On the theory of anadiabatic star pulsations: a continuation, extended and emended

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Hence:

$$J_2 = A_2 + X_2 + 2\sqrt{A_2 X_2}\cos(2w_1 - x_2)\,, \quad J_3 = A_3 + X_3 + 2\sqrt{A_3 X_3}\cos(3w_1 - x_3)\,.$$

Also:

$$\sqrt{J_2}\cos(2w_1 - w_2) = \sqrt{A_2} + \sqrt{X_2}\cos(2w_1 - x_2)\,, \quad \sqrt{J_3}\cos(3w_1 - w_3) = \sqrt{A_3} + \sqrt{X_3}\cos(3w_1 - x_3)\,.$$

Hence, to the degree of approximation involved in the determination of Q, the function R is determined by the equation:

$$R = -\left\{1 - \frac{k_{12}}{n_1}\sqrt{A_2} - \frac{3}{2}\frac{k_{13}}{n_1}\sqrt{A_3 z}\right\}\frac{n_2}{n_1}X_2 - \left\{1 - \frac{k_{12}}{n_1}\sqrt{A_2} - \frac{3}{2}\frac{k_{13}}{n_1}\sqrt{A_3 z}\right\}\frac{n_3}{n_1}X_3$$
$$+\frac{k_{12}}{n_1}\left[\frac{n_2}{n_1}\{X_2 + 2\sqrt{A_2 X_2}\cos(2w_1 - x_2)\} + \frac{n_3}{n_1}\{X_3 + 2\sqrt{A_3 X_3}\cos(3w_1 - x_3)\}\right]\sqrt{X_2}\cos(2w_1 - x_2)$$
$$+\frac{3}{2}\frac{k_{13}}{n_1}\sqrt{z}\left[\frac{n_2}{n_1}\{X_2 + 2\sqrt{A_2 X_2}\cos(2w_1 - x_2)\} + \frac{n_3}{n_1}\{X_3 + 2\sqrt{A_3 X_3}\cos(3w_1 - x_3)\}\right]\sqrt{X_3}\cos(3w_1 - x_3)\,.$$

From this relation the development of R as a goniometric function of w_1, x_2, x_3 with periods 2π may easily be derived, valid to the degree of approximation adopted. The term:

$$\left(\frac{k_{12}}{n_1}\frac{n_3}{n_1}\sqrt{A_3} + \frac{3}{2}\frac{k_{13}}{n_1}\frac{n_2}{n_1}\sqrt{A_2 z}\right)\sqrt{X_2 X_3}\cos(5w_1 - x_2 - x_3)$$

is especially important as the corresponding interaction allows the possibility of a secondary oscillation of arbitrary period. Suppose as a special example $\frac{dw_2}{dw_1} = 2 + \frac{1}{20}$, $\frac{dw_3}{dw_1} = 3 - \frac{1}{20}$, then the period of the χ-argument, hence of the secondary oscillation, is approximately equal to twenty times the fundamental period.

6. *Concluding remarks.*

The theory developed in the preceding sections is to a large extent general; it is not restricted to some terms singled out in the goniometric development of the function H; also it is independent from detailed considerations on the internal constitution of the star in its "normal" static state. However, if a numerical comparison with observation is intended, a definite construction of the "normal state" must be assumed: especially important are the evaluation of the ratio of the values of the function $s_i\,(r_n)$ at centre and surface of the star, and also the values of the damping-constants, closely related to the behaviour of these s-functions. If EDDINGTON's construction of the stellar interior is made the base of the numerical analysis, many results are available achieved several years ago[1]). However, the degree of concentration towards the central regions of the star is a fundamental factor in the determination of the functions s_i as well as in that of the damping-constants; hence it is necessary not to bind up this determination too closely with a special theory of the internal constitution of the star.

The general analysis has clearly shown which periods in the secondary oscillation may be expected: very long (secular) periods, comparable with the possibly-existing long period in ζ Geminorum and periods of moderate value, comparable with those observed in cluster-type variables.

A determination of the amplitude of the oscillation is not difficult if only a simple-periodic solution is considered; the analysis involved if the amplitudes of the secondary oscillations are to be computed is very intricate, though well within reach of the known methods of celestial mechanics. However, the existence of the upper-limit relation to a large extent replaces the results of these computations and allows insight in the relation between the amplitudes of the first and second harmonic in observed radial velocities.

On the theory of anadiabatic star-pulsations: a continuation, extended and emended, by *J. Woltjer Jr* † [2]).

1. Two systems of simultaneous linear differential equations are involved in the theory of the anadiabatic radial star-pulsations if only first-order quantities are to be considered. One system, the r-equations, principally determines the variations δr of the radius vector r, introduced by the two variables $\left(\frac{\delta r}{r}\right)_{\cos}$ and $\left(\frac{\delta r}{r}\right)_{\sin}$ by the equation exhibiting the dependency on the time-variable t:

$$\frac{\delta r}{r} = e^{-\alpha t}\left\{\left(\frac{\delta r}{r}\right)_{\cos}\cos n\,t + \left(\frac{\delta r}{r}\right)_{\sin}\sin n\,t\right\};$$

1) Cf. Miss H. A. KLUYVER's computations in *B.A.N.* No. 268, 276, 313.
2) See Note by the Editor on page 136.

α is the damping-constant, $\frac{2\pi}{n}$ is the period. The other system, the V-equations, principally determines the deviations from adiabatic conditions by the introduction of the variable V and the corresponding functions $V_{\cos}$ and $V_{\sin}$, V being the excess in the relative variation $\frac{\delta T}{T}$ of the temperature T over the value as function of the relative variation $\frac{\delta\rho}{\rho}$ of the density ρ resulting from the adiabatic relation. As the density ρ and the entropy η are the two independent variables to be used in the determination of the physical state of the matter, the variable V is connected with $\delta\eta$ by the relation

$$V = \frac{1}{T}\frac{\partial T}{\partial\eta}\,\delta\eta\,.$$

In a former paper [1]) the solution of the two systems has been treated of by approximation with regard to the anadiabatic parameter. However, the reaction of the solution of the r-equations on the V-equations had not been taken duly into account: the present note contains an evaluation of its effect and the corresponding reduction of the V-equations [2]). Moreover, the restriction of the "normal" state of the star to that usually denoted by the value 3 of the "polytropic index" has been removed and asymptotic values of the variables $V_{\cos}$ and $V_{\sin}$ at the surface of the star have been evaluated.

2. The r-equations are formed from the equation of variation corresponding to the fundamental hydrodynamical equation in the case of radial motion:

$$\frac{\partial P}{\partial r} = -\,g\rho - \rho\,\frac{\partial^2 r}{\partial t^2}\,;$$

P denotes the total pressure, g the acceleration of gravity.

Variation of the variables involved reduces this equation to the relation:

$$\frac{\partial\,\delta P}{\partial r} = 4\,g\rho\,\frac{\delta r}{r} - \rho\,\frac{\partial^2\delta r}{\partial t^2}\,,$$

and, as P is considered a function of ρ and η, to the relation:

$$\frac{\partial\left(\frac{\partial P}{\partial\rho}\,\delta\rho\right)}{\partial r} - 4\,g\,\rho\,\frac{\delta r}{r} + \rho\,\frac{\partial^2\delta r}{\partial t^2} = -\,\frac{\partial\left(\frac{\partial P}{\partial\eta}\,\delta\eta\right)}{\partial r}\,.$$

Multiplication by the factor r^3 reduces the equation to the form:

$$\frac{\partial}{\partial r}\left\{\rho\,\frac{\partial P}{\partial\rho}\,r^4\,\frac{\partial\frac{\delta r}{r}}{\partial r}\right\} + \left\{3\,\frac{\partial}{\partial r}\left(\rho\,\frac{\partial P}{\partial\rho}\right) + 4\,g\,\rho\right\} r^3\,\frac{\delta r}{r} - \rho\,r^4\,\frac{\partial^2\frac{\delta r}{r}}{\partial t^2} = r^3\,\frac{\partial}{\partial r}\left\{\frac{\partial P}{\partial\eta}\,\delta\eta\right\}.$$

Introduction of the assumed functional relation between $\frac{\delta r}{r}$ and t transforms this equation into two simultaneous equations with only r as independent variable and $\left(\frac{\delta r}{r}\right)_{\cos}$, $\left(\frac{\delta r}{r}\right)_{\sin}$, $V_{\cos}$, $V_{\sin}$ as dependent variables; moreover the quantities n and α are introduced, to be determined by the boundary conditions. If V and α are supposed to be zero the two equations are reduced to one ordinary differential equation involving the parameter n. A solution of this equation exists, $s(r)$, that satisfies the boundary condition at the surface of the star and by appropriate determination of the n-value also the boundary condition at the centre; a second solution exists $S(r)$ that is singular at the surface of the star as well as at the centre. These two functions are related by the relation:

$$\rho\,\frac{\partial P}{\partial\rho}\,r^4\left(s\,\frac{dS}{dr} - S\,\frac{ds}{dr}\right) = \text{constant};$$

hence, the singularities of S at the centre $r = 0$ and the stellar surface $r = r_0$ are apparent. The function $\rho\frac{\partial P}{\partial\rho}$, usually termed $P\gamma$, has the zeros of P; hence it is evident that S has the singularities r^{-3} and $(r-r_0)^{-3}$.

The functions s and S are appropriate means to connect the functions $\left(\frac{\delta r}{r}\right)_{\cos}$ and $\left(\frac{\delta r}{r}\right)_{\sin}$ with V, α and the excess of n over the value used in the construction of s and S. Here, only the dependency on V is needed; the corresponding contribution to $\frac{\delta r}{r}$ is:

$$\frac{S\displaystyle\int_{r_0}^{r} s\,r^3\,\frac{\partial}{\partial r}\left(\frac{\partial P}{\partial\eta}\,\delta\eta\right)dr - s\int_{r_0}^{r} S\,r^3\,\frac{\partial}{\partial r}\left(\frac{\partial P}{\partial\eta}\,\delta\eta\right)dr}{\rho\,\frac{\partial P}{\partial\rho}\,r^4\left(s\,\frac{dS}{dr} - S\,\frac{ds}{dr}\right)}\,.$$

This expression may be transformed by partial integration into:

$$\frac{-\,S\displaystyle\int_{r_0}^{r}\frac{\partial P}{\partial\eta}\,\delta\eta\,\frac{d\,(s\,r^3)}{dr}\,dr + s\int_{r_0}^{r}\frac{\partial P}{\partial\eta}\,\delta\eta\,\frac{d\,(S\,r^3)}{dr}\,dr}{\rho\,\frac{\partial P}{\partial\rho}\,r^4\left(s\,\frac{dS}{dr} - S\,\frac{ds}{dr}\right)}\,.$$

[1]) *B.A.N.* No. 282.

[2]) The reduced V-equations have already been used in the determination of the damping-constant, several years ago, by Miss H. A. Kluyver (*B.A.N.* No. 313).

Restriction to large values of the anadiabatic parameter further reduces this amount to the asymptotic value:

$$\frac{1}{\gamma r_0}\int_{r_0}^{r}\frac{1}{P}\frac{\partial P}{\partial\eta}\,\delta\eta\,dr.$$

This contribution to $\frac{\delta r}{r}$ belongs to a higher order of approximation; however in $\frac{d\frac{\delta r}{r}}{dr}$ and hence in $\frac{\delta\rho}{\rho}$ the contribution is effective; in $\frac{\delta\rho}{\rho}$ the value is:

$$-\frac{1}{\gamma}\frac{1}{P}\frac{\partial P}{\partial\eta}\,\delta\eta.$$

The contribution in $\frac{\delta T}{T}$ equals this amount multiplied by $\frac{\rho}{T}\frac{\partial T}{\partial\rho}$. Hence the whole V-term in $\frac{\delta T}{T}$ is equal to:

$$V-\frac{\rho}{T}\frac{\partial T}{\partial\rho}\left(\frac{1}{\gamma}\frac{1}{P}\frac{\partial P}{\partial\eta}\,\delta\eta\right),$$

hence to:

$$\left\{1-\frac{\frac{\rho}{T}\frac{\partial T}{\partial\rho}\frac{\partial P}{\partial\eta}}{\rho\frac{\partial P}{\partial\rho}\frac{1}{T}\frac{\partial T}{\partial\eta}}\right\}V,$$

hence to:

$$\frac{\frac{\partial P}{\partial\rho}\frac{\partial T}{\partial\eta}-\frac{\partial T}{\partial\rho}\frac{\partial P}{\partial\eta}}{\frac{\partial P}{\partial\rho}\frac{\partial T}{\partial\eta}}V.$$

As $\frac{\partial P}{\partial\rho}=\left(\frac{\partial P}{\partial\rho}\right)_T+\left(\frac{\partial P}{\partial T}\right)_\rho\frac{\partial T}{\partial\rho}$, $\frac{\partial P}{\partial\eta}=\left(\frac{\partial P}{\partial T}\right)_\rho\frac{\partial T}{\partial\eta}$, the coefficient of V is equal to: $\frac{\left(\frac{\partial P}{\partial\rho}\right)_T}{\frac{\partial P}{\partial\rho}}$,

hence, as $\left(\frac{\partial P}{\partial\rho}\right)_T=\frac{p}{\rho}$, p denoting the gas-pressure, the whole V-term in $\frac{\delta T}{T}$ is equal to $\frac{p}{P\gamma}V$.

The V-contribution in $\frac{\delta\rho}{\rho}$ is equal to $\frac{-\frac{1}{\gamma}\frac{1}{P}\frac{\partial P}{\partial\eta}}{\frac{1}{T}\frac{\partial T}{\partial\eta}}V$.

From the fundamental relation:

$$T\,d\eta=d(U_r+U_i)-\frac{P}{\rho^2}\,d\rho,$$

U_r and U_i being the energy of radiation and the thermal energy per unit mass, the derivative $\frac{\partial P}{\partial\eta}$ is seen to be equal to $\rho^2\frac{\partial T}{\partial\rho}$. Furthermore the ratio of $\frac{\partial T}{\partial\rho}$ to $\frac{\partial T}{\partial\eta}$ is equal to that of $-\left\{\frac{\partial(U_r+U_i)}{\partial\rho}\right\}_T+\frac{P}{\rho^2}$ to T. Hence the V-contribution in $\frac{\delta\rho}{\rho}$ is equal to

$$\frac{1}{\gamma}\left[\frac{\rho^2}{P}\left\{\frac{\partial(U_r+U_i)}{\partial\rho}\right\}_T-1\right]V.$$

As $\left\{\frac{\partial(U_r+U_i)}{\partial\rho}\right\}_T$ is equal to $-\frac{U_r}{\rho}$, hence to $-3\frac{(P-p)}{\rho^2}$, the V-contribution in $\frac{\delta\rho}{\rho}$ is equal to

$$-\frac{4-3\frac{p}{P}}{\gamma}V.$$

3. The formation of the V-equations requires the value of $\frac{\delta L_r}{L_r}$, L_r being the amount of energy passing per unit of time outwards through a sphere with radius r. The absorption-coefficient is supposed to be proportional to $\rho\,T^{-3}$. Hence:

$$\frac{\delta L_r}{L_r}=4\frac{\delta r}{r}-\frac{\delta\rho}{\rho}+7\frac{\delta T}{T}+\frac{T\,d\frac{\delta T}{T}}{dT}.$$

The whole V-term in $\frac{\delta L_r}{L_r}$ hence is equal to

$$T\frac{d}{dT}\left(\frac{1}{\gamma}\frac{p}{P}V\right)+\frac{4+4\frac{p}{P}}{\gamma}V.$$

The remaining part of $\frac{\delta L_r}{L_r}$ must be computed from the adiabatic values of $\frac{\delta r}{r}$, $\frac{\delta\rho}{\rho}$ and $\frac{\delta T}{T}$.

It is to be remembered that the preceding analysis only refers to asymptotic values of the quantities involved: the anadiabatic parameter is supposed to be large. The V-equation may now be formed from the equation that expresses the conservation of energy:

$$\varepsilon-\frac{\partial L_r}{\partial M_r}=T\frac{\partial\eta}{\partial t};$$

ε is the rate of generation of energy per unit of mass and time, M_r is the mass inside the sphere with radius r; the variation of ε with ρ and T will not be taken into account. Then:

$$\frac{\partial L_r}{\partial M_r}\frac{\delta L_r}{L_r} + L_r \frac{\partial \frac{\delta L_r}{L_r}}{\partial M_r} + T\frac{\partial\, \delta\eta}{\partial t} = 0.$$

The first term, being of no importance, may be omitted; then the V-equation results:

$$\frac{\partial \frac{\delta L_r}{L_r}}{\partial T} + \frac{4\pi\rho\ T^2 r^2}{L_r \frac{\partial T}{\partial \eta}}\frac{dr}{dT}\frac{\partial V}{\partial t} = 0.$$

Substitution of the functional relation between V and t reduces this equation to two simultaneous ordinary differential equations:

$$\frac{d\left(\frac{\delta L_r}{L_r}\right)_{\cos}}{dT} + \frac{4\pi\rho\ T^2 r^2}{L_r \frac{\partial T}{\partial \eta}}\frac{dr}{dT} n\, V_{\sin} = 0.$$

$$\frac{d\left(\frac{\delta L_r}{L_r}\right)_{\sin}}{dT} - \frac{4\pi\rho\ T^2 r^2}{L_r \frac{\partial T}{\partial \eta}}\frac{dr}{dT} n\, V_{\cos} = 0.$$

The variables $\left(\frac{\delta L_r}{L_r}\right)_{\cos}$, $\left(\frac{\delta L_r}{L_r}\right)_{\sin}$ consist of the two terms: a V- contribution and an adiabatic part. Substitution of the two terms and transposition of the adiabatic parts to the right-hand members reduces these equations to the form:

$$\frac{d}{dT}\left\{T\frac{d}{dT}\left(\frac{1}{\gamma}\frac{p}{P}V_{\cos}\right) + \frac{4+4\frac{p}{P}}{\gamma}V_{\cos}\right\} + \frac{4\pi\rho\ T^2 r^2}{L_r \frac{\partial T}{\partial \eta}}\frac{dr}{dT} n\, V_{\sin} = -\frac{d}{dT}\left\{\left(\frac{\delta L_r}{L_r}\right)_{\cos}\right\}_{\text{adiabatic}},$$

$$\frac{d}{dT}\left\{T\frac{d}{dT}\left(\frac{1}{\gamma}\frac{p}{P}V_{\sin}\right) + \frac{4+4\frac{p}{P}}{\gamma}V_{\sin}\right\} - \frac{4\pi\rho\ T^2 r^2}{L_r \frac{\partial T}{\partial \eta}}\frac{dr}{dT} n\, V_{\cos} = -\frac{d}{dT}\left\{\left(\frac{\delta L_r}{L_r}\right)_{\sin}\right\}_{\text{adiabatic}}.$$

4. The coefficient of $V_{\sin \atop \cos}$ in the left-hand members of the V-equations is asymptotically proportional to T^4, the value of $\frac{\partial T}{\partial \eta}$ being equal to

$$T : \left\{\frac{\partial (U_r + U_i)}{\partial T}\right\}_\rho.$$

The value of $\frac{1}{\gamma}\frac{p}{P}$ may be taken as constant, equal to the value in the outer part of the stellar interior. Hence, the equations may be made more homogeneous by multiplication with the factor $\gamma \frac{P}{p} T$; then the coefficient of $V_{\sin \atop \cos}$ is a dimensionless quantity. A new independent variable ξ may be introduced by equating this coefficient to $\frac{25}{2}\xi^2$:

$$\frac{4\pi\rho\ T^2 r^2 n}{L_r}\frac{dr}{dT}\gamma\frac{P}{p}\left\{\frac{\partial (U_r+U_i)}{\partial T}\right\}_\rho = -\frac{25}{2}\xi^2.$$

Denote the coefficient $4\left(\frac{P}{p}+1\right)$ by 5λ and introduce the new dependent variables $Z_{\cos}$ and $Z_{\sin}$ by the relation:

$$V = \xi^{-\lambda-\frac{1}{2}} Z.$$

Then the V-equations are transformed into the equations:

$$\frac{d^2 Z_{\cos}}{d\xi^2} - \left(\lambda^2 - \frac{1}{4}\right)\frac{Z_{\cos}}{\xi^2} - 2 Z_{\sin} = -\frac{2}{5}\gamma\frac{P}{p}\xi^{\lambda-\frac{1}{2}}\frac{d}{d\xi}\left\{\left(\frac{\delta L_r}{L_r}\right)_{\cos}\right\}_{\text{adiabatic}},$$

$$\frac{d^2 Z_{\sin}}{d\xi^2} - \left(\lambda^2 - \frac{1}{4}\right)\frac{Z_{\sin}}{\xi^2} + 2 Z_{\cos} = -\frac{2}{5}\gamma\frac{P}{p}\xi^{\lambda-\frac{1}{2}}\frac{d}{d\xi}\left\{\left(\frac{\delta L_r}{L_r}\right)_{\sin}\right\}_{\text{adiabatic}}.$$

The operator $\frac{d}{d\xi}$ in the right-hand members may be changed into $r\frac{d}{dr}$ by the relation:

$$\frac{d}{d\xi} = \left(\frac{1}{r}\frac{dr}{d\xi}\right) r\frac{d}{dr}.$$

As only asymptotic values are needed the derivative with regard to r is to be taken at the surface of the star: at $r = r_0$. The factor $\frac{1}{r}\frac{dr}{d\xi}$ is proportional to $\xi^{-\frac{3}{5}}$; the factor of proportionality depends on the anadiabatic parameter, hence is a small number.

Then the right-hand members of the differential equations are equal to:

$$\left[-\frac{2}{5}\gamma\frac{P}{p}\frac{\xi^{\frac{3}{5}}}{r}\frac{dr}{d\xi} r\frac{d}{dr}\left\{\left(\frac{\delta L_r}{L_r}\right)_{\cos \atop \sin}\right\}_{\text{adiabatic}}\right]_{r=r_0} \xi^{\lambda-\frac{11}{10}}.$$

The solution of the equations may be expressed in terms of the right-hand members with aid of the solutions of the homogeneous equations; the necessary relations have already been worked out in the former paper [1]).

5. The solution of the homogeneous equations may be constructed following the theory of the differential equation of BESSEL; only that solution is needed which has the singularity $\xi^{-\lambda+\frac{1}{2}}$. It may be made to depend on an auxiliary function $F(u)$ by the relation:

$$Z_{\cos} = \xi^{\frac{1}{2}} \int e^{-\xi u} \cos \xi u \; F(u) \; du\,,$$

$$Z_{\sin} = \xi^{\frac{1}{2}} \int e^{-\xi u} \sin \xi u \; F(u) \; du\,,$$

the limits of the integrals to be determined presently. Then the homogeneous differential equations are satisfied if the function $F(u)$ is a solution of the differential equation:

$$(u^2 - 1) \frac{d^2 F}{d u^2} + 3 u \frac{dF}{du} + (1 - \lambda^2) F = 0\,,$$

the integrations being performed from a positive value of u that makes $(u^2 - 1) F(u)$ and $(1 - u^2) \frac{dF(u)}{du} - uF(u)$ zero to the value $u \to \infty$.

The differential equation is satisfied by each of the functions $\frac{(\sqrt{u^2-1} + u)^{\pm\lambda}}{\sqrt{u^2-1}}$. The function

$$F(u) = \frac{(\sqrt{u^2-1} + u)^{\lambda} + (\sqrt{u^2-1} + u)^{-\lambda}}{\sqrt{u^2-1}}$$

and the lower limit of integration $u = 1$ satisfy the necessary conditions.

This solution of the homogeneous equations is at the surface of the star, as far as regards the terms of lowest degree in ξ, equal to:

$$2^{\lambda} \; \xi^{\frac{1}{2} - \lambda} \int_0^{\infty} e^{-u} \left.\begin{matrix}\cos\\ \sin\end{matrix}\right\} u \; u^{\lambda - 1} \, du\,.$$

The value of the integrals may be evaluated in the usual way: they are equal to

$$2^{-\frac{\lambda}{2}} \Gamma(\lambda) \left.\begin{matrix}\cos\\ \sin\end{matrix}\right\} \lambda \frac{\pi}{4}.$$

6. The computation of the values of $V_{\cos}$ and $V_{\sin}$ at the outer boundary involves the integrals

$$\int_0^{\infty} \xi^{\lambda - \frac{11}{10}} \, Z_{\substack{\cos\\ \sin}} \, d\xi\,;$$

the functions Z introduce the integration with regard to the variable u. If the order of integration is changed these integrals are reduced to the values:

$$\int_1^{\infty} F(u) \; u^{-\lambda - \frac{2}{5}} \, du \int_0^{\infty} e^{-x} \left.\begin{matrix}\cos\\ \sin\end{matrix}\right\} x \;\; x^{\lambda - \frac{3}{5}} \, dx\,;$$

as λ is positive the integrals are convergent at the limits $x \to 0$, $u \to \infty$.

The first integral may be evaluated in terms of the Γ-function:

$$\int_1^{\infty} F(u) \; u^{-\lambda - \frac{2}{5}} \, du = 2^{\lambda - \frac{3}{5}} \frac{\Gamma\left(\lambda + \frac{1}{5}\right) \Gamma\left(\frac{1}{5}\right)}{\Gamma\left(\lambda + \frac{2}{5}\right)}.$$

Hence:

$$\int_0^{\infty} \xi^{\lambda - \frac{11}{10}} \, Z_{\substack{\cos\\ \sin}} \, d\xi = 2^{\frac{\lambda}{2} - \frac{4}{5}} \, \Gamma\left(\frac{1}{5}\right) \Gamma\left(\lambda + \frac{1}{5}\right) \left.\begin{matrix}\cos\\ \sin\end{matrix}\right\} \left(\lambda + \frac{2}{5}\right) \frac{\pi}{4}.$$

7. The factor $\left(-\frac{\xi^{\frac{3}{5}}}{r} \frac{dr}{d\xi}\right)_{r=r_0}$ must be computed from the equation that defines the variable ξ:

$$\frac{4\pi\rho \, T^2 r^2 n}{L_r} \frac{dr}{dT} \gamma \frac{P}{p} \left\{ \frac{\partial (U_r + U_i)}{\partial T} \right\}_{\rho} = -\frac{25}{2} \xi^2.$$

This relation shows the value of the ratio $\xi^2 : \left(\frac{r_0 - r}{r_0}\right)^5$ at the boundary of the star to be equal to

$$\frac{8\pi}{25} \frac{\rho}{T^3} \frac{r_0^{\,7} n}{L} \left(\frac{dT}{dr}\right)^4 \gamma \frac{P}{p} \left\{ \frac{\partial (U_r + U_i)}{\partial T} \right\}_{\rho}\,;$$

this value is connected with the factor to be computed, by the equation:

$$\lim \xi^2 \left(\frac{r_0}{r_0 - r}\right)^5 = \left(-\frac{r}{\xi^{\frac{3}{5}}} \frac{d\xi}{dr}\right)^5_{r=r_0}.$$

The derivative $\frac{dT}{dr}$ in the outer part of the star is equal to $-\frac{1}{4} \frac{m}{R} \frac{p}{P} g$; m is the molecular weight, R the absolute gas-constant. Then the limiting value considered is equal to:

$$\frac{\pi}{200} \left(\frac{m}{R}\right)^4 a \, r_0^{\,7} \frac{n}{L} g^4 \gamma \frac{\beta^3}{1 - \beta} \left(1 - \frac{7}{8}\beta\right);$$

the limiting value of $\frac{p}{P}$ in the outer part of the star has been denoted by β; the energy per unit volume of the radiation in thermodynamic equilibrium has been denoted by aT^4. Denote the constant of gravitation

[1]) *B.A.N.* No. 282.