B.A.N. 17, no 3, 176–184; *Communication from the Observatory at Leiden*

THE ASYMMETRY OF THE SCATTERING DIAGRAM OF A SPHERICAL PARTICLE

WILLIAM M. IRVINE*

Received May 24, 1963

The asymmetry factor $\overline{\cos\theta}$ for scattering of unpolarized radiation by a particle is defined as the weighted mean over the sphere of the cosine of the scattering angle, with the particle phase function as weighting function. The asymmetry factor determines the radiation pressure exerted on a particle and is also important in the approximate theory of multiple scattering by large particles (aerosols or cloud droplets). It is calculated for spherical, dielectric particles in the following cases:

(1) refractive index $n = 1.20, 1.33, 1.50, 1.60, 2.00$ for various values of the size parameter x in the range $0.4 \leqslant x \leqslant 12.5$;
(2) $n \approx 1$ and $0.2 \leqslant x \leqslant 10$ (Rayleigh-Gans scattering);
(3) $x \to \infty$ and $1.0 \leqslant n \leqslant 5.0$ (very large spheres).

An altitude chart in the n–x plane is constructed which should make it possible to obtain $\overline{\cos\theta}$ by interpolation for other values of $n \leqslant 2.0$ and x.
New calculations of the total Mie scattering coefficient for $n = 2.00$ are also presented.

1. Introduction

If a spherical particle of radius r and index of refraction n is illuminated by unpolarized parallel radiation of wavelength λ, the resultant values of the total scattered intensity, when plotted versus scattering angle θ, form the particle-scattering diagram. When appropriately normalized, this scattering diagram is called the phase function $f(\theta)$ of the particle.

For general values of n and the particle-size parameter $x = 2\pi r/\lambda$, the scattering diagram is very complicated, containing a number of maxima and minima. There are applications for which it would be useful to characterize the phase function by a single parameter. This paper is concerned with the calculation of such a parameter, which may be called the *asymmetry factor* of the phase function. Defined as the weighted mean over the sphere of $\cos\theta$ with the phase function itself as weighting function, the asymmetry factor will be denoted by $\overline{\cos\theta}$. Thus

$$\overline{\cos\theta} = \tfrac{1}{2}\int_{-1}^{1} d(\cos\theta)\cos\theta f(\theta)\,. \qquad (1)$$

The previous explicit calculations of $\overline{\cos\theta}$ are contained in VAN DE HULST (1946 and 1953).** They are confined to the limiting cases $n \to \infty$, $n \to 1$, $x \to 0$, $x \to \infty$, with the exception of a few values obtained for $n = 2$ and $0 \leq x \leq 3.5$. The accuracy of the results is generally about $\pm$ 0.02.

Many values of $\overline{\cos\theta}$ for a wide range of x and n are contained implicitly in the tables of CHU, CLARK, and CHURCHILL (1957) discussed in section 3b below.

In a thorough paper on Mie scattering by interplanetary particles GIESE (1961) has calculated Q_{pr} (the efficiency factor for radiation pressure) for several complex values of n. Giese's results would suffice to determine $\overline{\cos\theta}$, but unfortunately he seems to have used an incorrect formula for Q_{pr} (see the footnote to our eq. (5)).

In the present paper $\overline{\cos\theta}$ is calculated to an accuracy of at least $\pm$ 0.0001 for the following cases:

1. $n = 1.20, 1.33, 1.50$ and $x = 0.4\ (0.2)\ 3\ (0.4)\ 5\ (1)\ 10$;
   $n = 1.60$ and $x = 1.0\ (1)\ 6\ (2)\ 10$;
   $n = 2.00$ and $0.5 \leq x \leq 12.5$ (various values);
2. Rayleigh–Gans scattering ($n \approx 1$, $x = 0.2\ (0.2)\ 10$);
3. Very large spheres ($x \to \infty$, $n = 1.05\ (0.05)\ 2\ (0.1)\ 2.5, 2.75, 3.0, 3.5, 4.0, 5.0$).

We thus confine our attention to dielectrics.

* Now at Harvard College Observatory and Smithsonian Astrophysical Observatory, Cambridge, Massachusetts, U.S.A.
** Hereafter referred to as VDH1 and VDH2, respectively.

2. Applications

a. Radiation pressure

The radiation pressure exerted on a scattering particle may be calculated if $\overline{\cos\theta}$, Q_{sca}, and Q_{ext} are known, where Q_{sca} and Q_{ext} are the particle efficiency factors for scattering and extinction (VDH2). The force exerted in the direction of propagation of the radiation is given for unit incident intensity by

$$F = \frac{\pi r^2}{c}(Q_{ext} - \overline{\cos\theta}\, Q_{sca}). \tag{2}$$

b. Interstellar scattering

The properties of interstellar dust grains may be studied from observations of the diffuse light in the Galaxy. Since the observations suffice only for a rough determination of the preponderance of forward over backward scattering by the grains, a very approximate theory is more adequate than an examination of detailed scattering diagrams. The natural parameters of the problem are the particle albedo and $\overline{\cos\theta}$ (e.g. VAN HOUTEN, 1961).

c. Radiative transfer

Application of the theory of radiative transfer to scattering media in which the particles have dimensions comparable with or greater than λ has been severely limited. To apply the theory, the phase function is usually expanded as a series of Legendre polynomials. For strongly asymmetric phase functions a large number of terms must be included in this series. Although the theoretical principles of the method are known, the solution to the equation of transfer in closed form when $\overline{\cos\theta} \neq 0$, *including tabulated functional values*, is available only for phase functions of the type

$$f(\theta) = a + bP_1(\cos\theta), \tag{3}$$

where a and b are constants, and even for this relatively simple case only for semi-infinite atmospheres (HARRIS, 1961). The function (3) has a maximum asymmetry on a scale from 0 to 1 of only

$$\overline{\cos\theta} = \tfrac{1}{3}.$$

This corresponds to a maximum size parameter of $x \approx 1.3$ (see section 4).

CHU, LEACOCK, CHEN and CHURCHILL (1962) have obtained numerical results for a phase function consisting of a five-term expansion in Legendre polynomials, but even here the value of the size parameter was only $x = 2.2$.

VAN DE HULST and IRVINE (1962) have suggested that the scattering diagram of large particles could usefully be approximated by means of the Henyey–Greenstein phase function

$$f(\theta) = \frac{(1 - g^2)}{(1 + g^2 - 2g\cos\theta)^{\frac{3}{2}}}, \tag{4}$$

where g is for the function (4) just equal to $\overline{\cos\theta}$. This approximation is thought to be particularly useful in connection with the "method of successive scattering", in which the intensity is found by adding the contributions of successive orders of scattering (VAN DE HULST and DAVIS, 1961).

In order to approximate the actual scattering diagram of a spherical particle by means of eq. (4), one must determine an optimum value of the parameter g. It is natural to assume that this value will be given by, or closely related to, the value of $\overline{\cos\theta}$ found from the true scattering diagram.

3. Procedure

a. New calculations from Mie theory

The scattering diagram of a spherical particle must in general be written as an infinite series involving the Mie coefficients a_m and b_m. $\overline{\text{Cos}\,\theta}$ is given in terms of a_m and b_m (VDH2, p. 128*) by

$$\overline{\cos\theta} = \frac{4}{x^2 Q_{sca}} \sum_{m=1}^{\infty} \left\{ \frac{m(m+2)}{m+1} \Big[R(a_m)R(a_{m+1}) + I(a_m)I(a_{m+1}) + R(b_m)R(b_{m+1}) + I(b_m)I(b_{m+1}) \Big] + \frac{2m+1}{m(m+1)} \Big[R(a_m)R(b_m) + I(a_m)I(b_m) \Big] \right\}, \tag{5}$$

* Complex conjugate bars have been omitted in VDH2. That equation (5) is correct may be seen from VDH1 (ch. 2) and DEBYE (1909, § 7).

where $R(a)$ and $I(a)$ signify the real and imaginary parts of a. a_m and b_m are defined by

$$a_m = \frac{S'_m(nx)S_m(x) - nS_m(nx)S'_m(x)}{S'_m(nx)Z_m(x) - nS_m(nx)Z'_m(x)},$$
$$b_m = \frac{nS'_m(nx)S_m(x) - S_m(nx)S'_m(x)}{nS'_m(nx)Z_m(x) - S_m(nx)Z'_m(x)}, \tag{6}$$

with S_m and Z_m, the Riccatti–Bessel functions, defined in terms of the Bessel functions of half-integral order by

$$S_m(x) = \left(\frac{\pi x}{2}\right)^{\frac{1}{2}} J_{m+\frac{1}{2}}(x)$$
$$Z_m(x) = \left(\frac{\pi x}{2}\right)^{\frac{1}{2}} \left[J_{m+\frac{1}{2}}(x) - iN_{m+\frac{1}{2}}(x)\right]. \tag{7}$$

The primes in eq. (6) denote differentiation.

The values of a_m, b_m, and Q_{sca} were taken for $n = 1.20$ from PANGONIS, HELLER and JACOBSON (1957), for $n = 1.33$ and 1.50 from PENNDORF and GOLDBERG (1956) and PENNDORF (1956), for $n = 1.60$ from GUMPRECHT and SLIEPCEVICH (1951), and for $n = 2.00$ from LOWAN (1949), KERKER and PERLEE (1953), and KERKER and COX (1955). The notation of each of these authors differs somewhat from that defined above. They all include an additional factor of $(-1)^m i$ in the definition of a_m and of $(-1)^{m+1} i$ in that of b_m. In addition all except PENNDORF and GOLDBERG include a factor of

$$\frac{(2m+1)}{m(m+1)}$$

in these definitions. Since we consider here the case of real n,

$$Q_{sca} = Q_{ext} = Q,$$

which is called K in the tables cited.

For $n = 1.20$, 1.33, 1.50, and 1.60, the values of $\overline{\cos\theta}$ were found by using a desk computer. All operations were carried out twice, once in reverse order, to eliminate errors. All computations were carried out to five significant figures. Table 1 gives the results to four significant figures. Apart from possible computational errors, all given digits should be correct for $n = 1.33$ and 1.50. For $n = 1.60$ the last digit may be slightly in error because of the stated inaccuracy of ± 1 in the fourth significant figure of Q. For $n = 1.20$ all figures should be significant for $x \geqslant 1.6$, while only three figures may be significant for smaller x.

TABLE 1

$\overline{\cos\theta}$ for $n = 1.20, 1.33, 1.50, 1.60$ (sec. 3 a, b)

x \ n	1.20	1.33	1.50	1.60
0.4	0.0278	0.02915	0.03140	
0.6	0.0624	0.06539	0.07015	
0.8	0.1116	0.1165	0.1249	
1.0	0.1761**	0.1845***	0.1989	0.2097
1.2	0.2581	0.2729	0.2992	
1.4	0.3578	0.3838	0.4296	
1.6	0.4701	0.5083	0.5625	
1.8	0.5789	0.6148	0.6286	
2.0	0.6613	0.6697	0.6259	0.5838
2.2	0.7045	0.6826	0.6272	
2.4	0.7240	0.6914	0.6643	
2.6	0.7372	0.7176	0.7145	
2.8	0.7581	0.7549	0.7370	
3.0	0.7860	0.7832	0.7343	0.6754
3.4	0.8299	0.8013	0.7388	
3.8	0.8483	0.8184	0.7601	
4.0	0.8561*	0.8272*	0.7502*	0.6449
4.2	0.8642	0.8346	0.7394	
4.6	0.8799	0.8457	0.7403	
5.0	0.8934	0.8454	0.7073	0.5725
6.0	0.9110	0.8479	0.6442	0.3892
7.0	0.9199	0.8791	0.5186	
8.0	0.9253	0.8040	0.4529	0.6378
9.0	0.9282	0.7601	0.6484	
10.0	0.9273	0.7125	0.7429	0.7482
15.0	0.8744*	0.7948*	0.7357*	0.7315*
20.0	0.8159*	0.7691*	0.7109*	0.7348*
25.0	0.8661*	0.8542*	0.7959*	0.7540*
30.0	0.8888*	0.8269*	0.8046*	0.7731*

* Taken from CHU *et al.* (1957).
** CHU *et al.* (1957) have 0.1757.
*** Checked by IBM 7090. CHU *et al.* (1957) have 0.1843.

For $n = 2.00$ the calculations were done on the Smithsonian Astrophysical Observatory's IBM 7090. Newly computed values of Q are listed with the corresponding values of $\overline{\cos\theta}$ in table 2. The results are given, after rounding off, to the same number of significant figures as the tabulated values of a_m and b_m. The values of Q agree with those obtained by PENNDORF and GOLDBERG (1956, part 6) to ± 0.0001 with one exception ($x = 2.5$, difference of 0.0002).

b. Previous calculations from Mie theory

A large number of values of $\overline{\cos\theta}$ are contained implicitly in a report by CHU, CLARK and CHURCHILL (1957). These authors expand the phase function $f(\theta)$

TABLE 2

$\overline{\cos\theta}$ and total scattering coefficient Q for $n = 2.00$ (sec. 3*a*, *b*)

x	$\overline{\cos\theta}$	Q
0.5	0.06264	0.04565
0.6	0.09031	0.09774
1.0	0.2762	0.7968
1.2	0.4507	1.609
1.3	0.5329	2.349
1.4	0.5493	3.466
1.5	0.4958	4.231
1.65	0.4299	3.986
1.8	0.4332	3.718
2.0	0.5056	4.768
2.05	0.5324	5.387
2.1	0.5542	5.744
2.2	0.5615	5.319
2.25	0.5687	4.952
2.3	0.5261	4.622
2.4	0.4712	4.108
2.5	0.4122	3.797
2.6	0.3576	3.925
2.7	0.3950	4.827
2.75	0.3784	4.539
2.8	0.4587	4.052
3.0	0.3907	3.035
3.3	0.15167	3.2406
3.6	0.1494	2.020
3.8	0.14965	1.6029
4.0	0.3237	1.747
4.2	0.25484	1.9299
4.4	0.29934	3.0289
4.6	0.66045	2.4403
4.8	0.5845	2.576
5.0	0.60618*	
5.5	0.49822	3.8010
6.0	0.66912*	
6.5	0.71367	2.7086
7.0	0.49635	2.1723
8.0	0.58511	2.3413
10.0	0.61796	2.0437
12.5	0.65891	2.5461

* Taken from CHU *et al.* (1957).

of a spherical particle as a series in the Legendre polynomials,

$$f(\theta) = \frac{1}{4\pi} \sum_{m=0}^{\infty} a_m^*(x, n) P_m(\cos\theta), \tag{8}$$

where the coefficients a_m^* may be expressed in terms of the Mie coefficients a_m and b_m. Because of the orthogonality properties of Legendre polynomials, we see from the definition (1) and eq. (8) that

$$\overline{\cos\theta} = \tfrac{1}{3} a_1^* . \tag{9}$$

The general formula for a_m^* given by CHU *et al.* does in fact reduce when $m = 1$ to our eq. (5) multiplied by three.

CHU *et al.* have computed the a_m^* for $n = 0.9$, 0.93, 1.05, 1.10, 1.15, 1.20, 1.25, 1.30, 1.33, 1.40, 1.44, 1.50, 1.55, 1.60, 2.00, ∞ and some or all of the values $x = 1\,(1)\,6\,(2)\,10\,(5)\,30$. Their results for $a_1^*/3$ agree with the corresponding manually computed values presented here to $\pm$ 0.0001, with the two exceptions mentioned in table 1. In the case $n = 2.0$ (machine computation) the results agree to five significant figures.

c. *Rayleigh–Gans scattering*

As the index of refraction $n \to 1$ for a given particle size parameter x, the scattering diagram simplifies considerably and may be written in closed form. More precisely, the conditions for the validity of Rayleigh–Gans scattering are $|n - 1| \ll 1$ and $2\,x\,|n - 1| \ll 1$. $\overline{\text{Cos}\,\theta}$ in this case is given (VDH2, § 2.3, 7.21) by

$$\overline{\cos\theta} = \frac{\int \cos\theta\,(1 + \cos^2\theta)\,G^2\,(2\,x \sin\tfrac{1}{2}\theta)\,d\omega}{\int (1 + \cos^2\theta)\,G^2\,(2\,x \sin\tfrac{1}{2}\theta)\,d\omega}, \tag{10}$$

where the integrals are over the sphere and

$$G(u) = \frac{3}{u^3}(\sin u - u\cos u). \tag{11}$$

The integral in the denominator of eq. (10) has been evaluated by RAYLEIGH (VDH2, § 7.22). The numerator also reduces to an elementary integration after the substitution

$$v = 4x\sqrt{\tfrac{1}{2}(1 - \cos\theta)}.$$

We obtain

$$\overline{\cos\theta} = \frac{\psi(x)}{\varphi(x)} \tag{12a}$$

with

$$\psi(x) = \frac{4}{x^4}\Bigg[\left(\frac{9}{128} - \frac{5x^2}{64}\right)(\cos 4x + 4x \sin 4x) + \frac{x^6}{2} + \frac{11x^4}{8} - \frac{31x^2}{64} - \frac{9}{128} + x^2(x^2 - \tfrac{3}{8})(-\gamma + \text{Ci}(4x) - \ln 4x)\Bigg] \tag{12b}$$

and

$$\varphi(x) = 2.5 + 2x^2 - \frac{\sin 4x}{4x} - \frac{7}{16x^2}(1 - \cos 4x) + \left(\frac{1}{2x^2} - 2\right)(\gamma + \ln 4x - \mathrm{Ci}(4x)), \quad (12c)$$

where γ = Euler's constant = 0.5772157... and

$$\mathrm{Ci}(x) = -\int_x^\infty \frac{\cos u}{u} du \quad (12d)$$

is the cosine integral (values of $\mathrm{Ci}(x)$ were taken from LOWAN (1940) for $x \leqslant 10.0$ and from *Mathematical Tables* (1931) for $x > 10.0$).

$\overline{\cos\theta}$ was computed for $x \geqslant 0.6$ from eqs. (12) with the IBM 7090. The results agree well with the rough estimates given in VDH2 (p. 90) when $x \leqslant 1.4$. For larger x there are significant differences. The present values fit smoothly into the altitude chart (figure 3), whereas the previous estimates do not.

For $x \leqslant 0.5$, $\psi \ll 1$, and the most convenient procedure is to expand ψ and φ in powers of x. The leading terms are

$$\psi(x) = 0.189630\,x^6 - 0.065016\,x^8 + \ldots$$
$$\varphi(x) = 1.18519\,x^4 - 0.47407\,x^6 + \ldots . \quad (13)$$

The tabulated values for x = 0.2, 0.4 were calculated from eq. (13).

d. Very large spheres

When $x \gg 1$ and $2x|n - 1| \gg 1$, the scattering diagram may be separated into a component resulting from reflection and refraction and a component arising from diffraction around the sphere. This separation greatly simplifies the calculation of $\overline{\cos\theta}$. We find (VDH2, § 12.5; the parameter $w = 1$ for dielectrics)

$$\overline{\cos\theta} = \tfrac{1}{2}(1 + \gamma), \quad (14)$$

where γ is the weighted mean of $\cos\theta$ when only reflected and refracted light are included in the scattered radiation (i.e., when the diffracted light, which is concentrated in a very narrow forward cone, is treated as unscattered). For some applications γ may be a more appropriate parameter than $\overline{\cos\theta}$ (VAN DE HULST and IRVINE, 1962, § III).

We may in turn divide γ into contributions from the two mutually perpendicular directions of polarization in the scattered light. If the index 1 refers to light whose electric vector is perpendicular to the scattering plane and the index 2 to light with electric vector in this plane,

$$\gamma = \tfrac{1}{2}(\gamma_1 + \gamma_2), \quad (15)$$

where*

$$\gamma_1 = \int_0^1 d\sigma (A_1 + B_1)/C_1, \quad (16)$$

$$A_1 = 4r_1^4(n^2 - \sigma)(2\sigma - 1)/n^2, \quad (17)$$

$$B_1 = (1 - r_1^2)^2 \left[(2\sigma - 1)(2\sigma/n^2 - 1) + 4\sigma\sqrt{1-\sigma}\sqrt{n^2 - \sigma}/n^2\right], \quad (18)$$

$$C_1 = 1 - 2r_1^2(2\sigma/n^2 - 1) + r_1^4, \quad (19)$$

and

$$r_1 = \frac{\sqrt{1-\sigma} - \sqrt{n^2-\sigma}}{\sqrt{1-\sigma} + \sqrt{n^2-\sigma}}. \quad (20)$$

The same expression (16) gives γ_2 with r_1 replaced by

$$r_2 = \frac{n\sqrt{1-\sigma} - \sqrt{1-\sigma/n^2}}{n\sqrt{1-\sigma} + \sqrt{1-\sigma/n^2}}. \quad (21)$$

The integrals (16) for γ_1 and γ_2 were evaluated on the IBM 7090 using Simpson's rule and an interval of 0.001. Values of the integrand were printed out at intervals of 0.025. The integrand is well behaved except at $\sigma = 1$, where its first derivative becomes infinite. From a comparison with check integrations using intervals of 0.002 and 0.004, we conclude that the results for $\overline{\cos\theta}$ and γ are accurate to $\pm$ 0.00005, while those for γ_1 and γ_2 have an uncertainty of $\pm$ 0.0001. The values obtained agree well ($\pm$ 0.02) with those quoted by VDH2 (p. 226).

4. Results

The results obtained in the present paper, as well as selected values from CHU *et al.* (1957) and VDH2, are listed in tables 1–4. Their accuracy is discussed in the preceding section.

A convenient graphical presentation of the data

* An exponent 2 is omitted from the factor $(1 - r_1^2)$ in the term corresponding to our B_1 by VDH2 (p. 225) and DEBYE (1909, eq. (156)). That eq. (18) is correct may be seen from VDH1 (p. 30) and DEBYE's eq. (154).

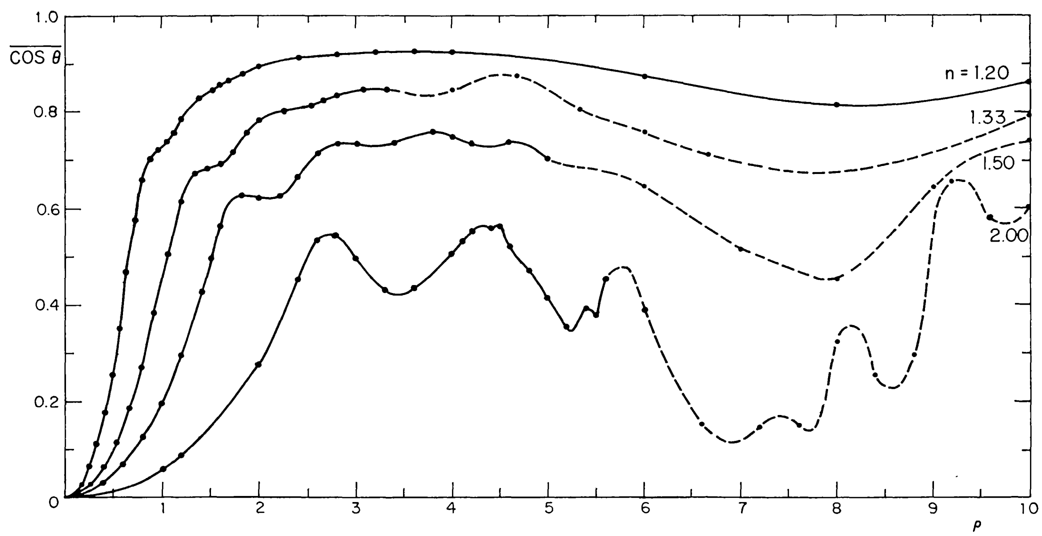

Figure 1. Asymmetry factor $\overline{\cos\theta}$ versus normalized particle size parameter $\rho = 2x\,(n - 1)$. Dashed portions omit minor oscillations (for $n = 2.00$, this portion of the curve is somewhat speculative).

proved to be a plot of $\overline{\cos\theta}$ versus ρ, where $\rho = 2x(n - 1)$ gives the phase shift experienced by a light-ray passing through the centre of the particle. VAN DE HULST (1946) and PENNDORF (1956) have shown that ρ, which may be called the normalized size parameter, possesses a basic significance for particles with n near 1. In figure 1 we have plotted $\overline{\cos\theta}$ versus ρ for $n = 1.20$, 1.33, 1.50, 2.00.

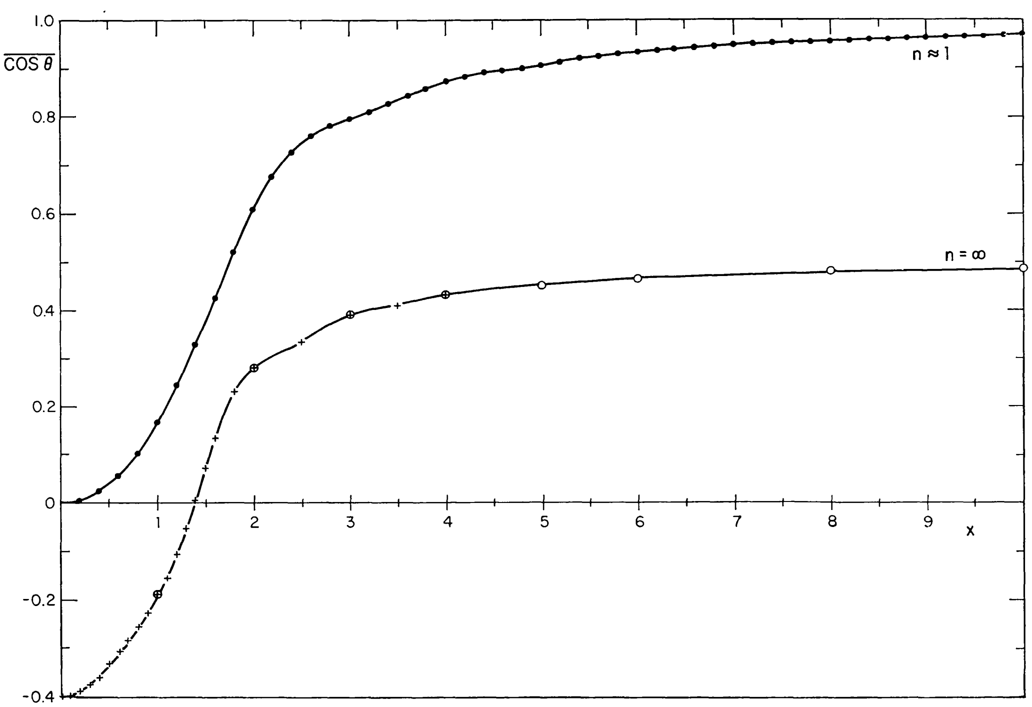

Figure 2. $\overline{\cos\theta}$ versus particle size parameter x for Raleigh–Gans scattering and for totally reflecting spheres. Crosses: VAN DE HULST (1957, sections 10.61, 10.62); open circles: CHU *et al.* (1957); closed circles: table 3.

Observe that $\overline{\cos\theta}$ is not a monotonic function of ρ for a given n. "Long-period" oscillations occur, the period of which is approximately the same for the various values of n considered. For $n = 1.33$ a similar oscillatory behavior has been found by DEIRMENDJIAN, CLASEN and VIEZEE (1961) for the absolute intensity at $\theta = 0^0$, and by PENNDORF (1961) for the ratio $i_1(0)/i_1(\pi)$.

The value of $\overline{\cos\theta}$ at the first maximum of the long-period oscillations decreases with increasing n. In addition, as n increases there is an increasing amount of "fine structure" in the curves. These features are what would be expected from the behavior of Q_{sca} with x and n (PENNDORF, 1956).

The solid portions of the curves in figure 1 should be accurate to better than $\pm$ 0.01 for $n \leqslant 1.50$. For $n = 2.00$ the appearance of small spikes (such as those at $\rho = 4.5$ and 5.4) may increase this uncertainty to perhaps $\pm$ 0.02.

Because of the appearance of the "fine structure", the curves are written as dashed lines when the computed points are too far apart to permit accurate interpolation. Since $\overline{\cos\theta}$ is an average quantity, however, there is no reason to expect violent fluctuations, at least for $n \leqslant 1.60$. We may thus expect that the main features of the behavior of $\overline{\cos\theta}$ with x, sufficient for applications like those of parts b and c of section 2, are shown even by the dashed portions of the curve.

The values of $\overline{\cos\theta}$ for Rayleigh–Gans scattering are shown in figure 2, using x as the abscissa. For completeness, we have also included the results for totally reflecting spheres ($n = \infty$) from VDH2 (§ 10.6). The latter case is qualitatively different from that for finite n.

5. Altitude chart of $\overline{\cos\theta}$

To facilitate the determination of $\overline{\cos\theta}$ for values of n and x for which explicit calculations have not been

TABLE 3

$\overline{\cos\theta}$ for Rayleigh–Gans scattering (sec. 3c)

x	$\overline{\cos\theta}$	x	$\overline{\cos\theta}$
0.2	0.00631	5.2	0.91468
0.4	0.0258	5.4	0.92100
0.6	0.05869	5.6	0.92665
0.8	0.10560	5.8	0.93116
1.0	0.16693	6.0	0.93453
1.2	0.24231	6.2	0.93718
1.4	0.32976	6.4	0.93967
1.6	0.42487	6.6	0.94240
1.8	0.52044	6.8	0.94543
2.0	0.60754	7.0	0.94854
2.2	0.67821	7.2	0.95138
2.4	0.72874	7.4	0.95371
2.6	0.76103	7.6	0.95553
2.8	0.78107	7.8	0.95701
3.0	0.79573	8.0	0.95843
3.2	0.81005	8.2	0.95998
3.4	0.82613	8.4	0.96171
3.6	0.84346	8.6	0.96349
3.8	0.86015	8.8	0.96512
4.0	0.87422	9.0	0.96649
4.2	0.88475	9.2	0.96758
4.4	0.89212	9.4	0.96849
4.6	0.89764	9.6	0.96939
4.8	0.90278	9.8	0.97038
5.0	0.90842	10.0	0.97147

TABLE 4

$\overline{\cos\theta}$ for very large spheres (sec. 3d)

n	$\overline{\cos\theta}$	γ	γ_1	γ_2
1.00	1.0	1.0	1.0	1.0
1.05	0.99030	0.98060	0.97799	0.98320
1.10	0.97308	0.94617	0.93701	0.95532
1.15	0.95376	0.90752	0.89000	0.92503
1.20	0.93407	0.86814	0.84193	0.89435
1.25	0.91475	0.82950	0.79512	0.86389
1.30	0.89612	0.79224	0.75067	0.83381
1.333	0.88427	0.76853	0.72287	0.81419
1.35	0.87830	0.75661	0.70905	0.80417
1.40	0.86134	0.72268	0.67039	0.77497
1.45	0.84521	0.69043	0.63463	0.74623
1.50	0.82990	0.65980	0.60165	0.71795
1.55	0.81535	0.63070	0.57125	0.69016
1.60	0.80153	0.60305	0.54325	0.66286
1.65	0.78838	0.57676	0.51743	0.63609
1.70	0.77587	0.55173	0.49361	0.60986
1.75	0.76395	0.52790	0.47160	0.58420
1.80	0.75259	0.50519	0.45125	0.55912
1.85	0.74176	0.48352	0.43240	0.53465
1.90	0.73142	0.46285	0.41490	0.51079
1.95	0.72155	0.44310	0.39864	0.48756
2.00	0.71212	0.42424	0.38351	0.46496
2.10	0.69448	0.38896	0.35623	0.42169
2.20	0.67833	0.35666	0.33236	0.38096
2.30	0.66353	0.32705	0.31135	0.34275
2.40	0.64994	0.29987	0.29275	0.30699
2.50	0.63745	0.27489	0.27619	0.27359
2.75	0.61040	0.22080	0.24187	0.19973
3.00	0.58832	0.17664	0.21511	0.13817
3.50	0.55525	0.11049	0.17625	0.04474
4.00	0.53254	0.06509	0.14948	−0.01931
5.00	0.50543	0.01087	0.11503	−0.09330

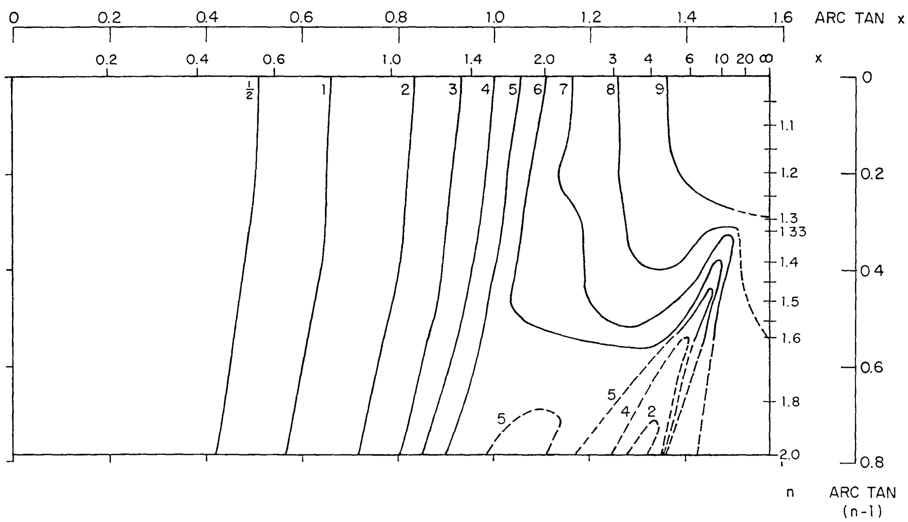

Figure 3. Altitude chart in the n–x domain. Figures along the curves denote 10 $\overline{\cos\theta}$. Dashed portions indicate approximate behavior.

made, it is convenient to construct the altitude chart (figure 3). Following VDH1 (p. 67), we plot the n–x plane with abscissa arctan x and ordinate arctan $(n - 1)$. We limit ourselves to values $1 \leqslant n \leqslant 2$.*

Along the left border of figure 3, $x \ll 1$ and $\overline{\cos\theta} = 0$ because Rayleigh scattering is symmetric. On the right border ($x \gg 1$) the values for very large spheres (table 4) apply. The upper edge of the altitude chart is the region of Rayleigh–Gans scattering (table 3). In compiling the remainder of figure 3, use has been made of tables 1–2 and figures 1–2, as well as all the other values of $\overline{\cos\theta}$ available from CHU, CLARK and CHURCHILL (1957).

For $n \gtrsim 1.3$ the oscillations in $\overline{\cos\theta}$ become apparent as "hills and valleys" for $2 \lesssim x < \infty$. An infinite number of such hills should be crowded into the region $15 < x < \infty$ in figure 3.

Over most of figure 3 we estimate that $\overline{\cos\theta}$ may be read to an accuracy of better than $\pm$ 0.03. Within the dashed portions the accuracy may be reduced to about $\pm$ 0.15, except when $x \geqslant 15$, in which case the long-period oscillations make the result still more uncertain.

* LÖSCH's (1954) tables were very helpful in the construction of figure 3.

Acknowledgments

I am grateful to Prof. H. C. van de Hulst for several helpful suggestions, to I. Robbers for carrying out most of the desk calculations at a time of personal stress, to Dr O. Gingerich for assistance with FORTRAN and to Susan Irvine for checking many of the computations.

I am also pleased to thank Prof. J. H. Oort for the hospitality extended to me during a year spent at the Sterrewacht, Leiden, where this work was begun, and the North Atlantic Treaty Organization for a Post-doctoral Fellowship.

References

C. M. CHU, G. C. CLARK and S. W. CHURCHILL, 1957, *Tables of Angular Distribution Coefficients for Light Scattering by Spheres*, Univ. Mich., Engineering Research Inst., Ann Arbor

C. M. CHU, J. A. LEACOCK, J. C. CHEN and S. W. CHURCHILL, 1962, "Exact Numerical Solutions for Multiple Anisotropic Scattering", *Proc. Interdisciplinary Conference on Electromagnetic Scattering*, Potsdam, New York. In press.

P. DEBYE, 1909, *Ann. Physik* **30** 57

D. DEIRMENDJIAN, R. CLASEN and W. VIEZEE, 1961, *J. Opt. Soc. Amer.* **51** 620

R. H. GIESE, 1961, *Z. Astrophys.* **51** 119

R. O. GUMPRECHT and C. M. SLIEPCEVICH, 1951, *Tables of Light Scattering Functions for Spherical Particles*, Univ. Mich., Engineering Research Inst., Ann Arbor

D. L. HARRIS, 1961, chap. 8, App. I, *Planets and Satellites*, ed. G. P. KUIPER and B. M. MIDDLEHURST, Univ. Chicago Press, Chicago

C. J. VAN HOUTEN, 1961, *Bull. Astron. Inst. Netherlands* **16** 1 (No. 509)

H. C. VAN DE HULST, 1946, *Recherches Astron. Obs. d'Utrecht* **11**, part 1

H. C. VAN DE HULST, 1953, *Light Scattering by Small Particles*, John Wiley, New York

H. C. VAN DE HULST and M. M. DAVIS, 1961, *Proc. Kon. Ned. Akad. Wet. Series B* **64** 220

H. C. VAN DE HULST and W. M. IRVINE, 1963, *Mémoires Soc. R. Sc. Liège* **7** 78

M. KERKER and A. L. COX, 1955, *J. Opt. Soc. Amer.* **45** 1080

M. KERKER and H. E. PERLEE, 1953, *J. Opt. Soc. Amer.* **43** 49

F. LÖSCH, 1954, *Siebenstellige Tafeln der elementaren transzendenten Funktionen*, Springer-Verlag, Berlin, Göttingen, Heidelberg

A. N. LOWAN, 1940, *Tables of Sine, Cosine and Exponential Integrals*, Vol. I, Federal Works Agency-Work Projects Administration (U.S.)

A. N. LOWAN, 1949, *Tables of Scattering Functions for Spherical Particles*, Nat. Bur. Standards, Appl. Math. Ser. 4, U.S. Govt. Printing Office, Washington, D.C.

Mathematical Tables, Vol. I, 1931, British Association for the Advancement of Science, London

W. J. PANGONIS, W. HELLER and A. JACOBSON, 1957, *Tables of Light Scattering Functions for Spherical Particles*, Wayne State Univ. Press, Detroit, Mich.

R. B. PENNDORF, 1956, *New Tables of Mie Scattering Functions for Spherical Particles, Part* 6, ASTIA Document No. AD-98772, U.S. Dept. of Commerce, Office of Technical Services

R. B. PENNDORF, 1961, "Research in Aerosol Scattering in the Infrared, Sci. Rep. 5," Tech. Rep. RAD-TR-61-32, U.S. Dept. of Commerce, Office of Technical Services

R. B. PENNDORF and B. GOLDBERG, 1956, *New Tables of Mie Scattering Functions for Spherical Particles, Parts 1-5*, ASTIA Documents Nos. AD-9876 to AD-98771, U.S. Dept. of Commerce, Office of Technical Services