A new determination of the mass of Titan by Hill's method

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A new determination of the mass of Titan by Hill's method *),

by *D. Brouwer*.

The determination of the mass of Titan, as given by G. W. HILL **), has lost much of its value as to the numerical result to which it leads. For the most part this is due to the important difference between HILL's excentricity of the osculating orbit of Hyperion in opposition ($e_0 = 0{,}1$) and that obtained by H. STRUVE ***) ($e_0 = 0{,}1043$), that is valid at present. It will be seen that the use of better elements has a considerable influence on the result.

The notation here adopted is:

a, semi-axis major; e, excentricity; n, mean motion; r, radius vector; v, true anomaly; m, mass.

a_0, e_0.... refer to the osculating orbit of Hyperion in opposition;

a',....... refer to Titan;

a'',....... refer to the sun;

T = synodic period of Hyperion-Titan; n = average daily motion of Hyperion.

The units are so chosen, that the constant of gravitation = 1 and Saturn's mass = 1; the unit of time is one mean solar day.

HILL supposes that, the orbit of Titan being considered as circular and Hyperion starting from opposition at $t = 0$, the motion of Hyperion can be represented by the periodic solution:

$$r = \Sigma A_i \cos i\,(n'-n)\,t;\; v = nt + \Sigma B_i \sin i\,(n'-n)\,t.$$

Consequently after half a synodic period:

$$v = n\frac{T}{2} \quad \text{and} \quad \frac{dr}{dt} = 0.$$

Now, if values m', n_0 and e_0 are chosen, the process of mechanical quadrature will lead to such values of

$$\bar{v} \quad \text{and} \quad \overline{\frac{dr}{dt}} \quad \ldots\ldots \left(t = \frac{T}{2}\right)$$

that generally

$$\bar{v} \neq n\frac{T}{2}; \quad \overline{\frac{dr}{dt}} \neq 0;$$

but if m', n_0 and e_0 are not too far from the right values,

$$\bar{v} - n\frac{T}{2} \quad \text{and} \quad \overline{\frac{dr}{dt}}$$

are small and the conditions that they shall be zero give rise to two linear equations, from which the values of m' and of a combination of n_0 and e_0, satisfying the periodic solution can be found.

Not only better elements of Titan and Hyperion have been used, but also perturbations due to the sun and the oblateness of Saturn have been taken into account. This was done by adding to the perturbative function the most important term due to the sun:

$$+\tfrac{1}{4}\frac{m'' r^2}{a''^3}$$

and by taking the following potential of Saturn:

$$V = \frac{1}{r}\left(1 + \tfrac{1}{3}\frac{K}{r^2} + \tfrac{1}{4}\frac{L}{r^4}\right);$$

$$\frac{K}{a_s^2} = 0{,}02438; \quad \frac{L}{a_s^4} = 0{,}00070;$$

$$2a_s = \text{aequ. diam. of Saturn} = 17''.500;$$
$$a_0 = 214'',$$

which leads to:

$$V = \frac{1}{r}\left(1 + 0{,}00001358629\frac{a_0^2}{r^2} + 0{,}00000000049\frac{a_0^4}{r^4}\right).$$

It is obvious, that these additions do not affect the possibillity of a periodic solution.

*) The computations contained in this paper were proposed to me by Dr. J. WOLTJER JR., as an exercise in the practice of numerical methods. Different checks have been applied to assure the trustworthiness of the results.

**) G. W. HILL. On the motion of Hyperion and the mass of Titan. *Astr. Journ.* VIII, 1888; *Works* II p. 135.

***) H. STRUVE. Beobachtungen am 30" Refraktor. *Poulkova Obs.* (2) XI (1898).

The following elements have been employed:
average daily motion of Hyperion $n = 16°,919983$
daily motion of Titan $n' = 22°,577012$
radius vector of Hyperion in opposition $a_0(1 - e_0) = 1{,}08476\, a'$
osc. excentricity of Hyperion in opposition $e_0 = 0{,}1043$
mass of Titan $m' = 1:4100$
sun's mass $m'' = 3501.6$
daily motion of sun $n'' = 120''.455$

From these data are deduced:
half synodic period $\frac{1}{2}T = 31^d.818822212$
motion of Titan in half syn. period $718°.3739309$
„ „ Hyperion „ „ „ „ $538°.3739309$
„ „ conj. line „ „ „ „ $= -1°,6260691 = -5853''.849$

In the units here used the constant radius vector of Titan is:

$$a' = [0{,}2696728],$$

the radius vector of Hyperion in opposition:

$$a_0(1 - e_0) = [0{,}30500065],$$

the mean distance in the osculating orbit in opposition:

$$a_0 = [0{,}3528439].$$

The mean motion of Hyperion in the osculating orbit in opposition, obtained by:

$$a_0^3\, n_0^2 = a'^3\, n'^2\, (1 + m')^{-1}$$

is:

$$n_0 = 60976''.0548.$$

The process of mechanical quadrature has been performed by means of the equations due to HANSEN.*) The interval was half a day in the beginning, Hyperion starting from opposition. At $27^d.75$ a reduction of the interval to one-fourth of a day was necessary, followed at $30^d.375$ by a reduction to one-eigth of a day.

For the argument: $t = 31^d.818822212$ were computed the perturbations:

$$\delta v_0 = -2538''.014$$
$$\delta \frac{dr_0}{dt} = -0{,}0011437005,$$

referred to the adopted osculating orbit in opposition.

In the undisturbed motion, we have for this value of t:

$$v_0 = 539°8'9''.664$$
$$\frac{dr_0}{dt} = 0.0010534240.$$

The periodic solution requires:

$$v = \tfrac{1}{2}nT = 538°22'26''.151$$
$$\frac{dr}{dt} = 0.$$

Now supposing that the mass of Titan needs to be multiplied by a factor $1 + \mu$ and that n_0 needs to be corrected by Δn_0, e_0 remaining unchanged, we arrive at:

$$v_0 + \Delta v_0 + (1 + \mu)\,\delta v_0 = v$$
$$\frac{dr_0}{dt} + \Delta\frac{dr_0}{dt} + (1 + \mu)\,\delta\frac{dr_0}{dt} = 0,$$

where Δ means the change of the elliptical values due to Δn_0.

These equations are true if the perturbations may be considered as varying proportionally to m', which is sufficiently exact when μ is a small quantity.

The equations become:

$$25.95055\,\Delta n_0 - 2538.014\,\mu + 205.499 = 0$$
$$87.82633\,\Delta n_0 + 11437.006\,\mu + 902.766 = 0$$

(the latter has been multiplied by -10^7).

The solution is:

$$\mu = -0.0103503$$
$$\Delta n_0 = -8''.9312.$$

The mass of Titan becomes:

$$m' = 1:4142.9\,^{*})$$

and the osculating elements of Hyperion in opposition:

$$n_0 = 60967''.1236$$
$$\log a_0 = 0.3528863$$
$$e_0 = 0.1043.$$

*) P. A. HANSEN. Auseinandersetzung.... I. § 2. *Abh. der Königl. Sächs. Ges. d. Wiss.* V. p. 82. (1859).

*) HILL's computation gave: $m' = 1:4714$.