = \begin{pmatrix} \frac{15}{4} \\ \frac{3}{4}\sqrt{15} \\ 1 \end{pmatrix} \tag{49}$$

$$\mathbf{e}_3 = \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} \tag{50}$$

and

$$\begin{aligned} \psi_1 &= \gamma\left(-\sqrt{\tfrac{3}{5}}\beta + 1\right) & \lambda_1 &= \gamma\left(-\sqrt{\tfrac{3}{5}} + \beta\right) \\ \psi_2 &= \gamma\left(\sqrt{\tfrac{3}{5}}\beta + 1\right) & \lambda_2 &= \gamma\left(\sqrt{\tfrac{3}{5}} + \beta\right) \\ \psi_2 &= \gamma & \lambda_2 &= \gamma\beta \end{aligned} \tag{51}$$

The dynamics of the moments can now be viewed in terms of advection of eigenmodes. The characteristic velocities are

$$v_i = \frac{\lambda_i}{\psi_i} \tag{52}$$

where, again, v_i is taken dimensionless (the actual velocity is cv_i). The C matrix mixes the various eigenmodes, an effect that can be ascribed to the differential motion of the underlying medium. Because the eigenvectors are independent of the velocity β one can diagonalize the moment equations, so that the hyperbolic nature of the problem becomes more transparent. If the C matrix and the source terms $\mathbf{s}$ would not be there, the eigenmodes would be entirely independent and the advection problem would be trivial. It is these extra terms that make the moment equations more complicated.

The characteristic velocities v_i are related to the roots of the Legendre polynomials (Turolla & Nobili 1988). In the frame of the fluid (i.e. for $\beta = 0$) they satisfy

$$P_{\bar{k}+1}(v_i) = 0 \tag{53}$$

where again $\bar{k}$ is the index of the highest moment. This equation hints that one can view an eigenmode with some v_i as being related to light traveling under an angle of θ_i with the x-axis, with θ_i defined as $\cos \theta_i = v_i$. Choosing a finite set of moment is therefore actually similar to choosing a finite set of angles along which one solves the Boltzmann equation. However, this moment method is a mathematically more appealing way of limiting this angular resolution.

5. Stationary solutions of the moment equations

In order to get a good understanding of the time dependent radiation transfer problem it is important to study stationary solutions first. There are several reasons for this, among which is the fact that for stationary velocity profiles any time dependent radiation transfer solution will eventually relax to a stationary one. Another reason is that stationary solutions often reveal a lot of the mathematical structure of the time dependent equations, especially considering the fact that the transfer equation is hyperbolic.

The type of solution I have in mind is a transfer solution along a relativistic radiation pressure dominated shock. Non-relativistic versions of such shocks have been studied by several authors (Becker 1988, Blandford & Payne 1981, Imshennik 1975, Belokon 1959). The velocity profile of such a shock is expected to be smooth because the dynamics is governed by the radiation pressure which can never acquire shock-like discontinuities. Of course a shock will only be radiation pressure dominated when enough photons are available for building up

the required radiation pressure by bulk-Comptonization, without heating the gas. These photons might be produced in the shock or provided by the upstream matter.

For the stationary case all time derivatives are taken zero, so that the tetrad derivatives become

$$\partial_{\hat{0}} = \gamma\beta\frac{\partial}{\partial x} \tag{54}$$

$$\partial_{\hat{1}} = \gamma\frac{\partial}{\partial x} \tag{55}$$

The moment equations therefore reduce to

$$\begin{aligned}
\gamma\beta\partial_x w^k &+ \gamma\partial_x(w^{k+1} + a_k w^{k-1}) \\
&+ \gamma^3[b_k\partial_x\beta \cdot w^{k-2} + c_k\beta\partial_x\beta \cdot w^{k-1} \\
&+ d_k\partial_x\beta \cdot w^k + e_k\beta\partial_x\beta \cdot w^{k+1} \\
&+ f_k\partial_x\beta \cdot w^{k+2}] = s_k
\end{aligned} \tag{56}$$

Its matrix form is

$$\mathsf{B}(\beta)\partial_x\mathbf{w} + \mathsf{C}(\beta, \partial_x\beta)\mathbf{w} = \mathbf{s} \tag{57}$$

Using a Henyey method one can solve this set of equations for a given $\beta(x)$ and with some set of boundary conditions (Nobili & Turolla 1988).

5.1. Eigenvector decomposition

The mathematical structure of stationary and time-dependent solutions to hyperbolic equations is most easily studied after making an eigenvector decomposition. Define the matrix Λ by

$$\Lambda \equiv (\mathbf{e}_1, \cdots, \mathbf{e}_{\bar{k}+1})^{-1} \tag{58}$$

and transform the matrices B and C and the vectors $\mathbf{w}$ and $\mathbf{s}$ to the eigenbasis

$$\tilde{\mathsf{B}} = \Lambda\mathsf{B}\Lambda^{-1} \tag{59}$$

$$\tilde{\mathsf{C}} = \Lambda\mathsf{C}\Lambda^{-1} \tag{60}$$

$$\tilde{\mathbf{w}} = \Lambda\mathbf{w} \tag{61}$$

$$\tilde{\mathbf{s}} = \Lambda\mathbf{s} \tag{62}$$

The matrix $\tilde{\mathsf{B}}$ then becomes

$$\tilde{\mathsf{B}} = \text{diag}(\lambda_1, \cdots, \lambda_{\bar{k}+1}) \tag{63}$$

and the equations of motion, Eq. (57), become

$$\begin{aligned}
\lambda_1\partial_x\tilde{w}^1 + \sum_{i=1}^{\bar{k}+1}\tilde{C}_i^1\tilde{w}^i &= \tilde{s}^1 \\
\vdots \quad &= \vdots \\
\lambda_{\bar{k}+1}\partial_x\tilde{w}^{\bar{k}+1} + \sum_{i=1}^{\bar{k}+1}\tilde{C}_i^{\bar{k}+1}\tilde{w}^i &= \tilde{s}^{\bar{k}+1}
\end{aligned} \tag{64}$$

For the sake of simplicity I shall assume the source to consist of only absorption/emission terms,

$$s^0 = j - \alpha_a w^0$$

$$\begin{aligned}
s^1 &= -\alpha_a w^1 \\
\vdots \;\; &= \;\; \vdots \\
s^{\bar{k}} &= -\alpha_a w^{\bar{k}}
\end{aligned} \tag{65}$$

where α_a is the absorption opacity and j is some externally given emission term. In the eigenbasis of Eq. (58) they become

$$\begin{aligned}
\tilde{s}^1 &= \tilde{j}^1 - \alpha_a\tilde{w}^1 \\
\vdots \;\; &= \;\; \vdots \\
\tilde{s}^{\bar{k}+1} &= \tilde{j}^{\bar{k}+1} - \alpha_a\tilde{w}^{\bar{k}+1}
\end{aligned} \tag{66}$$

The 'characteristic decomposition' of the equations reveals the structure of information flow in the time–dependent version of the equations. This is particularly useful for studying critical points and determining where and how to impose boundary conditions, as will be discussed below. It should be noted that in the stationary equations the notion of 'direction of information flow' does not have a well–defined meaning anymore. Yet, the source terms usually give meaning to the direction of time, even in stationary situation, because source terms carry in them aspects of thermodynamic irreversibility.

6. Boundary conditions

Boundary conditions determine the solution to a set of equations of the kind discussed here. How this works is discussed in this section. I assume that the solution asymptotically approaches some constant value for $x \to \pm\infty$. Then, for $x \to \pm\infty$, $\tilde{\mathsf{C}} \to 0$ so that the equations become

$$\begin{aligned}
\lambda_1\partial_x\tilde{w}^1 &= \tilde{j}^1 - \alpha_a\tilde{w}^1 \\
\vdots \quad &= \vdots \\
\lambda_{\bar{k}+1}\partial_x\tilde{w}^{\bar{k}+1} &= \tilde{j}^{\bar{k}+1} - \alpha_a\tilde{w}^{\bar{k}+1}
\end{aligned} \tag{67}$$

The asymptotic solutions are simple e-powers,

$$\tilde{w}^i = \frac{\tilde{j}^i}{\alpha_a} + A_i\exp\left(-\frac{\alpha_a}{\lambda_i}x\right) \tag{68}$$

where A_i are some constants. The condition must obviously be imposed at the diverging end of the e-power: for positive λ_i on the left side and for negative λ_i on the right side. This can be seen also in the light of the hyperbolic character of the time dependent version of the equations: for time dependent simulations the boundary conditions should specify the flux for the instreaming eigenmodes, not for the outgoing eigenmodes.

When the eigenvalues λ_i do not flip sign anywhere, one sees that each eigenmode has precisely one boundary condition. However, when a λ_i changes sign at a certain position in space it can happen that this eigenmode requires *two* or *no* boundary conditions. This fact has important implications for the discussion on critical points below.

7. Critical points

For certain specific values of β the determinant of B passes through zero and flips sign. This is associated with a flip of sign

of one of the eigenvalues λ_i, i.e. with a flip of sign of the direction of flow of this eigenmode. Information along this characteristic cannot flow through that point, and the differential part of the equation becomes singular: the equation has a critical point at that position. By solving $\det B = 0$ one finds that these points occur where β satisfies (Turolla & Nobili 1988)

$$P_{\bar{k}+1}(\beta_c) = 0 \tag{69}$$

One can either have diverging or converging critical points, depending on the direction of information flow:

$$\partial_x v_i > 0 \qquad \text{diverging critical point} \tag{70}$$
$$\partial_x v_i < 0 \qquad \text{converging critical point} \tag{71}$$

By the definition of the characteristic velocities v_i one sees that $\partial_x v_i > 0$ is equivalent to $\partial_x \beta > 0$, and $\partial_x v_i < 0$ is equivalent to $\partial_x \beta < 0$. So one has diverging critical points in diverging fluid flows (e.g. black hole accretion[3]), and converging critical points in converging fluid flows (radiative shocks).

The reason for the existence of these critical points is identical to the reason for the existence of sonic points in fluid dynamics. The characteristic velocities of the radiation are constant in the fluid frame, so that they are dragged along with the fluid. When the fluid velocity becomes larger than one of the fluid-frame characteristic velocities, the lab-frame value of this characteristic velocity must flip sign.

To study this critical point it is best to work in the diagonalized version of the equations, Eq. (64). The critical point is now characterized by the vanishing of the differential part of one of these equations. Let us take it to be the first equation,

$$\lambda_1 \partial_x \tilde{w}^1 + \sum_{i=1}^{\bar{k}+1} \tilde{C}_i^1 \tilde{w}^i = \tilde{s}^1 \tag{72}$$

The critical point decouples the function $\tilde{w}^1(x)$ to the left and the right of $x = x_{c(1)}$. At the critical point the first equation becomes a condition on the full set of eigenmodes $(\tilde{w}^1, \cdots, \tilde{w}^{\bar{k}+1})$:

$$\tilde{C}_1^1 \tilde{w}^1 + \cdots + \tilde{C}_{\bar{k}+1}^1 \tilde{w}^{\bar{k}+1} = \tilde{s}^1 \tag{73}$$

In general, this condition is not enough to prevent the $\tilde{w}^1(x)$ from becoming singular at that point.

Turolla & Nobili (1988) solved this problem by adding an extra regularity condition at this point

$$\partial_x \tilde{w}^1 = a \tag{74}$$

(for any value of a) to force $\tilde{w}^1$ to be well behaved there. However, this does not always solve the problem of such critical points, as I shall argue below.

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[3] The definition of 'diverging' used here refers to the radial velocity pattern. While black-hole accretion is converging in a global sense, the radial stream lines diverge: they accelerate towards the hole.

8. The nature of the critical points

The case of critical points in the problem at hand is interwoven with the question of where to put boundary conditions in the problem. When there is need to put in a regularity condition at a critical point, somewhere a boundary condition must be sacrificed, because when one solves $\bar{k} + 1$ coupled first order differential equations one may only impose $\bar{k} + 1$ conditions to pick out the solution of interest. In the section on boundary conditions it became clear that the total number of boundary conditions can differ from the amount of equations. The question is whether it will be possible to sacrifice a boundary condition whenever a critical point requires a regularity condition, and what to do with the critical points in case one requires more boundary conditions than the number of equations. To find this out, a detailed study of the critical points is necessary.

As before we take a medium of pure isotropic emission/absorption. Eq. (64), after substituting Eq. (66), becomes

$$\lambda_n \partial_x \tilde{w}^n + (\tilde{C}_n^n + \alpha_a)\tilde{w}^n + \sum_{i \neq n} \tilde{C}_i^n \tilde{w}^i = \tilde{j}^n \tag{75}$$

We proceed by singling out the first equation, $n = 1$ (which is by choice taken to be the equation that becomes critical), and we call it the 'critical equation'. Define $x_{c(1)}$ to be the position where this equation becomes critical, i.e. where $\lambda_1(x_{c(1)}) = 0$. The value of $\tilde{C}_1^1$ and the derivative of λ_1 determine the behavior of $\tilde{w}^1$ close to the critical point, as will be shown below. It is hard to derive a general expression for $\tilde{C}_1^1$, because the complexity of the C and Λ matrices. However, since Λ is a constant matrix, the form of Eq. (56) indicates that $\tilde{C}_1^1$ must be of the form

$$\tilde{C}_1^1 = (c_1 + d_1\beta)\gamma^3 \partial_x \beta \tag{76}$$

The c_1 and d_1 are numerical constants. Precisely *at* the critical point (i.e. for $x = x_{c(1)}$) this expression reduces to

$$\tilde{C}_1^1(x_{c(1)}) = \frac{\bar{k}+4}{2\bar{k}+3} \gamma_{c(1)} \partial_x \beta \tag{77}$$

where we define $\gamma_{c(1)} = \gamma(x_{c(1)})$ and $\partial_x \beta$ is evaluated at $x_{c(1)}$ as well. This expression was found by inspection and verified for a wide range of values of $\bar{k}$ (ranging from $\bar{k} = 1$ to $\bar{k} = 12$).

Close to the critical point one can approximate the fluid velocity β by a linear function $\beta(x) \simeq \beta_{c(1)} + \partial_x \beta \, (x - x_{c(1)})$. Then the eigenvalue λ_1 is simply approximated by

$$\lambda_1(x) \simeq \gamma_{c(1)} \partial_x \beta \, (x - x_{c(1)}) \tag{78}$$

The equation for $\tilde{w}^1$, Eq. (75) for $n = 1$, can now be written more explicitly. If we divide this equation by $\gamma_{c(1)} \partial_x \beta$, we obtain

$$(x - x_{c(1)})\partial_x \tilde{w}^1 - \xi_1(x)\tilde{w}^1 + \sum_{i \neq 1} \Omega_i^1(x) \, \tilde{w}^i \simeq K_1(x) \tag{79}$$

where $\xi_1(x)$, $\Omega_i^1(x)$ and $K_1(x)$ are defined as

$$\xi_1(x) = -(\tilde{C}_1^1(x) + \alpha_a(x))\gamma_{c(1)}^{-1}(\partial_x \beta)^{-1} \tag{80}$$
$$\Omega_i^1(x) = \tilde{C}_i^1(x)\gamma_{c(1)}^{-1}(\partial_x \beta)^{-1} \tag{81}$$
$$K_1(x) = \tilde{j}^1(x)\gamma_{c(1)}^{-1}(\partial_x \beta)^{-1} \tag{82}$$

The functions $\Omega_i^1(x)$, $K_1(x)$ and $\xi_1(x)$ are smooth differentiable functions of x. At the critical point itself, $\xi_1(x)$ has the value

$$\xi_1(x_{c(1)}) \equiv -\frac{\bar{k}+4}{2\bar{k}+3} - \frac{\alpha_a}{\gamma_{c(1)}\partial_x\beta}. \tag{83}$$

The equations for $\tilde{w}^2 \cdots \tilde{w}^{\bar{k}+1}$ remain as they are in Eq. (75), for $n \neq 1$. The values of $\lambda_2 \cdots \lambda_{\bar{k}+1}$ are non–zero at $x = x_{c(1)}$.

To study the behavior of solutions around $x = x_{c(1)}$ we seek approximate solutions of the set of equations. (75 with $n \neq 1$) combined with Eq. (79) in the limit of $x \to x_{c(1)}$. The most rapidly diverging approximate solution is of the form

$$\tilde{w}^1(x) = A(x - x_{c(1)})^{\xi_1} + f_1(x) \tag{84}$$
$$\tilde{w}^2(x) = A\eta_2(x - x_{c(1)})^{\xi_1+1} + f_2(x) \tag{85}$$
$$\vdots = \vdots \tag{86}$$
$$\tilde{w}^{\bar{k}+1}(x) = A\eta_{\bar{k}+1}(x - x_{c(1)})^{\xi_1+1} + f_{\bar{k}+1}(x) \tag{87}$$

where $f_i(x)$ are smooth differentiable functions, which we do not specify any further. For convenience we wrote $\xi_1 = \xi_1(x_{c(1)})$. The values of $\eta_2 \cdots \eta_{\bar{k}+1}$ follow by substitution into the equations and taking the limit $x \to x_{c(1)}$. The value A is the free parameter of the homogeneous part of the approximate solution. The value that it actually acquires depends on the boundary conditions of the real solution, so that this value remains undetermined within the scope of this analysis. But the shape of the approximate solution Eq. (84) can teach us something about the behavior of the real solution close to this critical point:

$$\begin{array}{ccccc} & \xi_1 & < & 0 & \text{Pole} \\ 0 & < \xi_1 & < & 1 & \text{Infinite gradient} \\ 1 & < \xi_1 & & & \text{Differentiable} \end{array} \tag{88}$$

These three cases will be discussed below.

8.1. Diverging critical points

For a diverging critical point (i.e. $\partial_x\beta > 0$) one has

$$\xi_1 < 0 \tag{89}$$

so it is expected that the solution may have a pole around the critical point (see Eq. [88]). In order to avoid this singularity (i.e. to force $A = 0$ here) one must add a regularity condition at this critical point, which is numerically something like

$$\tilde{w}_j^1 = \tilde{w}_{j+1}^1 \tag{90}$$

where the j index refers to the spatial mesh point such that the critical point is in between $x_j < x_{c(1)} < x_{j+1}$. But adding a regularity condition requires the removal of one of the boundary conditions. Fortunately, for diverging critical points the corresponding eigenmodes move outwards at both outer boundaries, so no boundary conditions are required for these eigenmodes (see Sect. 6), and everything is consistent. So we may conclude that diverging critical points require a consistency condition at the cost of one boundary condition. This was described by Nobili & Turolla (1988).

8.2. Converging critical points

For a converging critical point (i.e. $\partial_x\beta < 0$) the ξ_1 might be either positive or negative (see Eq. [83]). When $\partial_x\beta$ is sufficiently small compared to the inverse mean free path, the ξ_1 is positive and the solution remains regular at the critical point. This implies that no regularity condition needs to be imposed here since all solutions converge to the same value at the critical point. In fact, what remains of the differential equation precisely *at* the critical point (numerically the equation connecting the grid points x_j and x_{j+1}),

$$(\chi_{c(1)}\partial_x\beta + \alpha_a)\tilde{w}^1 = \\ -\tilde{C}_2^1\tilde{w}^2 - \cdots - \tilde{C}_{\bar{k}+1}^1\tilde{w}^{\bar{k}+1} \tag{91}$$

is a condition which has now become redundant. Any solution will converge to the same value, making Eq. (91) unnecessary. By removing this equation at this point, one can add an extra boundary condition to the edges of the spatial domain. In fact, this is even *required* for converging fluid flows, as was argued in Sect. 6, so consistency is once again guaranteed.

The behavior of the converging critical point changes when one has large velocity gradients

$$-\gamma_{c(1)}\frac{\partial\beta}{\partial x} > \frac{2\bar{k}+3}{\bar{k}+4}\alpha_a \tag{92}$$

Then one has $\xi_1 < 0$ and the 'solution' Eq. (84) can become singular again. A regularity condition would be required, just as in the case of diverging critical points, but this time it is not possible to remove a condition from the boundaries. The chance that that by good luck the amplitude of the singular mode will be precisely zero is slim, since by integrating towards the critical point an arbitrary small perturbation will grow and will eventually go to infinity when one reaches the critical point. *It is therefore very likely that no regular solution can be found for such strong converging velocity gradients.* Note that increasing $\bar{k}$ might help a bit, but for very strong gradients, $-\gamma\partial_x\beta > 2\alpha_a$, the situation is hopeless.

A further interpretation of this peculiar phenomenon is given in Sect. 9.

8.3. Converging critical points for scattering media

The above analysis can also be done for purely scattering media. This involves one extra element. The source terms are now

$$\begin{aligned} s^0 &= 0 \\ s^1 &= -\alpha_s w^1 \\ \vdots &= \vdots \\ s^{\bar{k}} &= -\alpha_s w^{\bar{k}} \end{aligned} \tag{93}$$

where α_s is the scattering opacity. Due to the zero in the first equation the transformation to the eigenbasis is not trivial anymore. The source terms in the eigenbasis become

$$\begin{aligned} \tilde{s}^1 &= -\Pi_1^1\alpha_s\tilde{w}^1 - \cdots - \Pi_{\bar{k}+1}^1\alpha_s\tilde{w}^{\bar{k}+1} \\ \vdots &= \vdots \end{aligned}$$

$$\tilde{s}^{\bar{k}+1} = -\Pi_1^{\bar{k}+1}\alpha_s\tilde{w}^1 - \cdots - \Pi_{\bar{k}+1}^{\bar{k}+1}\alpha_s\tilde{w}^{\bar{k}+1} \tag{94}$$

If we take $\tilde{w}^1$ again to be the one that goes critical at the point we focus on, then the 'critical' equation again becomes like Eq. (79), but now the ξ_1 is given by

$$\xi_1 \equiv -\frac{\bar{k}+4}{2\bar{k}+3} - \frac{\Pi_1^1\alpha_s}{\gamma_{c(1)}\partial_x\beta} \tag{95}$$

So the condition for the solution to become singular is

$$-\gamma_{c(1)}\frac{\partial\beta}{\partial x} > \frac{2\bar{k}+3}{\bar{k}+4}\Pi_1^1\alpha_s \tag{96}$$

The values of Π_i^i are between $\frac{1}{2} \leq \Pi_i^i < 1$, and can be easily computed using symbolic manipulation programs.

8.4. Non-trivial closure

In case on adopts a non-trivial closure, the above analysis should be redone, but goes quite similarly. Consider the case of two moments w^0 and w^1 and a closure assumption

$$w^2(x) = f(x)w^0(x) \tag{97}$$

In this analysis I take the function $f(x)$ to be a given function of x. If one were to take f to be a non-linear function of w^1/w^0 then the analysis would be much more complicated. At a converging critical point the ξ_1 (corresponding to the left-moving characteristic in a fluid moving to the right) will have the form

$$\xi_1 = -1 - \frac{1}{\gamma_{c(1)}\partial_x\beta}\left[\alpha_a - \frac{\gamma_{c(1)}}{2\beta_{c(1)}}\frac{\partial f}{\partial x}\right] \tag{98}$$

So the solution goes singular when

$$-\gamma_{c(1)}\frac{\partial\beta}{\partial x} > \alpha_a - \frac{\gamma_{c(1)}}{2\beta_{c(1)}}\frac{\partial f}{\partial x} \tag{99}$$

and/or when

$$\frac{\gamma_{c(1)}}{2\beta_{c(1)}}\frac{\partial f}{\partial x} > \alpha_a \tag{100}$$

9. Interpretation

The equations for the PSTF moments apparently have problems dealing with strongly convergent flows such as strong relativistic radiation dominated shocks. An attempt to solve the moment equations by using some numerical relaxation scheme might lead to singular solutions, even in cases where the real solution to the transfer equation is regular.

The reason behind this peculiar mathematical phenomenon is the fact that the fluid frame velocity artificially enters the left–hand side of the transfer equation Eq. (5), when one expands this equation in moments (Eq. [19]). In principle, photons do no care about the fluid frame velocity, until they have actual interaction with the medium (which is the right–hand side of the transfer equation). But in the moment expansion, the fluid velocity artificially influences the radiation field, unless an infinite amount of

moments are used. It is this artificial interaction that can cause the solutions for the moments to become singular.

A qualitative understanding of these singularities can be obtained by considering the propagation of (radiative) information in the moment formalism. In this formalism, radiation is described by moments as seen from a fluid frame observer. The characteristic propagation velocities of these moments are measured with respect to this fluid frame. When the fluid has a velocity gradients, the lab–frame characteristic velocities of these moments can flip sign at a critical point. In a converging flow ($\partial_x\beta < 0$) the characteristic velocities point towards the critical point. Radiative information may therefore artificially 'pile up' at this critical point, leading to a singularity in the stationary solution. One could say that the radiation field gets 'shocked' at the critical point. Only a sufficiently strong absorption or scattering can prevent such a catastrophic pile–up, by removing radiative information faster than it is being transferred towards the critical point.

In reality the information propagates with a constant velocity $v = c\cos\theta$, where θ is the angle of the photon propagation with respect to the 1–D fluid flow. This characteristic velocity never changes sign, so in reality these critical points (and the corresponding pile–up of photons) do not occur. The singularities are therefore purely mathematical artifacts of the moment expansion.

10. Conclusion

In this paper I examined the nature of the stationary frequency integrated moment equations for radiative transfer in relativistically moving media. A plane–parallel geometry was chosen for simplicity. A generalization to other geometries is straightforward, and most likely does not influence the results significantly. I reviewed the analysis of Turolla & Nobili (1988) from a new perspective and extended it to include an alternative kind of critical point, which can appear in relativistic fluid flows with a negative velocity gradient (e.g. radiation pressure dominated relativistic shocks with a width of the order of a photon mean free path). This new type of critical point requires a different treatment in numerical relaxation schemes.

When the velocity gradients are very strong, the solutions might exhibit singular behavior around these critical points, while the real solutions of the radiative transfer equation remain regular. This pathological behavior is a consequence of the fluid–frame Legendre expansion of the transfer equation, and is a purely mathematical artifact. The problem might be solved by choosing the observer frame independent from the fluid frame, and requiring this new frame to have a sufficiently small velocity gradient. The introduction of this non-fluid-frame observer, however, might come at a price.

I conclude that the moment method, though very powerful and appealing for most problems, might have difficulty in dealing with problems involving negative relativistic velocity gradients.

Appendix A: notation and conventions

In this paper I use full units, and retain h and c. I use a Minkowski metric in a plane parallel setting of the form

$$ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2 \tag{A1}$$

I denote tensor indices as $ct = x^0, x = x^1, y = x^2, z = x^3$, and I use Greek symbols $\alpha\beta\mu\nu = 0, 1, 2, 3$ as four-index and Latin symbols $a, b, c, d = 1, 2, 3$ as spatial indices. The metric can thus be written as $g_{\mu\nu} = \text{diag}(-1, +1, +1, +1)$.

I define a tetrad by the symbol $e^\mu{}_{\hat\alpha}$. The hat on top of a symbol means that the index is taken in the tetrad frame whereas the ordinary symbols are taken in the coordinate frame. Conjugate to $e^\mu{}_{\hat\alpha}$ is $e^{\hat\alpha}{}_\mu$. They are related by

$$e^{\hat\alpha}{}_\mu e^\mu{}_{\hat\beta} = \delta^{\hat\alpha}_{\hat\beta} \qquad \text{and} \qquad e^\nu{}_{\hat\alpha} e^{\hat\alpha}{}_\mu = \delta^\nu_\mu \tag{A2}$$

The tetrad obeys $g_{\mu\nu} e^\mu{}_{\hat\alpha} e^\nu{}_{\hat\beta} = \eta_{\hat\alpha\hat\beta}$, where the $\eta_{\hat\alpha\hat\beta}$ is the conventional notation of the metric in the tetrad frame, but in this Minkowski plane-parallel case it is identical to $g_{\mu\nu}$, namely $\eta_{\hat\alpha\hat\beta} = \text{diag}(-1, +1, +1, +1)$.

The Pfaffian derivative $\hat\nabla_\mu$ working on a vector $v^{\hat\alpha}$ in the tetrad frame is defined by $\hat\nabla_\mu v^{\hat\alpha} = \partial_\mu v^{\hat\alpha} + \omega^{\hat\alpha}{}_{\mu\hat\beta} v^{\hat\beta}$. Here $\omega^{\hat\alpha}{}_{\mu\hat\beta}$ are the Ricci rotation coefficients, defined in this flat spacetime, as

$$\omega^{\hat\alpha}{}_{\mu\hat\beta} \equiv e^{\hat\alpha}{}_\kappa \partial_\mu e^\kappa{}_{\hat\beta} \tag{A3}$$

In the present plane parallel Minkowskian setting I define the dimensionless velocity $\beta = v/c$ and the Lorentz factor γ as $\gamma \equiv (1 - \beta^2)^{-1/2}$.

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