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COMMUNICATION FROM THE OBSERVATORY AT LEIDEN.

On the most probable values of some astronomical constants, first paper, constants connected with the earth, by *W. de Sitter.*

1. On various occasions I have found myself confronted with the question which would be the best value to adopt for one or the other fundamental astronomical constant. We have, of course, the set of constants officially adopted in the national ephemerides. But for some of these more recent determinations are now available, which are undoubtedly better than those on which the adopted values are based. Also some of the adopted constants are inconsistent with each other. Thus e. g. the adopted length of the year and the constant of precession are contradictory, and similarly the mass of the earth and the solar parallax. The discrepancies are, however, small, and it is very far from me to suggest that the time should already have come to revise the internationally adopted system. NEWCOMB's great work *) is now more than thirty years old, but it still stands unsurpassed as an example of sound critical discussion. The time is not yet ripe, in my opinion, for a repetition of this work. To mention only a few points, it would certainly be premature to make a new determination either of the constant of precession or of the elements of the planets, or to construct a new fundamental system of star places and proper motions, before the results of the reduction of HORNSBY's observations are available, or before the present uncertainty regarding the systematic errors in the declinations, and the proper motions in declination, of the stars has been cleared up.

Thus, if I have decided to publish some of the conclusions to which I have come on the occasions referred to above, it should be understood that I do not wish to propose that the values of the constants here given should replace the generally adopted ones. I have only been led by the considerations that on the one hand my results may perhaps be useful to others in similar circumstances, and on the other hand that a critical survey like the present may be helpful to guide the efforts of astronomers to those points where they are most needed.

No attempt has been made at completeness. In many cases I have deliberately left out of consideration all but the most reliable determinations, but, though, of course, it is possible that I may have overlooked one or more important researches, I have tried to include the most recent data. Some of the results have been published before. It has been considered convenient, in those cases, to repeat the principal data. The probable errors given for the final results, are in all cases intended to convey my judgment that the probability of the real error being smaller than the amount stated is one half. Those of the individual determinations have in most cases been quoted as given by their respective authors.

*) The elements of the four inner planets and the fundamental constants of astronomy, 1895.

2. *The mean radius of the earth.*

The mean radius is the radius in the geographical latitude φ_1 defined by $3 \sin^2 \varphi_1 = 1$. The value arrived at is the same as found in 1915 *). The data on which it is based are:

European arcs, reduced with $\varepsilon^{-1} = 299{\cdot}15$

STRUVE's arc	$b = 6378{\cdot}455 \pm {\cdot}085$	$r_1 = 6371{\cdot}387$
West-European arc	$6377{\cdot}935 \pm {\cdot}100$	$6370{\cdot}868$
Parallel $+ 52°$	$6378{\cdot}057 \pm {\cdot}070$	$6370{\cdot}990$
Parallel $+ 47\frac{1}{2}°$	$6377{\cdot}350 \pm {\cdot}435$	$6370{\cdot}284$

The mean with the weights 6, 4, 9, $\frac{1}{2}$ is:

$$r_1 = 6371{\cdot}069 \pm {\cdot}080.$$

The probable error has been derived from the residuals.

South African arc, reduced with $\varepsilon^{-1} = 298{\cdot}3$

$$b = 6378{\cdot}307 \pm {\cdot}120 \quad r_1 = 6371{\cdot}219$$

*) On the mean radius of the earth, the intensity of gravity, and the moon's parallax, *Proceedings Amsterdam*, XVII, p. 1291.

Indian arcs, reduced with $\varepsilon^{-1} = 298{\cdot}3$

$$b = 6378{\cdot}358 \pm {\cdot}120 \qquad r_1 = 6371{\cdot}270$$

HAYFORD's *and* TITTMAN's *discussion of North-American arcs, reduced with* $\varepsilon^{-1} = 297{\cdot}0$

$$b = 6378{\cdot}388 \pm {\cdot}030 \qquad r_1 = 6371{\cdot}268$$

Combining the results with the weights

Europe	1½
India	1
South Africa	1
North America	8

we find the mean

$$\mathbf{r_1 = 6371{\cdot}238 \pm {\cdot}030.}$$

The probable error derived from the deviations of the individual values from this mean is ± ·026, which has been increased to ± ·030. The value of the equatorial radius derived from this value of r_1 and the flattening $\varepsilon^{-1} = 297{\cdot}0 \pm 0{\cdot}10$ is

$$b = 6378.355 \pm {\cdot}030.$$

If the probable error of ε^{-1} corresponding to the *geodetic* determinations of that constant had been used, the probable error of b would become ± ·045.

3. *The acceleration of gravity at mean latitude.*

The fundamental determination at Potsdam by KÜHNEN and FURTWÄNGLER is still the most accurate. It is

$$g_P = 981{\cdot}274 \pm {\cdot}002.$$

This result has not been reduced to the geoid, and it seems possible that the local disturbances at Potsdam, as well as the topographical and isostatic reductions, may be subject to comparatively large uncertainty. Dr. VENING MEINESZ *) has executed a very precise comparison between Potsdam and de Bilt, giving

$$g_B - g_P = -{\cdot}0083 \pm {\cdot}0003,$$

from which, since the observations at Potsdam require a small correction because VENING MEINESZ's observations were taken at an altitude 4 meters less than those of KÜHNEN and FURTWÄNGLER, we find

$$g_B = 981{\cdot}267 \pm {\cdot}002.$$

At de Bilt the topographic and isostatic corrections are certainly very small. **) With HELMERT's formula of 1915 this would give at mean latitude

$$g_1 = 979{\cdot}772 \pm {\cdot}002.$$

*) Observations de pendule dans les Pays-Bas, p. 141.

**) An isostatic reduction was carried out by Dr. BOWIE for Wolberg, where the correction was found negligible. For de Bilt it may be expected to be still smaller.

For comparison we have

HELMERT (1915)	$g_1 = 979{\cdot}769 \pm {\cdot}003$
HAYFORD and BOWIE	$979{\cdot}761 \pm {\cdot}006$

Combining these with the weights 8,3 and 1 we have

$$\mathbf{g_1 = 979{\cdot}770 \pm {\cdot}002}$$

This, with the coefficients of $\sin^2\varphi$ and $\sin^2 2\varphi$ which will be derived below, gives for the acceleration of gravity at the equator

$$g_0 = 978{\cdot}052$$

and at de Bilt

$$g_B = 981{\cdot}266.$$

4. *The constant of precession.*

This important fundamental constant, which defines the relation of the system of coordinates to which our observations made on the earth are necessarily referred, to a system defined by the fixed stars, deserves a very thourough discussion. At present I will restrict myself to the derivation of a preliminary mean from the most reliable determinations. Those taken into account are the following.

NEWCOMB's classical determination of 1897, with corrections by HOUGH and HALM to allow for the unequal distribution of the stars over the two streams; *)

KAPTEYN's determination of 1901 from the proper motions of AUWERS—BRADLEY; **)

DYSON and THACKERAY from Groombridge stars; ***)

LEWIS BOSS from the proper motions of the *P.G.C.*; ****)

C. DE JONG from a comparison of KÜSTNER's catalogue (Bonn **10**) with the A. G. zones Leipzig I, Berlin A, Berlin B and Leiden; *****)

C. DE JONG from a comparison of KÜSTNER with BESSEL's zones; *****)

J. H. OORT from proper motions in galactic latitude. This last investigation has not yet been published. A preliminary result has been communicated to me by the author.

The values found by these astronomers, with the weights assigned to them, are, for 1850·0:

		weight
NEWCOMB, corrected	$p = 50''{\cdot}2486$	2
KAPTEYN	·2453	1½
DYSON and THACKERAY	·2503	2
BOSS	·2511	2
DE JONG, I	·2503	1
DE JONG, II	·2399	½
OORT	·2480	2

*) *M. N.* lxx, 587 (1910).

**) *A. N.* 3721/22 (1901).

***) *M. N.* lxv, 118 (1905).

****) *A. J.* 612/14 (1910).

*****) *Dissertation*, Leiden 1917.

The mean by weights is

$$\mathbf{p} = 50''\cdot 2486 \pm \cdot 0010.$$

The probable error derived from the residuals is $\pm$ ·0008, which has been slightly increased on account of the provisional nature of the investigation.

For the planetary precession I adopt

$$\lambda = 0''\cdot 1228 \pm \cdot 0012,$$

giving for the lunisolar precession:

$$p_1 = 50''\cdot 3714 \pm \cdot 0016.$$

Of this 0″·0191 is due to the orbital motion of the earth by EINSTEIN's theory of general relativity ('geodetic precession', SCHOUTEN—FOKKER), leaving for the true lunisolar precession due to the attraction of the sun and moon on the flattened earth:

$$p_0 = 50''\cdot 3523 \pm \cdot 0016.$$

5. *The constant of nutation.*

NEWCOMB has in *Astronomical Constants* derived from all data then available:

$$N = 9''\cdot 210 \pm ''\cdot 008 \quad (m.\ e.)$$

PRZYBYLLOK has made a new determination from the material of the international latitude observations *). He finds

$$N = 9''\cdot 207 \pm ''\cdot 003 \quad (m.\ e.)$$

The mean error derived from the interagreement of the different stations is, however, $\pm$ ″·005, which would make the weight 200 on NEWCOMB's system (m. e. of unit weight $\pm$ ″·07), instead of 538, as given by PRZYBYLLOK. The larger mean error seems to me to be more nearly representative of the true reliability of the result. The weight of NEWCOMB's value is 72. The relative weights of the two results would thus be 1 and 3. I think a ratio of 1 to 2 is nearer to the truth. Even then one might feel that too much weight is given to one method. The question of the weights is, however, not very important. We may adopt:

$$N = 9''\cdot 208 \pm \cdot 003 \quad (p.\ e.)$$

6. *The lunar inequality.*

The observed values are

NEWCOMB, from the sun	$L = 6''\cdot 485$	$\pm$ ″·012 (p .e.)
GILL, from Victoria	6·443	$\pm$ ·007
HINKS, from Eros	6·461	$\pm$ ·002

Assigning the weights 1, 3 and 10 respectively we find

$$L = 6''\cdot 459 \pm ''\cdot 005.$$

*) *Zentralbureau der Internationalen Erdmessung, Neue Folge der Veröffentlichungen*, 36, 1920.

7. *The solar parallax.*

For the solar parallax I have adopted

$$\pi_\odot = 8''\cdot 8032 \pm ''\cdot 0013.$$

The publication of the derivation of this value may be postponed to a future occasion.

8. *The mass of the moon and the ratio of the moments of inertia of the earth.*

As a first approximation for the mass of the moon I take HINKS's value

$$\mu = 1/81\cdot 53 \pm \cdot 05.$$

Then from the value of p_0 just derived we find

$$H = \frac{C-A}{C} = 0\cdot 0032769 \pm \cdot 0000015.$$

With these the computed values of the constant of nutation and of the lunar inequality become

$$N = 9''\cdot 2104$$
$$L = 6\cdot 4576$$

Putting now

$$H = 0\cdot 0032769 + x.\ 10^{-5}$$
$$\mu^{-1} = 81\cdot 53 + y,$$

we find from the observed values of the constant of precession, the constant of nutation, and the lunar inequality, the equations of condition

		weight	Resid.
$+0\cdot 154\,x - 0\cdot 417\,y = \Delta p_0 =$	·0000 $\pm$ .0016	10	+.0001
$+\ \cdot 028\ \ -\ \cdot 112\ = \Delta N = -\cdot 0024 \pm$	30	3	$-$ 23
$0\ \ -\ \cdot 078\ = \Delta L = +\cdot 0014 \pm$	50	1	+ 16

Solving these by least squares, we find

$$x = + \cdot 005 \pm \cdot 055$$
$$y = + \cdot 002 \pm \ 20,$$

the probable errors corresponding to the weights and a probable error $\pm$ ·0050 for unit weight. The residuals are practically equal to the original left hand members. The true probable errors must be larger, and I will adopt

$$\mathbf{H} = \cdot 0032770 \pm \cdot 0000010$$
$$\mu^{-1} = 81\cdot 53 \pm .04$$

Treating the probable errors as independent of each other, we would find from these

$$p_0 = 50''\cdot 3522 \pm 0''\cdot 0227$$
$$N = 9\cdot 2103 \pm \cdot 0053$$
$$L = 6\cdot 4574 \pm \cdot 0033,$$

the computed probable error of L including that of $\pi_\odot$. Comparison with the observed values shows that only in the case of L the computed probable error is smaller than that of the observed value. For N it would also be smaller, if we had not increased the adopted probable errors of H and μ to twice the computed values. Thus an increase of the accuracy of our knowledge of the mass of the moon requires an accurate determination of the lunar inequality L and, less urgently, of the constant of nutation. The best method of determining the lunar inequality is from the observation of minor planets, and I cannot help thinking that this determination, which will be derived as a by-product of the Eros campaign of 1931, will be almost more important than that of the solar parallax itself. At all events it will be very desirable to arrange the observations so as to get the largest possible weight for the mass of the moon.

9. *The flattening of the earth and related constants.*

From the adopted value of H we find by the formulas of *B. A. N.* 55

$$\varepsilon = 0{\cdot}0033675$$
$$\varepsilon^{-1} = 296{\cdot}96 \pm 0{\cdot}10$$

We take for the depression at 45° in the meridian of the geoid the same value as adopted in *B. A. N.* 55, viz:

$$\varkappa = {\cdot}0000005.$$

The effect on the surface, a depression of 3·2 meters, is, of course, entirely irrelevant. But if the earth is, below the isostatic layer, in hydrostatic equilibrium, corresponding terms must be present not only in the radius but also in the acceleration of gravity and in the coefficients of the gravitational potential. Although, of course, the exact value of $\varkappa$ is unknown, the adopted value is a plausible one, and is certainly nearer the truth than $\varkappa = 0$. It thus seems preferable to include the effect of $\varkappa$ in the computed values of the various derived quantities.

From the adopted values of r_1 and g_1 we find

$$\rho_1 = {\cdot}0034499.$$

Then we have

$$q = 0{\cdot}50009,$$

and the factors determining the coefficients of the second and fourth harmonics in the gravitational potential

$$V = \frac{fM}{r}\left[1 - \frac{2}{3}\frac{Jb^2}{r^2}P_2(\sin\delta) + \frac{4}{15}\frac{Kb^4}{r^4}P_4(\sin\delta)\right]$$

become

$$J = {\cdot}0016388, \qquad K = {\cdot}0000109.$$

The acceleration of gravity is now

$$g = 978{\cdot}052\,[1 + {\cdot}0052884 \sin^2\varphi - {\cdot}0000075 \sin^2 2\varphi],$$

φ being the geographical latitude. The radius in the latitude φ becomes

$$r = 6378{\cdot}359\,[1 - {\cdot}0033675 \sin^2\varphi + {\cdot}0000066 \sin^2 2\varphi].$$

The difference between geocentric and geographical latitude is

$$\varphi - \varphi' = 695''{\cdot}77 \sin 2\varphi - 0''{\cdot}96 \sin 4\varphi.$$

If $\varkappa$ were neglected the coefficients of the last terms would become $+{\cdot}0000071$ and $-1''{\cdot}17$ respectively. The differences are small, and in the case of $\varphi - \varphi'$ undistinguishable from local deviations of the vertical.

It should be remembered that all these constants have been derived from the adopted values of $r_1, g_1, H, \varkappa$ by the formulas of the theory of CLAIRAUT. It was proved in *B. A. N.* 55 that these are applicable to the actual earth if the isostatic compensation is perfect over the whole earth. If the compensation is reasonably approximate, the effect on the derived constants will not exceed a few units of the last decimal place. If there were no compensation, the deviations might run up to two or more units in the last decimal but one.

Thus even in the extreme case of no isostatic compensation the value of ε^{-1} derived here would be much more trustworthy than any derived from geodetic operations, or from the motion of the moon.

The determination of ε from geodetic arcs, of course, also presupposes a very close agreement of the geoid with an ellipsoid, i. e. small deviations from hydrostatic equilibrium, and the same is true of the determination of ε from β (the coefficient of $\sin^2\varphi$ in g) and from J, this latter being derived from the motion of the moon.

10. *The lunar parallax.*

The best determination of the parallax of the moon still is that from the observations made at Greenwich and the Cape in 1905—10, and discussed by Dr. CROMMELIN in *M. N.* lxxi, p. 526. The result derived there is

$$\Delta\pi_{☾} = +0''{\cdot}29 \pm 0''{\cdot}06 - 0''{\cdot}057\,\delta\varepsilon^{-1}$$

The probable error $\pm 0''{\cdot}06$ is derived from the residuals. There are 100 separate determinations giving 49 residuals smaller than, 49 exceeding and 2 equal to $\pm 0''{\cdot}63$. The distribution of the residuals is in excellent agreement with the law of errors. We can thus adopt the p. e. $\pm 0''{\cdot}06$ as a reliable measure of the accuracy apart from systematic errors. Such errors may easily have been introduced by the different aspect of the crater Moesting A as seen from the north and from the south. As Dr. CROMMELIN points out, the use of a reversing prism, by which this effect might have been eliminated, would have presented very considerable difficulties.

The correction should be applied to the parallax on which the ephemeris of the crater in the *Berliner Jahrbuch* is based, which, according to that publication, is

3422″·27 corresponding to $\pi'_☾ = \sin\pi_☾/\sin 1'' = 3422''\cdot 11$, or 0″·04 larger than HANSEN's parallax according to NEWCOMB *).

From the adopted values of r_1, g_1, ε and μ we have

$$\pi'_☾ = \mathbf{3422''\cdot 519} \pm \cdot 009$$
$$+ 0''\cdot 18\, \delta r_1 - 1''\cdot 16\, \delta g_1 - 0''\cdot 013\, \delta\varepsilon^{-1} + 0''\cdot 170\, \delta\mu^{-1},$$

the correction to r_1 being expressed in kilometers, and to g_1 in cm/sec².

Comparing with the observed value we find thus:

$$O-C = -0''\cdot 12 \pm ''\cdot 06 =$$
$$+0''\cdot 18\, \delta r_1 - 1''\cdot 16\, \delta g_1 + 0''\cdot 044\, \delta\varepsilon^{-1} - 0''\cdot 170\, \delta\mu^{-1}$$
$$+0''\cdot 422\, \delta r_{Gw} + 0''\cdot 301\, \delta r_{Cp} + 0''\cdot 0104\, \delta\varphi'_{Gw} - 0''\cdot 0138\, \delta\varphi'_{Cp}$$
$$+0''\cdot 0168\, \delta h,$$

where δr_{Gw}, δr_{Cp} are corrections to the adopted radii of Greenwich and the Cape, expressed in kilometers, $\delta\varphi'_{Gw}$ and $\delta\varphi'_{Cp}$ are corrections to the adopted reductions from geographical to geocentric latitude at the two stations, expressed in seconds of arc, and δh is the correction to the adopted selenocentric distance of the crater, expressed in seconds of arc. This latter is, according to the *B. J.*, 934″·71. The adopted radii r_{Gw} and r_{Cp} and reductions $(\varphi'-\varphi)_{Gw}$ and $(\varphi'-\varphi)_{Cp}$ respectively are, of course, those corresponding to the ellipsoid with the adopted flattening ε. The corrections δr_{Gw}, δr_{Cp}, $\delta\varphi'_{Gw}$, $\delta\varphi'_{Cp}$, if real, would therefore be due to deviations from the geoid produced by want of isostatic compensation, or deviations of the geoid from the ellipsoid. It need hardly be mentioned that the altitudes were duly taken into account, but, so far as I can ascertain, no deviations of the vertical. The term with $\varkappa$ was also neglected, but its effect is too small to require consideration. It will be seen that the observed $O-C$ can be explained e. g. by assuming a depression of the geoid below the ellipsoid of 400 meters at the Cape, or of 280 meters at Greenwich. The most probable explanation is, however, by systematic errors of pointing on Moesting A. The computed value of $\pi'_☾$ is in any case more reliable than the observed one.

11. *The motions of the lunar perigee and node.*

The annual motions of the lunar perigee and node corresponding to the value of J derived above are

$$d\varpi = +6''\cdot 386 \qquad d☊ = -5''\cdot 977$$

*) Researches on the motions of the moon, II; *Astronomical Papers*, Vol IX, p. 42.

Further we have by BROWN's theory

	$d\varpi$		$d☊$	
Principal term	+	146426″·92	—	69672″·04
mass of the earth	—	·68	+	·19
planetary terms	+	2·54	—	1·38
flattening of the earth	+	6·385	—	5·98
	+	146435·165	—	69679·21

The observed sidereal motions by Dr. JONES's discussion in *M. N.* lxxxv, p. 36, are, for 1850, with the constant of precession derived above:

$$d\varpi = +146435''\cdot 315 \qquad d☊ = -69679''\cdot 398$$

There thus remains for the figure of the moon:

$$d\varpi = +0''\cdot 15 \pm 0''\cdot 06 \qquad d☊ = -0''\cdot 19 \pm 0''\cdot 04$$

The probable errors are due for the greater part to the theoretical value. The uncertainty of the planetary masses introduces an uncertainty of about ± 0″·01 or ± 0″·02, but the principal term is uncertain to a much larger extent by the neglect of terms of higher orders in the developments. Prof. BROWN estimates the maximum uncertainty from this source as ± 0″·10 and ± 0″·05 respectively. *) Taking half of these and combining them with the other sources of error of the theoretical and the observed values, we find the probable errors given.

The theoretical expressions are **)

$$d\varpi = \left[-832'' + 1222''\, f\right]\left(J' + \tfrac{1}{2}K'\right)$$
$$d☊ = -470''\left(J' + \tfrac{1}{2}K'\right)$$

We find, therefore

$$J' + \tfrac{1}{2}K' = \cdot 000406 \pm 000085$$
$$f = 0\cdot 98 \pm 0\cdot 14.$$

For the ratio corresponding to q in the earth we now find

$$g' = 0\cdot 65 \pm 0\cdot 13.$$

The conclusions reached in the quoted paper of 1915 remain unaltered. It is impossible to derive a reliable value of J, and through it of ε, from the motions of the lunar perigee and node, on account of the uncertainty inherent in the parts of these motions due to the figure of the moon, which is not, like the earth, approximately in hydrostatic equilibrium.

*) *M. N.* lxiv, p. 532.

**) See: DE SITTER, The motions of the lunar perigee and node, and the figure of the moon, *Proceedings Amst.* XVII, p. 1309 (1915). The notations used here are the same as in that paper.