On the origin of smoke particles in the interstellar gas (Errata: 10 146)

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On the origin of smoke particles in the interstellar gas, by *D. ter Haar.*

This paper attempts to give a theory of the formation of solid particles from the interstellar gas. After a brief outline in section 1 of the present data pertaining to the interstellar gas and smoke [1]) in the region of the sun, a formula for the temperature of the smoke particles is derived in section 2, supposing the smoke particles to be heteropolar crystalline particles.

In section 3 it is shown that if the density in the gas clouds is smaller than a certain *characteristic density* the problem must be attacked differently from the case where the density is larger. In section 4 the case $\rho < \rho_{\text{char}}$ is treated. The probability for radiation capture, important in that case, is discussed in section 5. In section 6 the case $\rho > \rho_{\text{char}}$ is treated by a method used by BECKER and DÖRING in their theory of the formation of liquid drops in a supersaturated vapour. Now, the velocity of smoke formation decreases very rapidly with increasing ρ. The result of sections 4 and 6 is that there exists an optimum density for the smoke formation; this density will nearly coincide with the characteristic density.

Finally section 7 gives an estimate of the maximum size of the smoke particles and discusses the total smoke density in our part of the galactic system on the assumption that the condensation has stretched over a period of the order of 10^9 years. The density of the diatomic molecules is discussed also.

List of symbols used in the following; the values used in numerical calculations are inserted also.

$\rho(\omega)$: density of radiation at the frequency ω.
ω_0: fundamental frequency of the crystal; $\frac{\omega_0}{c} = 1000$ cm^{-1}.
m: reduced mass of an ion in the crystal; $m = 2{\cdot}5.10^{-23}$ g (nitrogen atom).
e: effective charge of an ion in the crystal; $e = 10^{-10}$ e.s.u. [2]).
σ: surface of an atom; $\sigma = 2.10^{-15}$ cm^2 (surface of a graphite sphere, divided by the $^2/_3$-power of the number of atoms contained in it).
i: number of atoms in a crystal.
T_i: temperature of a particle consisting of i atoms.
T_{rad}: temperature of the stellar radiation; $T_{\text{rad}} = 10000°$ [3]).
T_g: temperature of the interstellar gas; $T_g = 10000°$ [3]).
v: mean velocity of the hydrogen atoms in the interstellar gas; $v = 1{\cdot}5.10^6$ cm sec^{-1} [3]).
ρ, ρ': number of atoms per cm^3 in the interstellar gas.
g: dilution factor of the radiation: $g = 10^{-14}$ [3]).
u_i: binding energy of an atom in a particle of i atoms; $u_i = 1$ eV.
N_i: number of crystals consisting of i atoms.
O_i: surface of one crystal of i atoms; $O_i \cong i^{2/3}\,\sigma$.
Z_i: total surface of all crystals of i atoms; $Z_i = N_i O_i$.
$2\tau_c$: collision time; $\tau_c \cong 10^{-13}$ sec.
γ_i, γ_i': probability for an impinging atom to stick to the surface of a crystal.
ΔE: dissociation energy of a diatomic molecule.
$c_1, c_2 \ldots$: numerical factors of order 1.
$K, K', K'', \ldots$: non-specified factors.
c: velocity of light; $c = 3.10^{10}$ cm sec^{-1}.
h: PLANCK's constant; $h = 6{\cdot}6.10^{-27}$ erg sec.
$\hbar$: DIRAC's constant; $\hbar = \frac{h}{2\pi}$.
k: BOLTZMANN's constant; $k = 1{\cdot}4.10^{-16}$ erg degree^{-1}.
a_i: number of colliding atoms which in 1 sec stick to 1 cm^2 of the surface of a particle of i atoms.
b_i: number of atoms which in 1 sec evaporate from 1 cm^2 of the surface of a particle of i atoms.

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[1]) The term *smoke* for the solid interstellar particles was suggested by Mr VAN DE HULST. Considering the probable process of formation this name appears to be more adequate than that of *dust*, by which they have usually been indicated.
[2]) We have taken about 1/5 of the elementary charge, corresponding to DENNISON's figure for HCl (*Phys. Rev.* **31**, 503, 1928). For the actual crystals e may rather be expected to be a good deal smaller.
[3]) A. S. EDDINGTON, *Proc. Roy. Soc.* **A 111**, 424, 1926.

1. *Introduction.* Only a part of the mass of our galactic system is taken up by the stars. The mass of the interstellar gas and of the smoke particles in that gas taken together is of the same order as the mass of the stars [1]). After EDDINGTON's original investigations (l.c.) many papers have been published about the composition, temperature, density, etc. of the interstellar gas [1])[2])[3])[4]). Hydrogen is by far the most abundant element.

LINDBLAD [5]) has discussed the possibility that the solid particles should have been formed by condensation from the interstellar gas. He found that the maximum size of the particles which follows from the law for the light-scattering in the smoke clouds (λ^{-1}-law [6])) can be explained on the simple hypothesis that there exist particles acting as nuclei for condensation and that every atom falling on the surface of such a nucleus is caught by it, on account of the low temperature (3° K) of the latter. He did not inquire into the problem of the formation of the nuclei.

That problem will be investigated in the present paper, which is partly an elaboration, partly a completion of the author's lecture at an interacademic colloquium in Utrecht [7]) [8]).

2. *Temperature of the crystalline particles.* If there should be a temperature equilibrium between the atoms of the interstellar gas ($T_g = 10000°$) and the diatomic, triatomic and polyatomic particles, the number of particles consisting of i atoms would decrease rapidly with increasing i, and condensation would be entirely out of the question. The density of the radiation in space, however, is small [9]); therefore the emission of radiation of a polyatomic particle will not be counterbalanced by the absorption of radiation and the temperature of the particles will decrease with increasing i. As a consequence the conditions for the process of accretion will gradually become more favourable, till, for sufficiently large i, the temperature has become so low that every atom (except probably the H and He atoms [10])) falling upon a smoke particle is caught by it.

[1]) TH. DUNHAM JR., *Proc. Am. Phil. Soc.* **81**, 277, 1939.
[2]) B. STRÖMGREN, *Ap. J.* **89**, 529, 1939.
[3]) P. SWINGS a.o., *Ann. d'Ap.* **1**, 1, 1938.
[4]) E. SCHÖNBERG, *Ergebn. Ex. Naturw.* **19**, 1, 1940.
[5]) B. LINDBLAD, *M. N.* **95**, 20, 1934; *Nature* **135**, 133, 1935.
[6]) See for instance, HALL, *Ap. J.* **85**, 145, 1937.
[7]) J. H. OORT a.o., *Ned. Tijdschr. Natk.* **10**, 238, 1943.
[8]) In 1941 the University of Leiden offered a prize for a solution of the problem, whether, in the time of about 10^9 years during which the stellar system has probably existed, an appreciable number of solid particles could have been formed in the interstellar clouds. The present paper has developed from one of the answers in that competition.
[9]) EDDINGTON, l.c.
[10]) OORT, l.c.

As a model for the smoke particles we will use heteropolar crystals which are small compared to the wavelength of the stellar radiation. In as much as there are only slight deviations from harmonic binding between the atoms, these will approximately emit and absorb radiation as one large harmonic oscillator (only the fundamental frequencies contribute). A harmonic oscillator with mass M, charge ε and frequency ω absorbs and emits per sec the following energies (assuming $kT << h\omega$; viz. only a small fraction of the radiation oscillators is in the first excited quantum state, whereas higher excitations can be neglected):

$$E_{\text{abs}} = \frac{\pi \varepsilon^2}{M} \rho(\omega) \quad \text{and} \quad E_{\text{em}} = \frac{8\pi^2 \, h\omega^3 \, \varepsilon^2}{M c^3} \, e^{-\frac{h\omega}{kT}} \qquad (1)\ ^{1})$$

In our case we must use for particles of i atoms: $\varepsilon = i \times e$, $M = i \times m$, $T = T_i$ and $\omega = \omega_0$. The radiation in interstellar space is approximately a diluted PLANCK radiation [2]); therefore the formulae (1) become:

$$E_{\text{abs}} = i \, \frac{8\pi^2 e^2 h g \omega_0^3}{m c^3} \left(e^{\frac{h\omega_0}{kT_{\text{rad}}}} - 1 \right)^{-1}$$

$$\text{and} \quad E_{\text{em}} = i \, \frac{8\pi^2 \, h \, \omega_0^3 \, e^2}{mc^3} \, e^{-\frac{h\omega_0}{kT_i}} . \qquad (2)$$

The atoms colliding with the particles and those evaporating from them are practically all hydrogen atoms; they carry with them per sec the following energies:

$$E_{\text{on}} = c_1 \, i^{2/3} \, \sigma \, \rho \, v \, kT_g \quad \text{and} \quad E_{\text{off}} = c_2 \, i^{2/3} \, \sigma \, \rho \, v \, kT_i .$$

The energy balance yields the following equation:

$$c_1 \, \sigma \, \rho \, v \, kT_g \, i^{2/3} + i \, \frac{8\pi^2 e^2 h g \omega_0^3}{m c^3} \left(e^{\frac{h\omega_0}{kT_{\text{rad}}}} - 1 \right)^{-1} =$$

$$c_2 \, \sigma \, \rho \, v \, kT_i \, i^{2/3} + i \, \frac{8\pi^2 \, h \, \omega_0^3 \, e^2}{m c^3} \, e^{-\frac{h\omega_0}{kT_i}} .$$

On the left-hand side the ratio of the coefficient of i to the coefficient of $i^{2/3}$ is approximately 10^{-9}; on the right-hand side their ratio is larger than 10^4 for $T_i \cong 100°$ K.

Hence, the following equation holds approximately:

[1]) For the significance of the various symbols see the list at the beginning of the paper.
[2]) EDDINGTON, l.c.

$$e^{-\frac{h\omega_0}{kT_i}} = \frac{c_3 \sigma \rho v m c^3 k T_g}{8\pi^2 e^2 \omega_0^3 h} i^{-1/3} = \rho e^{-A} i^{-1/3}. \quad (3)$$

The quantities ρ and v in formula (3) may be taken to refer to the hydrogen atoms in the interstellar gas, while σ, m, e and ω_0 are quantities characteristic for the composition of the crystals (which can be safely assumed to contain no or little hydrogen). Using for all these quantities the values given in the list and using for ρ 2 cm^{-3} [1]) we obtain from formula (3):

$$T_i \cong \frac{83}{1 + \frac{1}{50} \log i}. \quad \left(\log = {}^e\log\right) \quad (4)$$

If we had taken for ρ a value 100 times larger and for e a value 10 times smaller we would get: $T_i \cong \dfrac{182}{1 + \frac{1}{24}\log i}$. We are therefore certain that the temperature pertaining to the internal degrees of freedom of the particles is very low, even if i is not more than 2 or 3. In the region of large i the formulae for our crystal model are clearly rather different from those holding for a black body, but it seems physically justified to use this model for the determination of T_i in the region of i-values where this temperature is significant for our problem.

3. *The condensation process; characteristic density.* Assuming that a quasi-stationary situation has been established, so that the number of particles containing a given number of atoms remains practically constant, we can express this situation by the equation:

$$j = a_i Z_i - b_{i+1} Z_{i+1}. \quad (5)$$

A similar equation also occurs in BECKER and DÖRING's [2]) theory on the formation of liquid drops in a supersaturated vapour. The quantity j is the number of particles "flowing through" the various particle sizes; the number of new smoke particles formed per second will clearly depend on j.

Using the condensation-evaporation picture we obtain the following formulae for a_i and b_i [3]) [4]):

$$a_i = c_4 \gamma_i \rho' v' = \gamma_i e^{\alpha} \quad (6a)$$

$$b_i = c_5 \gamma'_{i-1} \frac{m (kT_i)^2}{h^3} e^{-\frac{u_i}{kT_i}} = \gamma'_{i-1} e^{\beta_i - \frac{u_i}{kT_i}} \quad (6b)$$

1) STRÖMGREN, l.c.
2) R. BECKER and W. DÖRING, *Ann. d. Phys.* (V) **24**, 719, 1935.
3) ρ' in formula (6a) is the density of the atoms constituting the crystal; ρ' differs therefore from the ρ in formula (3) and will be much smaller.
4) In b_i γ'_{i-1} appears (instead of γ'_i), since the probability of evaporation is dependent on the number of impinging atoms sticking to a particle of $i-1$ atoms. Except for small i $\gamma'_i = \gamma_i$. For the deduction of the formula for b_i see for instance SCHAEFER, *Einf. Theor. Phys.* II, Berlin-Leipzig 1929, 609.

We see that, apart from the factor γ_i, a_i is independent of i. We may take β_i and u_i practically independent of i. The quantity b_i is still very much dependent on i through T_i appearing in the exponent.

The ratios $\dfrac{b_{i+1}}{a_i}$ decrease gradually to values smaller than 1 when i increases. We can now distinguish between two cases, viz. the case where these ratios decrease from a value originally larger than 1, and the case where these ratios even from $i = 1$ onwards are always smaller than 1.

Assuming all quantities appearing in our formulae to remain constant, with the exception of ρ, the first or the second case is realized according as ρ is larger or smaller than a certain *characteristic density*. We can determine this characteristic density by calculating T_i from the equation $a_i = b_{i+1}$ and then solving ρ from equation (3), putting $i = 1$. The result is:

$$\log \rho_{\text{char}} = A - \frac{h\omega_0}{u}(\beta - \alpha) \quad (7)$$

Using different values for u, $\frac{\omega_0}{c}$ and e, we have computed the following table for ρ_{char}:

TABLE 1

u in eV	$\frac{\omega_0}{c}$ in cm^{-1}	e in e.s.u.	ρ_{char}
2	1000	10^{-10}	4.10^6
		10^{-11}	4.10^4
	2000	10^{-10}	4.10^5
		10^{-11}	4.10^3
1	1000	10^{-10}	4.10^4
		10^{-11}	4.10^2
	2000	10^{-10}	200
		10^{-11}	2
½	1000	10^{-10}	20
		10^{-11}	0·2
	2000	10^{-10}	2.10^{-5}
		10^{-11}	2.10^{-7}

The value we choose for u depends on the assumption of the nature of the evaporation. If molecules evaporate from the crystal, u may be a good deal smaller than in the case where there evaporate only atoms. Since ρ_{char} will nearly coincide with the optimum density for the smoke formation (cf. section 6), the conditions will be most favourable for this formation if ρ nearly coincides with ρ_{char}.

From Table 1 we see that the order of magnitude of ρ_{char} depends first of all on the choice of u. Since particles for which u is small will not be formed on account of their large evaporation (cf. the argument for the absence of hydrogen in the smoke particles [1])), it seems probable that u has approximately the value of 1 eV. Since the composition of the smoke particles is still less known than that of the gas, our choice for

1) OORT, l.c.

ω_0/c and e necessarily remains somewhat arbitrary [1]). It seems probable to us that the value of ρ_{char} is between 20 and 200. Since in our galactic system in the neighbourhood of the sun ρ is of the order of 2 or 3 [2]), we are here probably beneath the characteristic density. It is possible, however, that in some nebulae densities occur of about 10^2 or 10^3.

In the following sections we will treat the two cases ($\rho < \rho_{char}$ and $\rho > \rho_{char}$) as if they could be separated rigorously, although there will be of course a certain transition range of ρ-values.

4. *First case*: $\rho < \rho_{char}$. Assuming in equation (5) the terms $b_{i+1}\, Z_{i+1}$ to be negligible we get:

$$j = a_i Z_i = c_6\, \sigma\, \rho'^2\, v'\, \gamma_1\,, \quad i = 1, 2, 3, \ldots \qquad (8)$$

As in formula (6a), ρ' is the density of the atoms constituting the crystal.

Using formula (8) we obtain:

$$N_i = \frac{j}{a_i\, O_i} = \frac{a_1\, Z_1}{a_i\, O_i} = \frac{\gamma_1\, N_1}{\gamma_i\, i^{2/3}}\,. \qquad (9)$$

Except for small values of i we may put $\gamma_i = 1$ and formula (9) becomes:

$$\boxed{N_i = \gamma_1\, N_1\, i^{-2/3}}\,. \qquad (10)$$

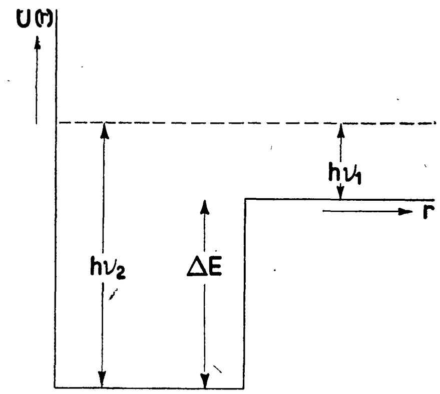
We see the importance of γ_1, the probability that, when a collision of two atoms occurs, a diatomic molecule is formed. In ordinary chemical kinetics the formation of diatomic molecules takes place through a three body collision. The ratio of the number of three body collisions to that of two body collisions is, however, only of the order 10^{-24} in interstellar space, whereas the ratio required to explain the observed smoke density would be much larger. It is therefore necessary to explain the formation of diatomic molecules by radiation capture (i.e. association accompanied by emission of radiation) [3]).

5. *Estimate of* γ_1. I am much indebted to Prof. H. A. Kramers for the following estimates of γ_1, i.e. the probability of the formation of a diatomic molecule by radiation capture.

We will first consider the following rough model. When the two atoms that are to form the molecule approach each other, the system will during the collision possess a dipole moment which varies with the time, and radiation will therefore be emitted. This emission will be calculated on a rough correspondence basis analogous to Kramers' [1]) treatment of X-ray transitions.

For this purpose we assume the potential of the interaction forces of the two atoms, considered as a

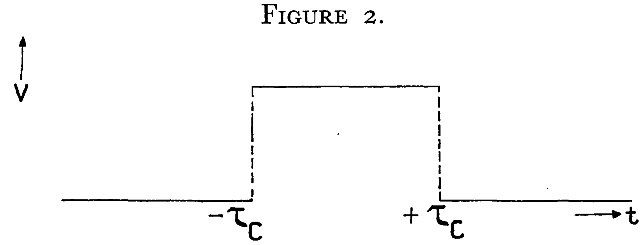
Figure 1.



function of their distance apart, to have the shape shown in Figure 1 and we wish to calculate the probability of a radiation capture in the two particle system so defined.

The relative velocity of the atoms will be a function of the time. A representative type of this function is shown in Figure 2. The corresponding acceleration

Figure 2.



will be a function consisting of two δ-functions [2]):

$$a(t) = \sqrt{\frac{2\Delta E}{m}}\,\{\delta\,(t+\tau_c) - \delta\,(t-\tau_c)\},$$

with the Fourier analysis:

$$a(t) = \sqrt{\frac{2\Delta E}{m}} \int_{-\infty}^{+\infty} 2\; e^{2\pi i \nu t} \sin 2\pi\nu\tau_c\, d\nu.$$

The total classical emitted radiation will formally be given by:

$$\frac{2e^2}{3c^3}\int |a(t)|^2\, dt = \frac{2e^2}{3c^3}\,\frac{16\Delta E}{m}\int_0^\infty \sin^2 2\pi\nu\tau_c\, d\nu$$

(e is an effective charge entirely comparable with the effective charge in the previous sections).

[1]) The sublimation heats of respectively H_2, H_2O, Ca and BaO are 0·22, 11·26, 43 and 90 Cal, which correspond to u-values of about 0·01, 0·5, 1·9 and 4 eV. It must be borne in mind, however, that an exact theory should demand a thorough sorting out of the different possibilities of the building up of a smoke particle.

[2]) Cf. Eddington, l.c., Strömgren, l.c.

[3]) Swings (*Ap. J.* **95**, 270, 1942) has also urged the necessity of taking radiation capture into consideration, but for a different purpose.

[1]) H. A. Kramers, *Phil. Mag.* **46**, 836, 1923.

[2]) P. A. M. Dirac, *The Principles of Quantum Mechanics*, Oxford 1935, 72.

Using KRAMERS' [1]) application of the correspondence principle we obtain an estimate for the average energy actually emitted by quantum processes by replacing the boundaries of the integral by ν_1 and ν_2.

Because $\nu_2 - \nu_1 \cong 10^{15}$ sec^{-1} and $\tau_c \cong 10^{-13}$ sec, it is permissible to replace $\sin^2 2\pi\nu\tau_c$ by its mean value $\frac{1}{2}$. At every molecule formation approximately $\frac{1}{2}h\,(\nu_1+\nu_2)$ is emitted. We obtain therefore:

$$\gamma_1 \cong \frac{2e^2}{3c^3}\,\frac{8\Delta E}{m}\,\frac{\nu_2-\nu_1}{\frac{1}{2}h\,(\nu_2+\nu_1)}$$

or

$$\gamma_1 \cong \frac{16}{3\pi}\,\frac{\nu_2-\nu_1}{\nu_2+\nu_1}\,\frac{e^2}{\hbar c}\,\frac{\Delta E}{mc^2}. \qquad (11)$$

For the formation of CH molecules we obtain for instance approximately $\gamma_1 \cong 10^{-13}$.

It should be borne in mind that the mechanism of radiation capture actually may be more complicated. In fact, when two atoms approach each other, two or more electronic states may be realized and a radiation capture may occur which is accompanied by electron transitions.

According to the considerations and figures of MULLIKEN [1]) this will hold for CH, CN, N_2^+, CO^+ and BH (and therefore presumably also for CH^+; data of HERZBERG, cited by SWINGS [2]) confirm this conclusion), but not for NH, OH, NO and NaH. In that way γ_1 may be much larger. A calculation of γ_1 taking into account this possibility of electronic transitions yields for CH the approximate value of 10^{-10}. It seems, however, that for other compounds γ_1 will be much smaller.

If $A(r)$ is the EINSTEIN probability for the transition between the two possible electronic states of the molecule, it is easily shown that, taking into account the different possibilities during a collision, the cross section for a radiation capture will be given by:

$$Q_{\rm rad} = Q_{\rm coll}\,\gamma_1 = f\int 4\pi\, r^2 A\,(r)\, e^{-\frac{U(r)}{kT}} \left\{ F\left(\sqrt{-\frac{U'}{kT}}\right) - F\left(\sqrt{-\frac{U}{kT}}\right)\right\} dr. \qquad (12)$$

In this formula $U(r)$ is the energy of the molecule in the upper state, $U'(r)$ that in the lower state (U and U' tend to zero for $r \to \infty$), T is the temperature of the atoms involved in the capturing process, $F(r)$ is a function which for imaginary argument is equal to zero, whereas for real positive argument it is defined by:

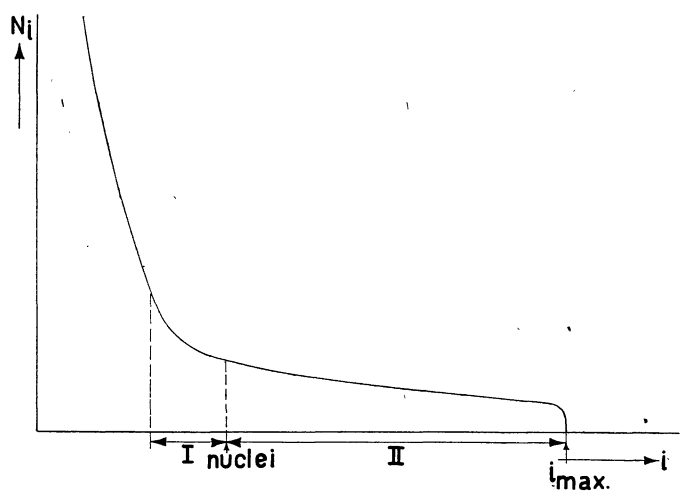
$$F(x) = \frac{4}{\sqrt{\pi}}\int_0^x e^{-x^2}\, x^2\, dx,$$

while f is the probability that, when the two atoms meet, they will find themselves in the upper electronic state under consideration.

In the case of CH the ground state $^3P(C) + {}^2S(H)$ splits into $^4\Pi$ ($g = 8$), $^2\Sigma$ ($g = 2$), $^4\Sigma$ ($g = 4$), $^2\Pi$ ($g = 4$) (g are the weights of the respective states). The radiation capture corresponds to $^2\Sigma \to {}^2\Pi$ transitions. Thus f in this case equals $2/18 = 1/9$. From an inspection of the curves for $U(r)$ ($^2\Sigma$) and $U'(r)$ ($^2\Pi$) one infers that for $T \cong 10000°$ the factor $e^{-\frac{U}{kT}} \times (F' - F)$ equals about 1 between $r = 0.5$ and $r = 1.5$ Å.U., whereas it can be neglected for other r-values. Thus $Q_{\rm rad}$ becomes $\cong 4.10^{-25}$ A ($r = 1$ Å.U.). With $A \cong 10^6$ sec^{-1} and $Q_{\rm coll} \cong 1.5.10^{-9}$, this gives for γ_1 the estimate 10^{-10} mentioned above.

6. *Second case*: $\rho > \rho_{\rm char}$. In this case we are in the same situation as BECKER and DÖRING [2]). The ratios $\frac{b_{i+1}}{a_i}$ decrease gradually with increasing i from a value larger than 1 to values smaller than 1. In this case a quasi-stationary situation will be established where at first N_i will decrease rapidly as

FIGURE 3.



I: critical range; II: pure condensation range.

function of i till we have reached a *critical range* of i-values (this critical range appears for $\frac{b_{i+1}}{a_i} \cong 1$). After that, N_i will decrease according to the formula: $N_i = K\, i^{-2/3}$ (cf. formula (10)); in the latter range

[1]) L.c.
[2]) L.c.

[1]) R. S. MULLIKEN, *Rev. Mod. Phys.* **2**, 60, 1930.
[2]) L.c.

the condensation predominates largely over the evaporation. The particles at the end of the critical range may indeed be considered as the nuclei for further (easy) smoke formation (see Figure 3). Of course, the curve representing N_i will break off somewhere in the pure condensation range, because the period over which the condensation has stretched (presumably about 10^9 years), has been too short for the formation of larger particles.

We will now eliminate, as BECKER and DÖRING have done (cf. their formula 7a and the calculations following thereupon) $Z_2, Z_3, \ldots, Z_{n-1}$ from equations (5) by writing them in the following form:

$$\frac{j}{a_1} = Z_1 - Z_2 \frac{b_2}{a_1}$$

$$\frac{j\,b_2}{a_2 a_1} = Z_2 \frac{b_2}{a_1} - Z_3 \frac{b_3 b_2}{a_2 a_1}$$

$$\frac{j\,b_3 b_2}{a_3 a_2 a_1} = Z_3 \frac{b_3 b_2}{a_2 a_1} - Z_4 \frac{b_4 b_3 b_2}{a_3 a_2 a_1}$$

....

$$\frac{j\,b_{n-1} \ldots\ldots b_2}{a_{n-1} a_{n-2} \ldots\ldots a_1} = Z_{n-1} \frac{b_{n-1} \ldots b_3 b_2}{a_{n-2} \ldots a_2 a_1} - Z_n \frac{b_n \ldots\ldots b_2}{a_{n-1} \ldots a_1}$$

Summation of these equations gives:

$$j = \frac{Z_1 - \frac{b_n \ldots\ldots b_2}{a_{n-1} \ldots\ldots a_1} Z_n}{\sum_{i=1}^{n-1} R_i} \cong \frac{Z_1}{\sum_{i=1}^{\infty} R_i} \qquad (13)$$

with $$R_1 = \frac{1}{a_1},\ R_i = \frac{b_2}{a_1} \cdot \frac{b_3}{a_2} \ldots\ldots \frac{b_{i-1}}{a_{i-2}} \cdot \frac{1}{a_{i-1}}. \qquad (14)$$

Since the ratios $\frac{b_i}{a_{i-1}}$ are at first larger than 1 and finally smaller than 1, R_i reaches a maximum for $i = i_o$, where i_o is that particular value of i, for which $\frac{b_i}{a_{i-1}} = 1$.

Using for a_i, b_i and T_i the following formulae:

$$a_i = \gamma_i\ e^{\alpha}, \qquad (6a)$$

$$b_i = \gamma'_{i-1}\ e^{\beta - \frac{u}{kT_i}}, \qquad (6b)$$

$$e^{-\frac{h\omega_o}{kT_i}} = \rho\, e^{-A}\, i^{-1/3}, \qquad (3)$$

we get for R_i approximately (cf. BECKER and DÖRING l.c.):

$$R_i = \frac{1}{\rho' v'} e^{C(i-1)\log i_o - Ci\log i + Ci} \qquad (15)$$

where $C = \frac{u}{3\,h\omega_o}$.

For $\sum_{i=1}^{\infty} R_i$ we get thus approximately:

$$\sum_{i=1}^{\infty} R_i = \frac{c_7}{\rho' v'}\, i_o^{-C}\, e^{Ci_o} \qquad (16)$$

Since i_o and ρ_{char} satisfy the following equations:

$$\frac{1}{3} \log i_o = \frac{h\omega_o}{u} (\beta - \alpha) - A + \log \rho \qquad (17)$$

and $$\log \rho_{char} = A - \frac{h\omega_o}{u} (\beta - \alpha), \qquad (7)$$

we get:

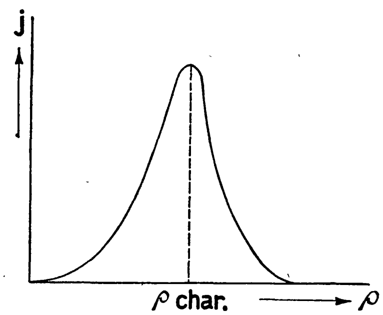
$$\boxed{i_o = \left(\frac{\rho}{\rho_{char}}\right)^3}, \qquad (18)$$

so that we obtain for j the following formula:

$$\boxed{j = c_8\, \sigma \rho'^2 v' \left(\frac{\rho}{\rho_{char}}\right)^{\frac{u}{h\omega_o}} e^{-\frac{u}{3\,h\omega_o}\left(\frac{\rho}{\rho_{char}}\right)^3}}\ . \qquad (19)$$

From formula (19) it follows that above the characteristic density the velocity of smoke formation will *decrease* with *increasing* density. Since $\frac{u}{3\,h\omega_o}$ is of the order of 1, we arrive at the conclusion that smoke formation will be practically absent in a system where the gas density were only a few times larger than the characteristic density. Together, formulae (8) and (19) present us with j as a function of ρ (see Figure 4). There

FIGURE 4.



will exist an optimum density, which will only differ from the characteristic density by a factor 2 or 3.

7. *Maximum size of the smoke particles; smoke density; density of the diatomic molecules.* Equation (5) is of course not suited for the determination of the maximum size of the smoke particles reached in a given interval of time. We have instead to use the equations:

$$\frac{dN_i}{dt} = a_{i-1} Z_{i-1} - a_i Z_i - b_i Z_i + b_{i+1} Z_{i+1}. \qquad (20)$$

(We assume for the sake of simplicity that the exhaustion of the interstellar gas is negligible.)

In the condensation range we may neglect terms with b and, putting $\tau = \sigma\rho' v' t$, $\delta_i^{-1} = \gamma_i i^{2/3}$, we obtain:

$$\frac{dN_i}{d\tau} = \delta_{i-1}^{-1} N_{i-1} - \delta_i^{-1} N_i. \qquad (21)$$

These simultaneous differential equations can be solved (cf. RUTHERFORD [1])). For the initial conditions at $t = 0$: $N_1 = N_1°$, $N_2 = N_3 \ldots\ldots = 0$, the solution is:

$$N_i = N_1° \prod_{j=1}^{i-1} \delta_j^{-1} \sum_{j=1}^{i} \frac{e^{-\delta_j^{-1}\tau}}{\prod_{k\neq j} (\delta_k^{-1} - \delta_j^{-1})} \qquad (22a)$$

or

$$N_i = N_1° \delta_i \frac{\sqrt{-1}}{2\pi} \oint \frac{e^{-x\tau}}{\prod_{j=1}^{i} (1 - x\,\delta_j)} dx. \qquad (22b)$$

where the integration is counter-clockwise round the poles of the integrand. We see from (22a) that terms with $\delta_i^{-1}\tau >> 1$ will not materially contribute to N_i. $\delta_1^{-1}\tau$ is certainly small compared with 1 (a period $t = 10^9$ years corresponds to $\tau \cong 1000$, using for ρ' the approximate density of oxygen [2])). Since diatomic molecules in their lowest vibration state (from formula (4) follows $T_2 \cong 83°$ K) will have a chance of capturing a third atom in a purely mechanical way, γ_2 must certainly be expected to be much larger than γ_1. For the moment we shall assume $\delta_3^{-1}\tau$, $\delta_4^{-1}\tau$, etc. to be $>> 1$. About $\delta_2^{-1}\tau$ we shall decide nothing at first ($\delta_2^{-1}\tau << 1$ means that the density of the diatomic molecules and therefore also the stream of larger particles developing from these molecules, is still increasing with the time). We then obtain approximately in the region, where $|x|\ \delta_3 << 1$:

$$N_i \cong \frac{-N_1° \delta_i}{2\pi\sqrt{-1}} \oint \frac{e^{x\left(\frac{1}{3} i^{1/3} - \tau\right)}}{(1-\delta_1 x)(1-\delta_2 x)} dx. \qquad (23)$$

It now follows from (23) (assuming $\gamma_2 >> \gamma_1$ and $\gamma_1 << 1$):

$$N_i \cong \frac{\gamma_1}{\gamma_i} i^{-2/3} N_1° \left(1 - e^{\gamma_2\left(\frac{1}{3} i^{1/3} - \tau\right)}\right); 1 < i < (3\tau)^3 \qquad (24a)$$

$$N_i = 0 \qquad i > (3\tau)^3 \qquad (24b)$$

From (24a) we get:

$$i < i_{max} \begin{cases} N_i \cong \dfrac{\gamma_1}{\gamma_i} i^{-2/3} N_1 & \gamma_2\tau >> 1 \quad (25a) \\ N_i \cong \dfrac{\gamma_1\gamma_2}{\gamma_i} i^{-2/3} N_1 (\tau - \frac{1}{3} i^{1/3}) & \gamma_2\tau << 1 \quad (25b) \end{cases}$$

The maximum size is given by the following formula, which is practically equivalent with LINDBLAD's [1]) results:

$$\boxed{i_{max} = (3\tau)^3 = (3\sigma\rho' v' t)^3}\ . \qquad (26)$$

Using the values $\rho' = 10^{-4}$ cm^{-3} (approximate density of O atoms [2])), $t = 3.10^{16}$ sec, $\sigma = 10^{-15}$ cm^2, we get $i_{max} = 10^9$, which does not contradict the observed extinction law (cf. STRUVE [3])), since it corresponds to a particle size of about 1000 atomic diameters, i.e. of about 2000 Å.U.

The total smoke density now becomes:

$$\rho_{sm} = \sum_{i=2}^{i_{max}} N_i\, i\, m,$$

or

$$\rho_{sm} \cong \frac{3}{4} m\ \gamma_1\ N_1\ i_{max}^{4/3} \qquad \gamma_2\tau >> 1 \qquad (27a)$$

$$\rho_{sm} \cong \frac{1}{20} m\,\gamma_1\gamma_2\,N_1\ i_{max}^{5/3} \qquad \gamma_2\tau << 1 \qquad (27b)$$

For the density of the diatomic molecules we get from formulae (25):

$$N_2 \cong \frac{\gamma_1}{\gamma_2} N_1 \qquad \gamma_2\tau >> 1 \qquad (28a)$$

$$N_2 \cong \gamma_1 N_1 \tau \qquad \gamma_2\tau << 1 \qquad (28b)$$

These simple formulae involve the assumption that the smoke formation may be considered as due to the agglomeration of heavy atoms, all of one kind, and that this simplified picture holds even for the first steps in the process, where diatomic and triatomic molecules are formed.

If we wish on this basis to explain the observed smoke density of 10^{-26} g cm^{-3} [4]) we find from (27) assuming for the density of the heavy atoms $N_1 \cong 10^{-4}$ (corresponding to $\tau \cong 10^3$):

$$\gamma_1 \cong 10^{-12},\ \gamma_2 >> 10^{-3} \qquad (29a)$$

or

$$\gamma_1\gamma_2 \cong 10^{-14},\ \gamma_2 << 10^{-3}, \text{say}\ \gamma_1 \cong 10^{-10}, \gamma_2 \cong 10^{-4}. \qquad (29b)$$

These values of γ_1 would both be theoretically allowable (see section 5). An estimate of γ_2 has not been performed, but $\gamma_2 \cong 10^{-2}$ or larger seems not very likely.

For the density of the diatomic molecules we would get:

$$N_2 << 10^{-13} \qquad (30a)$$

$$\text{or}\ N_2 \cong 10^{-11}. \qquad (30b)$$

In both cases this density would be so small, as to be practically unobservable.

Our conclusions would substantially be the same, if we had chosen N_1 to be larger by a factor 10, but this

1) E. RUTHERFORD, *Radioactivity*, Cambridge 1905, 331.
2) OORT, l.c.

1) L.c.
2) OORT, l.c.
3) O. STRUVE, *Ann. d'Ap.* 1, 143, 1938.
4) SCHÖNBERG, l.c.

would make the linear dimensions of the maximum size of the particles also 10 times larger.

An important question is how these results can be reconciled with the observed presence of the diatomic molecules CH, CH^+ and CN in interstellar space, with densities of the order 10^{-6}. Of all the hydrides of the atoms in the beginning of the periodic system, the probability of the formation of CH and CH^+ by radiation capture outweighs, according to theory, by far that of the formation of the others like NH, OH, NaH (see section 5). This is of course a satisfactory result and it seems not improbable that the first step in the coagulation process is precisely the formation of CH by radiation capture. For the corresponding γ_1 the theoretical estimate gives approximately 10^{-10}. The next step would be either the mechanical capture of a second H atom or that of some heavy atom, the former process being more probable because of the larger H concentration. The corresponding γ_2 is very difficult to estimate theoretically, but from the observed CH density one would find the approximate value of 10^{-7}. Indeed, for N_{CH} we have:

$$\frac{dN_{CH}}{dt} = \gamma_1 N_C \sigma N_H v - \gamma_2 N_{CH} \sigma N_H v, \quad \text{giving}$$

$$N_{CH} = \frac{\gamma_1}{\gamma_2} N_C \left(1 - e^{-\gamma_2 \sigma N_H v_H t}\right)$$

$$= \frac{\gamma_1}{\gamma_2} N_C \left(1 - e^{-1\cdot 35 . 10^8 \gamma_2}\right).$$

Taking DUNHAM's [1]) values: $N_{CH} \cong 2.10^{-6}$, $N_C \cong 3.10^{-3}$, we find that the exponential term can practically be neglected and that $\gamma_2 \cong 1\cdot5.10^{-7}$. (All data considered this value is rather uncertain.)[2])

If C → CH → CH_2 actually should be the initial steps in the smoke formation our rough interpretation of the observed smoke density will no longer hold. First of all we must consider more closely the subsequent steps. It is clear that these will certainly not consist in the capture of H atoms only, since these will evaporate again. Heavy atoms must necessarily now be added; a certain amount of hydrogen atoms may be added too, but their number will at its highest be of the same order as the number of other atoms[3]). In a detailed theory one might try to take the capture and evaporation of hydrogen into account, but it seems safe to use the simplified picture that after CH_2 (or perhaps after CH_3 or CH_4; for simplicity we will, however, start directly after CH_2) only atoms are captured whose density is of the same order as that of carbon.

We can now use again the differential equations (21), but with this difference that the values of δ_1^{-1} and δ_2^{-1} are no longer equal to γ_1 and $2^{2/3}\gamma_2$, but larger by a factor N_H/N_C, for which we will take 3.10^{-4}, corresponding to $\rho' \cong 10^{-4}$ used in (26); the value of τ, however, remains the same (viz. $\tau \cong 10^3$). If now $\delta_3^{-1}\tau$, $\delta_4^{-1}\tau$, etc. are all large compared with unity, we would from formula (27) using $\gamma_1 \cong 10^{-10}$, clearly get a smoke density which is too large by a factor 10^6.

This difficulty can be overcome if we remember that the first steps of the capture of heavy atoms by the CH_2 molecule need by no means be easy, so that even if $\delta_2^{-1}\tau$ is large compared with 1, the quantities $\delta_3^{-1}\tau$, $\delta_4^{-1}\tau$, etc. may eventually be small compared with 1. This means that for instance the probability for the capture of a C atom by CH_2 or CH_3 or C_2H_2 during a collision is still small compared with 10^{-3}, as may well be the case. It is clear that this will have a diminishing effect on the final smoke density. With $\delta_3^{-1}\tau$ and $\delta_4^{-1}\tau << 1$, $\delta_5^{-1}\tau$, $\delta_6^{-1}\tau$, etc. $>> 1$, we get from formula (22) for the total smoke density:

$$\rho_{sm} = \left(3^5\, m\, N_1 \tau^3 \frac{\gamma_1 \tau}{4}\right)\left(\frac{\gamma_3 3^{2/3}\tau}{5}\right)\left(\frac{\gamma_4 4^{2/3}\tau}{6}\right). \quad (31)$$

The factor between the first parentheses is the same as the expression for ρ_{sm} in formula (27a). The factors $\gamma_3\tau$, $\gamma_4\tau$ and perhaps one or two more, may easily account for the required factor 10^{-6}.

As long as we are not better informed about the capture probabilities γ_3, γ_4, etc., it will be difficult to predict a precise value for ρ_{sm}, but, anyhow, there is as yet no particular reason to fear that the theoretical picture is in contradiction with the observations.

Some remarks may still be made about CN. This molecule is rather easily formed by radiation capture. The γ_1 corresponding to this process should be smaller than in the case of CH, perhaps by a factor 10 or so. The total number of CN molecules formed in the course of 10^9 years from C and N in the interstellar gas becomes, however, many powers of 10 smaller than the observed density ($\sim 10^{-6}$) and their formation by direct capture seems therefore to play an insignificant part. Its presence in space should perhaps be ascribed to evaporation from particles originally built up along the CH line. If this is the case, other diatomic molecules should be expected likewise.

I wish to express my thanks to Prof. Dr H. A. KRAMERS for his manifold advice and constant assistance and to Prof. Dr J. H. OORT for much helpful criticism [1]).

[1]) TH. DUNHAM Jr., *P.A.A.S.* **10**, 123, 1941.

[2]) Prof. OORT has pointed out that the probability of photo-dissociation of the CH molecules by star light might be so large that it outweighs by far their formation by radiation capture. Calculation shows that this difficulty will arise if the frequency of the corresponding absorption transitions is well below 10 eV. From the data at hand it seems hardly probable that the frequency would be so low.

[3]) The fact that H and He atoms can only partake to a limited extent in the building up of interstellar particles has first been pointed out by Mr VAN DE HULST (cf. OORT, l.c.).

[1]) This work has been carried out in the "Instituut voor Theoretische Natuurkunde", University of Leiden.