BULLETIN OF THE ASTRONOMICAL INSTITUTES OF THE NETHERLANDS.

1937 October 30. Volume VIII. No. 303.

COMMUNICATION FROM THE OBSERVATORY AT LEIDEN.

On the critical values of the amplitudes required to maintain the pulsation in Cepheid-variation, by *J. Woltjer Jr.*

Introduction. Consider a star oscillating with infinitesimal amplitude; suppose the damping constant corresponding to this state of vibration to be negative. The amplitude will increase with the time and the forced oscillations induced in the remaining degrees of freedom will complicate the motion. However, only intensified by resonance, these forced oscillations seem to be of importance in actual Cepheids[1]). Hence, the consideration is restricted to that one with period approximately commensurable in the ratio 1 : 2 with the original period. This forced oscillation dissipates energy nearly in the same way as the corresponding free oscillation in this degree of freedom would do; suppose its damping constant to be positive. If the commensurability is not too close, the resulting contribution to the dissipation of energy of the pulsation of the star is proportional to the fourth power of the amplitude; as the oscillation originally present dissipates proportionally to the square of the amplitude and the partial dissipations considered have opposite signs, a balance appears possible at a critical value of the amplitude; then, the total dissipation of energy of the pulsation is zero [2]). If the commensurability is too close, a more detailed investigation is necessary.

However, things may become more complicated; if the interaction between the motions in the two degrees of freedom is considered, a reaction of the original oscillation on a hypothetical free oscillation in the degree of freedom that till now was only in forced oscillation appears possible in such a way as to make this free oscillation overstable. If the corresponding critical value of the amplitude is larger than the critical value considered above, the free oscillation will not be excited; however, if this value is smaller, then the star, while increasing its amplitude to reach a dissipation of energy of the motion equal to zero, is likely to excite this free oscillation. A balance seems possible; however, the result is not a star pulsating with one period, but with two nearly commensurable periods, qualitatively resembling RR Lyrae; quantitatively however, the required amplitude of the secondary oscillation is far too large.

1) Cf. Miss H. A. Kluyver's investigation of the second order terms, *B.A.N.* Nos. 268 and 276.

2) *Nature*, **140**, p. 195.

1. *The equations of motion.* The solutions s_i of the differential equation that determines the first order terms in the radius vector in adiabatic star pulsations constitute a system of functions that satisfy the relation of orthogonality:

$$\int r_n^4 \rho_n \, s_i s_j \, dr_n = 0 \quad i \neq j;$$

r denotes the radius vector, ρ the density; the index n denotes the "normal" state of the star. The relative displacement, a function of r_n and the time t, quite generally may be developed in the series

$$\frac{r - r_n}{r_n} = \sum_1^\infty C_i s_i;$$

the coefficients C are functions of t only; the functions s_i are adjusted so as to satisfy the relation:

$$\int r_n^4 \rho_n \, s_i^2 \, dr_n = 1.$$

The equations of motion of the variables C may be reduced to the canonical form if only adiabatic changes of state are considered [1]); if this restriction is not made, it is necessary to consider the possibility of this reduction more closely.

The internal energy of the star is equal to

$$\int U \, dM_r;$$

U is the internal energy (thermal and radiation energy) per unit mass, M_r is the mass contained within the sphere of radius r. U is a function of the density ρ and the entropy η. The density ρ is a function of r_n and the variables C, as is evident from the equation of continuity; the entropy η will be

1) Miss H. A. Kluyver, *B.A.N.* No. 276.

considered to be a function of r_n and the time t explicitly. Hence, the integral is a function of the variables C and the time t. As

$$dU = \frac{P}{\rho^2}\, d\rho + T d\eta,$$

(P denoting the total pressure, T the temperature), the derivative with respect to one of the variables C of the function U, which depends on the variables C_i, r_n, t, is equal to:

$$\frac{\partial U}{\partial C_i} = \frac{P}{\rho^2}\frac{\partial \rho}{\partial C_i}.$$

Hence:

$$\frac{\partial}{\partial C_i}\int U\, dM_r = -\int P \frac{\partial \frac{1}{\rho}}{\partial C_i}\, dM_r.$$

As

$$\rho r^2\, dr = \rho_n r_n^2\, dr_n,$$

$$\frac{\partial \frac{1}{\rho}}{\partial C_i} = \frac{1}{3}\frac{1}{\rho_n}\frac{1}{r_n^2}\frac{\partial^2 r^3}{\partial C_i \partial r_n}.$$

Then, the last integral may be transformed as follows:

$$\int P \frac{\partial \frac{1}{\rho}}{\partial C_i} dM_r = 4\pi \int P r_n^2 \rho_n \frac{\partial \frac{1}{\rho}}{\partial C_i} dr_n = \frac{4\pi}{3}\int P \frac{\partial^2 r^3}{\partial C_i \partial r_n}\, dr_n = -\frac{4\pi}{3}\int \frac{\partial r^3}{\partial C_i}\, dP = -4\pi \int r^2\, r_n\, s_i\, dP = -4\pi \int \rho_n \frac{r_n^3}{\rho}\, s_i \frac{\partial P}{\partial r}\, dr_n \text{ [1]}.$$

The gravitational energy is equal to

$$-\int f \frac{M_r}{r}\, dM_r;$$

f denotes the constant of gravitation. Hence:

$$-\frac{\partial}{\partial C_i}\int f\frac{M_r}{r}\, dM_r = \int f \frac{M_r}{r^2}\, r_n\, s_i\, dM_r = 4\pi \int g r_n^3 \rho_n s_i\, dr_n;$$

g is the acceleration of gravity. The derivative of the sum of internal and gravitational energy is equal to

$$\frac{\partial}{\partial C_i}\left\{\int U\, dM_r - \int f\frac{M_r}{r}\, dM_r\right\} = 4\pi \int r_n^4\, \rho_n\, s_i \left\{\frac{1}{r_n}\left(\frac{1}{\rho}\frac{\partial P}{\partial r} + g\right)\right\} dr_n.$$

The right-hand member may be reduced by means of the hydrodynamical equation to the form:

$$-4\pi \int r_n^4\, \rho_n\, s_i \frac{d^2}{dt^2}\left(\frac{r - r_n}{r_n}\right) dr_n = -4\pi\, \frac{d^2 C_i}{dt^2}.$$

If the sum of kinetic, thermal and gravitational energy be denoted by $4\pi H$, then

$$H = \frac{1}{2}\sum_{i=1}^{\infty}\left(\frac{dC_i}{dt}\right)^2 + \frac{1}{4\pi}\int U\, dM_r - \frac{1}{4\pi}\int f\frac{M_r}{r}\, dM_r,$$

and the differential equations of motion of the variables C_i are:

$$\frac{dC_i}{dt} = \frac{\partial H}{\partial \dot{C}_i} \qquad \frac{d\dot{C}_i}{dt} = -\frac{\partial H}{\partial C_i},$$

equations of the same form as derived by Miss H. A. Kluyver, now however proved to be valid also if the restriction to adiabatic changes of state is removed. In the case considered by Miss Kluyver, H is a function of the variables C_i, $\dot{C}_i$ only; here, H is a function of the variables C_i, $\dot{C}_i$ *and* the time t explicitly. Hence it is evident that these equations do not *determine* the non-adiabatic motion: the functional relation between the entropy η and the variables r_n, t, here supposed a given relation, must be derived from another equation.

These equations generally now do not possess the integral H = constant; only the relation

$$\frac{dH}{dt} = \frac{\partial H}{\partial t}$$

may be derived. As $\frac{\partial U}{\partial \eta} = T$, this relation may be expressed so:

$$\frac{dH}{dt} = \frac{1}{4\pi}\int T\frac{d\eta}{dt}\, dM_r;$$

in this form it is related to the equation used by Eddington in determining the dissipation of energy[2].

2. *The particular solution.* Consideration of adiabatic motion in case of commensurability 1 : 2 is effected [3]) by transformation to the canonical variables J_i, w_i by means of the equations:

$$C_i = \sqrt{\frac{2J_i}{n_i}}\cos w_i;\ \dot{C}_i = -\sqrt{2 J_i n_i}\sin w_i,\ i = 1, \ldots \infty$$

and restriction of H to the form:

$$H = n_1 J_1 + n_2 J_2 + k J_1 \sqrt{J_2}\cos(2w_1 - w_2);$$

the quantities n_i ($i = 1, \ldots \infty$) and k are constants, n_i is always positive.

The resulting equations of motion,

$$\frac{dJ_i}{dt} = -\frac{\partial H}{\partial w_i}, \quad \frac{dw_i}{dt} = \frac{\partial H}{\partial J_i}$$

(also valid, if no restriction is imposed on the function H), now may be treated according to the methods of celestial mechanics.

Though this restricted problem may be solved generally, it is more convenient to start from the solution:

[1]) The independent variable r must be distinguished from the variable r that is a function of r_n and t.

[2]) A. S. Eddington, *The internal constitution of the stars*, p. 198.

[3]) Cf. *M.N.* 95^3 p. 260 and Miss Kluyver's treatment in *B.A.N.* No. 276.

J_1, J_2 constants; w_1, w_2 linear functions of the time.

The values of w_1 and w_2 are connected by the relation

$$2\, w_1 - w_2 = \theta_0,$$

θ_0 being equal either to 0 or to π. The constant values of J_1, J_2 must satisfy the relation:

$$2\,\frac{\partial H}{\partial J_1} - \frac{\partial H}{\partial J_2} \equiv 2n_1 - n_2 + k\cos\theta_0 \left\{ 2\sqrt{J_2} - \frac{J_1}{2\sqrt{J_2}} \right\} = 0.$$

The value of J_1 is arbitrary; the corresponding value of J_2 is determined by the relation:

$$\sqrt{J_2} = \frac{\dfrac{J_1}{4}}{\sqrt{\dfrac{J_1}{4} + \left(\dfrac{2\, n_1 - n_2}{4\, k \cos\theta_0}\right)^2} + \dfrac{2\, n_1 - n_2}{4\, k\cos\theta_0}}.$$

If the commensurability is not very close, the right-hand member may be developed in a power series with argument J_1. Then the proportionality of J_2 to the square of J_1 and the influence of the small divisor $2\, n_1 - n_2$ are apparent. However, as this restriction must be avoided here, the relation must be used as it stands.

The general solution may be approximated to by considering the variations with regard to this particular solution, the terms of the first degree only being taken into account. These variations satisfy the relations:

$$\frac{d\,\delta J_1}{dt} = -2\frac{\partial^2 H}{\partial\theta^2}\delta\theta,\quad \frac{d\,\delta J_2}{dt} = \frac{\partial^2 H}{\partial\theta^2}\delta\theta,\quad \frac{d\,\delta w_1}{dt} = \frac{\partial^2 H}{\partial J_1^2}\delta J_1 + \frac{\partial^2 H}{\partial J_1 \partial J_2}\delta J_2,\quad \frac{d\,\delta w_2}{dt} = \frac{\partial^2 H}{\partial J_1\partial J_2}\delta J_1 + \frac{\partial^2 H}{\partial J_2^2}\delta J_2,$$

$$2\, w_1 - w_2 \equiv \theta = \theta_0 + \delta\theta.$$

Hence:

$$\frac{d\,\delta\theta}{dt} = \left(2\frac{\partial^2 H}{\partial J_1^2} - \frac{\partial^2 H}{\partial J_1\partial J_2}\right)\delta J_1 + \left(2\frac{\partial^2 H}{\partial J_1\partial J_2} - \frac{\partial^2 H}{\partial J_2^2}\right)\delta J_2,\qquad \frac{d^2\delta\theta}{dt^2} = \left(-4\frac{\partial^2 H}{\partial J_1^2} + 4\frac{\partial^2 H}{\partial J_1\partial J_2} - \frac{\partial^2 H}{\partial J_2^2}\right)\frac{\partial^2 H}{\partial\theta^2}\delta\theta.$$

The solution of this equation is: $\delta\theta = \varkappa\sin\tau$, $\tau = \nu t + \text{const.}$, ν being determined from the equation:

$$\nu^2 = \left(4\frac{\partial^2 H}{\partial J_1^2} - 4\frac{\partial^2 H}{\partial J_1\partial J_2} + \frac{\partial^2 H}{\partial J_2^2}\right)\frac{\partial^2 H}{\partial\theta^2} = 2\, J_1\left(1 + \frac{J_1}{8J_2}\right)k^2;$$

$\varkappa$ is a constant of integration. The sign of ν is fixed by the convention: $\dfrac{\nu}{k\cos\theta_0} > 0$.

The values of the variables may now be derived by quadrature:

$$\delta J_1 = 2\frac{\partial^2 H}{\partial\theta^2}\frac{\varkappa\cos\tau}{\nu},\qquad \delta J_2 = -\frac{\partial^2 H}{\partial\theta^2}\frac{\varkappa\cos\tau}{\nu},$$

$$\delta w_1 = \left(2\frac{\partial^2 H}{\partial J_1^2} - \frac{\partial^2 H}{\partial J_1\partial J_2}\right)\frac{\partial^2 H}{\partial\theta^2}\frac{\varkappa\sin\tau}{\nu^2},\qquad \delta w_2 = \left(2\frac{\partial^2 H}{\partial J_1\partial J_2} - \frac{\partial^2 H}{\partial J_2^2}\right)\frac{\partial^2 H}{\partial\theta^2}\frac{\varkappa\sin\tau}{\nu^2}.$$

The resulting values of δC_1, δC_2 are:

$$\delta C_1 = -\varkappa\sqrt{\frac{2J_1}{n_1}\frac{J_2}{J_1}}\,\frac{1}{1 + 8\dfrac{J_2}{J_1}}\left\{\left(-1 + \sqrt{1 + 8\frac{J_2}{J_1}}\right)\cos(\omega + \tau) + \left(1 + \sqrt{1 + 8\frac{J_2}{J_1}}\right)\cos(\omega - \tau)\right\}$$

$$\delta C_2 = \frac{\varkappa}{2}\sqrt{\frac{2J_2}{n_2}}\,\frac{\cos\theta_0}{1 + 8\dfrac{J_2}{J_1}}\left\{\left(-1 - 4\frac{J_2}{J_1} + \sqrt{1 + 8\frac{J_2}{J_1}}\right)\cos(2\omega + \tau) + \left(1 + 4\frac{J_2}{J_1} + \sqrt{1 + 8\frac{J_2}{J_1}}\right)\cos(2\omega - \tau)\right\};$$

ω is the linear function of the time that is equal to w_1 in the solution with $\varkappa = 0$.

The quantities J_1, J_2 are constants of integration connected by the quadratic equation already mentioned; hence, only J_1 is an arbitrary constant.

If the solution of the differential equations with the restricted function H had been effected generally, the variables might have been expanded in the series:

$$\theta = \sum_1^\infty \theta^{(i)}\sin i\tau \qquad w_1 = \omega + \sum_1^\infty w_1^{(i)}\sin i\tau$$

$$J_2 = \sum_0^\infty J_2^{(i)}\cos i\tau \qquad J_1 = \sum_0^\infty J_1^{(i)}\cos i\tau;$$

ω and τ being two linear functions of the time and the coefficients depending on two constants of integration. Then, a canonical Delaunay-transformation might have been constructed [1]) from the set $\left(\frac{J_1\, w_1}{J_2\, w_2}\right)$ to the set $\left(\frac{I_\omega\, \omega}{I_\tau\, \tau}\right)$, the quantities I_ω, I_τ being functions of the arbitrary constants involved in the coefficients of the goniometric series, defined by the relations:

$$I_\omega = J_1^{(0)} + 2\, J_2^{(0)},$$

$$I_\tau = -\frac{1}{2} \sum_{1}^{\infty} i\, J_2^{(i)}\, \theta^{(i)}.$$

In the solution developed in this section only first-order terms in $\varkappa$ have been considered. To the corresponding degree of approximation, the result may be stated that the variables canonically conjugate to the arguments ω and τ are:

$$I_\omega = J_1 + 2\, J_2,$$

$$I_\tau = \frac{1}{2} \frac{\varkappa^2}{\nu} \left(\frac{\partial^2 H}{\partial \theta^2}\right)_0 = -\frac{\varkappa^2 J_2}{\sqrt{1 + 8\frac{J_2}{J_1}}}.$$

It is to be remembered that in the right-hand members J_1, J_2 denote the constants of integration connected by the quadratic equation.

3. *The non-adiabatic perturbation.* The function H considered in section 1 depends explicitly on the time t only in as far as the entropy η is a function of t. Hence, if the adiabatic problem has been solved, the complete solution may be approximated to by adding to the value η_n, that is only a function of r_n, used in this solution, the variation $\delta\eta$, function of r_n and t, first-order terms in this quantity only being taken account of. The resulting addition to the function H is equal to

$$\frac{1}{4\pi} \int \frac{\partial U}{\partial \eta}\, \delta\eta\; dM_r.$$

As $\frac{\partial U}{\partial \eta} = T$, the addition is equal to:

$$\frac{1}{4\pi} \int T\, \delta\eta\; dM_r.$$

If only the terms to the second degree in C_i inclusive are taken into account in the adiabatic motion, each of the principal modes of vibration oscillates independently and the solution has the simple form:

J_i is constant, w_i is a linear function of the time with period $\frac{2\pi}{n_i}$.

Then, the addition of the non-adiabatic terms induces a variation of the quantity J_i determined by the equation:

$$\frac{dJ_i}{dt} = -\frac{1}{4\pi} \int \frac{\partial T}{\partial w_i}\, \delta\eta\; dM_r.$$

The secular part of the right-hand member only results from the combination of a term in T with the corresponding terms in $\delta\eta$ of the same period. Hence, this right-hand member consists of a sum of terms, each referring to one mode of vibration and proportional to the square of the amplitude a_i of the vibration. As $a_i^2 = \frac{2J_i}{n_i}$, the equation, as far as regards the secular terms, breaks up into an infinity of equations of the form:

$\frac{dJ_i}{dt}$ is equal to J_i multiplied by a constant.

These separate equations determine the secular variations of J_i; the multiplying constants may be denoted by $-2\alpha_i$, α_i being the damping constant of the mode of vibration with index i.

If the adiabatic equations are solved by restriction of the function H as in section 2, the problem of actual Cepheid-variation seems closely to be approximated to and the secular variations of the amplitudes may be determined. The solution considered consists in three terms in the variable C_1, all of nearly the period belonging to the first mode of vibration, and three terms in the variable C_2, all of nearly the period belonging to the second mode of vibration. By transformation to the canonical variables τ, ω, I_τ, I_ω the problem is reduced to the equations:

$$\frac{dI_\omega}{dt} = -\frac{1}{4\pi} \int \frac{\partial T}{\partial \omega}\, \delta\eta\; dM_r,$$

$$\frac{dI_\tau}{dt} = -\frac{1}{4\pi} \int \frac{\partial T}{\partial \tau}\, \delta\eta\; dM_r.$$

The secular terms in the right-hand members consist of sums of terms, each referring to one mode of vibration and one periodic term only, proportional to the squares of the amplitudes of these periodic terms. The factors of proportionality are closely related to the damping constants; however, the fact that T is differentiated not with regard to the argument of each periodic term, but with regard to a constituent part of this argument, introduces some complications. As demonstrated above, if the differentiation had to be performed with regard to the argument of the periodic term, then the factor would be equal to $-\alpha_i\, n_i$. Now however, if the argument has the form $m_1\, \omega + m_2\, \tau$ (m_1, m_2 integers) and, expressed in t, it has the coefficient of t approximately equal to n_i, then in the derivative $\frac{\partial T}{\partial \omega}$ the extra factor m_1 is introduced, in $\frac{\partial T}{\partial \tau}$ the factor m_2. As the amplitudes have been derived in the preceding section, the equations that determine

[1]) Cf. *M.N.* 81⁹ p. 603.

the secular variations of I_ω, I_τ may be written down at once:

$$\frac{dI_\omega}{dt} = -2\alpha_1 J_1 - 4\alpha_2 J_2 - \frac{\varkappa^2 J_2}{\left(1+8\frac{J_2}{J_1}\right)^{\frac{3}{2}}}\left\{8\alpha_1\frac{J_2}{J_1}\left(1+4\frac{J_2}{J_1}\right)+2\alpha_2\left[\left(1+8\frac{J_2}{J_1}\right)+\left(1+4\frac{J_2}{J_1}\right)^2\right]\right\},$$

$$\frac{dI_\tau}{dt} = \frac{\varkappa^2 J_2}{\left(1+8\frac{J_2}{J_1}\right)^{\frac{3}{2}}}\left\{8\alpha_1\frac{J_2}{J_1}+2\alpha_2\left(1+4\frac{J_2}{J_1}\right)\right\}.$$

As I_ω, I_τ are functions of J_1, $\varkappa$, the right-hand members are functions of I_ω, I_τ.

4. *The stationary values of the amplitudes.* First, consider the case $\varkappa = 0$. Then the second equation is satisfied and the first equation may be written:

$$\frac{dI_\omega}{dt} = -\frac{2\alpha_1 J_1 + 4\alpha_2 J_2}{J_1 + 2J_2} I_\omega.$$

Hence the damping constant is the (variable) mean of α_1, α_2 with weight factors approximately proportional to the energies of the components of the motion [1]). A stationary state requires the value of J_1 to satisfy the equation

$$\frac{J_2}{J_1} = -\frac{1}{2}\frac{\alpha_1}{\alpha_2} \text{ (ω-limit).}$$

As each damping constant consists of terms corresponding to positive dissipation of energy and negative dissipation derived from the energy sources, it is easily conceivable that α_1 is negative and α_2 positive in actual Cepheids. Then, if $\varkappa$ remains zero the star must adjust the amplitude of its pulsation so as to satisfy this relation.

Secondly, consider the case $\varkappa \neq 0$. Then I_τ is constant if

$$8\alpha_1\frac{J_2}{J_1} + 2\alpha_2\left(1+4\frac{J_2}{J_1}\right) = 0,$$

$$\frac{J_2}{J_1} = -\frac{1}{4}\frac{\alpha_2}{\alpha_1+\alpha_2} \text{ (τ-limit).}$$

If this limit is larger than the ω-limit, then, if the star is working up to the ω-limit, $\frac{dI_\tau}{dt}$ will have a positive value (α_2 being again positive); hence $\frac{1}{I_\tau}\frac{dI_\tau}{dt}$ is negative and the τ-oscillation will not be excited. However, if the τ-limit is smaller, the τ-oscillation becomes overstable if this limit is reached and the τ-oscillation generally must be excited.

Then a stationary state requires the value of $\frac{dI_\omega}{dt}$ to be zero if $\frac{J_2}{J_1}$ is equal to the τ-limit. This condition requires $\varkappa^2$ to have a determinate value, that may be expressed in terms of the τ-limit $\frac{J_2}{J_1}$ by the equation

$$\varkappa^2 = \frac{1}{4}\left(1+8\frac{J_2}{J_1}\right)\frac{1+4\frac{J_2}{J_1}-8\left(\frac{J_2}{J_1}\right)^2}{\left(\frac{J_2}{J_1}\right)^2}\left(\frac{J_2}{J_1} = \tau\text{-limit}\right).$$

If the right-hand member is to be positive, $\frac{J_2}{J_1}$ must satisfy the inequality

$$\tau\text{-limit} \leq \frac{1+\sqrt{3}}{4}.$$

This condition is equivalent to the condition

$$\tau\text{-limit} \leq \omega\text{-limit}.$$

Expressed in terms of the ratio $\frac{\alpha_1}{\alpha_2}$ the condition is

$$-\frac{\alpha_1}{\alpha_2} \geq \frac{1+\sqrt{3}}{2}.$$

Hence, if $\quad -\frac{\alpha_1}{\alpha_2} > \frac{1+\sqrt{3}}{2}$

the τ-oscillation is likely to be excited and the star has a secondary period (RR Lyrae).

If $\quad -\frac{\alpha_1}{\alpha_2} < \frac{1+\sqrt{3}}{2}$

the τ-oscillation cannot be maintained and the star is strictly periodic.

The requirement of a definite value of $\frac{J_2}{J_1}$ generally determines J_1 by the quadratic equation connecting the two constants of integration:

$$2n_1 - n_2 + k\cos\theta_0\left\{2\sqrt{J_2} - \frac{J_1}{2\sqrt{J_2}}\right\} = 0.$$

For, if the ratio $\sqrt{\frac{J_2}{J_1}} = \lambda$ is to be realised, this equation affords the value of J_1:

$$\sqrt{J_1} = \frac{2n_1 - n_2}{k\cos\theta_0}\frac{2\lambda}{1-4\lambda^2}.$$

The fact that $2n_1 - n_2$ is small compared with

1) Cf. *Nature* l.c.

n_1 generally leads to a small value of the amplitude. From the relation it is evident that $\lambda = \frac{1}{2}$ is the maximum value attainable if only that solution is considered in which J_2 becomes zero together with J_1. The other solution seems of no astronomical importance, at least at present.

5. *Concluding remarks.* The preceding developments are independent from the construction of the normal state of the star. However, if they are applied to actual stars, it is necessary to know the values of the dissipation constants α_1, α_2 and something about the properties of the functions s_1, s_2. Nothing definite can be said about these data. Hence it is necessary to proceed in the reverse direction.

The ratio $\frac{J_2}{J_1}$ is related to observational data in a simple way: $2\left(\frac{s_2}{s_1}\right)_0 \sqrt{\frac{J_2}{2J_1}}$ is approximately equal to the ratio of the amplitude of the second and the first harmonic in the radial-velocity curve, the index zero referring to values at the outer boundary of the star. If an empirical value of this ratio equal to unity is assumed, then

$$\frac{J_2}{J_1} = \frac{1}{2}\left(\frac{s_1}{s_2}\right)_0^2.$$

The behaviour of the functions s is determined by the construction of the normal state of the star; nothing definite can be said about the value of the ratio $s_1 : s_2$ at the outer boundary. As a numerical illustration suppose $\frac{J_2}{J_1}$ to be equal to $\frac{1}{8}$. If this is the ω-limit, then the τ-limit exceeds the ω-limit and the star will oscillate with one period only. However, if the value $\frac{1}{8}$ refers to the τ-limit, the τ-oscillation will be excited; then, the value of $\varkappa$ required to maintain the pulsation is about 7; apart from the consideration that the development of the solution in powers of $\varkappa$ does not admit so large a value, the fact that $\varkappa$ is about equal to the ratio of the amplitude of the free oscillation to that of the second harmonic is prohibitive. A small value of $\varkappa$ might have been found by a value of $\frac{J_2}{J_1}$ in the vicinity of $\frac{1+\sqrt{3}}{4}$; however, as stated above, this value belongs to that solution of the quadratic relation between $\sqrt{J_1}$ and $\sqrt{J_2}$, that does not become zero together with J_1.

The developments of this paper only refer to the case of approximate commensurability in the ratio 1 : 2, which seems to correspond to actual Cepheids; then, a stationary oscillation, either strictly periodic, or with a secondary relatively long period, may be arrived at by self-adjustment of the intensity of the oscillation; only a moderate amplitude of the principal oscillation appears to be required; however, if the secondary oscillation is present, the derived value of its amplitude seems inadmissibly large.