A family of triaxial mass models with central cusps

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Accepted 1996 March 11. Received 1996 February 22; in original form 1995 December 18

ABSTRACT

A simple family of triaxial mass models with a central density cusp is described, for which the projected surface density can be calculated easily. It can show ellipticity variations and isophote twists. These models are flattened versions of the popular spherical γ -models with density profile $\rho(r) \propto r^{-\gamma}(r_0 + r)^{\gamma-4}$. They are constructed by the addition of two spherical harmonic terms to the potential of the spherical model. The convolution of the surface brightness with a given point spread function is given. The intrinsic shape of the mass density, as a function of the viewing angles, can be derived from the observed ellipticities and the major axis position angles at small and large radii. This allows a convenient comparison with observations of real galaxies.

For a large range of flattenings and cusp slopes, the simple closed orbits in these potentials are accurately described by first-order epicyclic theory. This formalism is therefore well-suited for modelling, for example, the velocities of the ionized gas discs detected in the nuclei of many elliptical galaxies, the mass distributions of which are well-represented by the triaxial γ -models.

Key words: galaxies: elliptical and lenticular, cD – galaxies: kinematics and dynamics – galaxies: structure.

1 INTRODUCTION

High-resolution imaging has revealed that in all elliptical galaxies the surface brightness profile $\Sigma(R)$ rises as $R^{-\alpha}$ at small projected radii R , with $\alpha > 0$ (e.g. Lauer et al. 1991, 1992, 1995). This means that in all ellipticals the intrinsic luminosity profiles monotonically rise at small radii r as $r^{-\gamma}$, with $\gamma > 0$: there are no ‘isothermal’ cores! The value of γ is correlated with the total luminosity: low-luminosity ellipticals have steep cusps, $\gamma \gtrsim 1.3$, while high-luminosity galaxies have rather shallow cusps, $\gamma \lesssim 1.3$ (Lauer et al. 1995). Dynamical arguments based on the numerical construction of fully self-consistent triaxial models (Kuijken 1993; Schwarzschild 1993; Merritt & Fridman 1996) suggest that low-luminosity, rotation-supported elliptical galaxies are nearly oblate rotators, while the massive, pressure-supported ellipticals have a range of triaxial shapes (Merritt 1996; de Zeeuw & Carollo 1996). This conclusion is based on a limited number of construction experiments, and it is essential to explore a larger range of triaxial shapes and cusp slopes. In this paper we introduce a convenient set of mass models that can be used for this purpose. The models are triaxial generalizations of a set of well-studied spherical models in which the luminous mass

density $\rho(r) \propto r^{-\gamma}(r_0 + r)^{\gamma-4}$, where r_0 is a scalelength (Jaffe 1983; Hernquist 1990; Saha 1993; Dehnen 1993; Carollo 1993; Tremaine et al. 1994; Carollo, de Zeeuw & van der Marel 1995).

The spherical γ -models can be made triaxial by taking the expression for $\rho(r)$ and replacing r^2 by $m^2 = x^2 + y^2/p^2 + z^2/q^2$, where (x, y, z) are the usual Cartesian coordinates, and p and q are parameters (e.g. de Zeeuw & Pfenniger 1988; Merritt & Fridman 1996). This leads to models in which the density profile is that of the spherical γ -models, but with a distribution that is stratified on similar concentric ellipsoids with constant axial ratios p and q . The projected surface density is then stratified on similar concentric and aligned ellipses. Ellipticity variations and isophote twists can be incorporated by letting p and q vary with radius. A drawback of these models is that the calculation of the gravitational potential invariably leads to cumbersome numerical integrals.

Here we use a different method for flattening the γ -models. We follow a classical scheme used with success by Schwarzschild (1979), who took a spherical model with a modified Hubble profile $\rho(r) \propto (1 + r^2)^{-3/2}$ and made it triaxial by adding two low-order spherical harmonics, each multiplied by a simple radial function, to the potential (see

de Zeeuw & Merritt 1983). Hernquist & Quinn (1989) used the same method to construct triaxial models with profiles $\rho(r) \propto (1+r^2)^{-2}$. The advantage of this approach is that the potential is known explicitly, so that many calculations, including orbit integrations, can be carried out quickly. A disadvantage might be that the use of only the lowest-order spherical harmonic terms generally leads to models that become peanut-shaped for large values of the flattening. However, the addition of higher spherical harmonic terms leads to more nearly ellipsoidal shapes (e.g. Schwarzschild 1993). In this scheme, the γ -models are made triaxial by a simple choice for the radial functions $v(r)$ and $w(r)$ that multiply the two lowest-order spherical harmonics. These functions have four free parameters, which are related to the axial ratios of the triaxial density distribution at small and large radii. The resulting models have central cusps, and the projected surface density can have ellipticity variations and isophote twists.

In Section 2 we describe the intrinsic characteristics of the triaxial γ -models, and derive some properties of the simple closed orbits in the epicyclic approximation. In Section 3 we calculate the projected surface density of general models with the kind of potential described above, and then consider the specific projected properties of the γ -models. We show how the observed ellipticity of the isophotes and the position angle of the major axis – at large and at small radii – can be used to constrain the intrinsic parameters of the model, and we consider the convolution of the surface density with a point spread function (PSF). We summarize our conclusions in Section 4.

2 MASS MODEL

2.1 Potential and density

We consider triaxial potentials of the form

$$V(r, \theta, \phi) = u(r) - v(r)Y_2^0(\theta) + w(r)Y_2^2(\theta, \phi), \quad (2.1)$$

where (r, θ, ϕ) are spherical coordinates defined such that $x = r \cos \phi \sin \theta$, $y = r \sin \phi \sin \theta$, $z = r \cos \theta$, with the functions $Y_2^0 = (3/2) \cos^2 \theta - (1/2)$ and $Y_2^2 = 3 \sin^2 \theta \cos 2\phi$ the usual spherical harmonics, and $u(r)$, $v(r)$ and $w(r)$ three radial functions. We take $u(r)$ to be the potential of the spherical γ -models, defined by the choice

$$u(r) = \begin{cases} \frac{GM}{r_0} \ln \frac{r}{r+r_0}, & \text{for } \gamma = 2, \\ \frac{GM}{(2-\gamma)r_0} \left[\left(\frac{r}{r+r_0} \right)^{2-\gamma} - 1 \right], & \text{for } \gamma \neq 2, \end{cases} \quad (2.2)$$

where M is the total mass of the model, r_0 is the scalelength, and we take $0 \leq \gamma < 3$. We choose the functions $v(r)$ and $w(r)$ as follows:

$$v(r) = -\frac{GM r_1 r^{2-\gamma}}{(r+r_2)^{4-\gamma}}, \quad w(r) = -\frac{GM r_3 r^{2-\gamma}}{(r+r_4)^{4-\gamma}}, \quad (2.3)$$

where $r_1, \dots, r_4$ are constants.

The associated density distribution $\rho(r, \theta, \phi)$ follows from Poisson's equation

$$\rho(x, y, z) = f(r) - g(r)Y_2^0(\theta) + h(r)Y_2^2(\theta, \phi), \quad (2.4)$$

where

$$\begin{aligned} f(r) &= \frac{Mr_0(3-\gamma)}{4\pi} \frac{1}{r^\gamma(r+r_0)^{4-\gamma}}, \\ g(r) &= \frac{Mr_1}{4\pi} \frac{[4r^2 + 4(6-\gamma)r_2 + \gamma(5-\gamma)r_2^2]}{r^\gamma(r+r_2)^{6-\gamma}}, \\ h(r) &= \frac{Mr_3}{4\pi} \frac{[4r^2 + 4(6-\gamma)r_4 + \gamma(5-\gamma)r_4^2]}{r^\gamma(r+r_4)^{6-\gamma}}. \end{aligned} \quad (2.5)$$

The four ratios $r_1/r_0, \dots, r_4/r_0$ can be expressed in terms of the axial ratios of the density distribution at small and at large radii, where the surfaces of constant density are approximately ellipsoidal, i.e. $\rho \sim \rho(m^2)$ with $m^2 = x^2 + (y^2/p^2) + (z^2/q^2)$. When $\gamma > 0$ the density diverges $\propto r^{-\gamma}$ at small r , and we find

$$\begin{aligned} r_1 &= Ar_0, & r_2^{4-\gamma} &= \frac{A}{C} r_0^{4-\gamma}, \\ r_3 &= Br_0, & r_4^{4-\gamma} &= \frac{B}{D} r_0^{4-\gamma}, \end{aligned} \quad (2.6)$$

where

$$\begin{aligned} A &= \frac{(3-\gamma)(p_\infty^4 - 2q_\infty^4 + 1)}{4(1+p_\infty^4 + q_\infty^4)}, \\ B &= \frac{(3-\gamma)(1-p_\infty^4)}{8(1+p_\infty^4 + q_\infty^4)}, \\ C &= \frac{(3-\gamma)(p_0^4 - 2q_0^4 + 1)}{\gamma(5-\gamma)(1+p_0^4 + q_0^4)}, \\ D &= \frac{(3-\gamma)(1-p_0^4)}{2\gamma(5-\gamma)(1+p_0^4 + q_0^4)}, \end{aligned} \quad (2.7)$$

and (p_0, q_0) and (p_∞, q_∞) are the axial ratios of the surfaces of constant density at small and at large radii, respectively. When $\gamma = 0$ the central density is finite and equal to $3Mr_0/4\pi$, but the central density gradient does not vanish, as it does in models with, e.g., a modified Hubble profile. In this case we have

$$r_2^5 = \frac{A}{C} r_0^5, \quad r_4^5 = \frac{B}{D} r_0^5, \quad (2.8)$$

with A and B as in equation (2.7), but now

$$C = \frac{2p_0 - q_0 - p_0 q_0}{2(p_0 + q_0 + p_0 q_0)}, \quad D = \frac{q_0(1-p_0)}{4(p_0 + q_0 + p_0 q_0)}. \quad (2.9)$$

In all these models the density falls off as $1/r^4$ at large radii.

With appropriate choices of the constants $r_0, \dots, r_4$ our potentials can be made to correspond to approximately ellipsoidal mass models with realistic density profiles, similar to those observed in, for example, giant elliptical galaxies. In the following, we will concentrate on cases where $\gamma > 1$, so that the models have a cusp in projection.

2.2 Simple closed orbits

Settled cold/warm gas in elliptical galaxies is expected to occupy the simple closed orbits in these systems (e.g. Tohline, Simonson & Caldwell 1982). In the absence of figure rotation, the triaxial models defined in the above support two families of such closed orbits, one in the plane perpendicular to the short axis and the other in the plane perpendicular to the long axis of the model. Their properties generally have to be found by numerical orbit integration. However, for small orbital ellipticities, an epicyclic treatment suffices. The results are very useful to model the observed large-scale kinematics of cold gas (e.g. Franx, van Gorkom & de Zeeuw 1994; Pizzella et al. 1996), and the kinematics of the nuclear discs of ionized gas, detected in many elliptical galaxies (e.g. Ford et al. 1994; Harms et al. 1994; van den Bosch & van der Marel 1995). The special case $\gamma = 1$ was used previously by Quillen et al. (1992) to investigate the molecular gas motions in the dust ring of Centaurus A.

In the (x, y) -plane the gravitational potential is of the form

$$V(R, \phi) = V_0(R) + V_2(R) \cos 2\phi, \quad (2.10)$$

where

$$V_0(R) = u(R) + \frac{1}{2}v(R), \quad V_2(R) = 3w(R). \quad (2.11)$$

If the potential is close to axisymmetric, so that $|V_2(R)| \ll |V_0(R)|$, then we can use the first-order epicyclic approximation to calculate the properties of the simple closed orbits. The basic relations are given by Gerhard & Vietri (1986) and Binney & Tremaine (1987).

The sign of $v(r)$ determines whether the z -axis is the long or the short axis of the model. As a result, we can describe the two sets of stable simple closed orbits with the same epicyclic equations. Specifically, we consider two cases for the axial ratios p and q :

$$1 > p > q > 0, \quad q > 1 > p > 0. \quad (2.12)$$

In the first case the orbits in the (x, y) -plane are perpendicular to the short axis of the model, while in the latter case they are perpendicular to the long axis of the system.

To first order, the closed orbits are ellipses elongated opposite to the elongation of the mass models, and are given by

$$R = R_0(1 - \frac{1}{2}\varepsilon_R \cos 2\phi), \quad (2.13)$$

where ε_R is the orbital ellipticity $1 - R_{\min}/R_{\max}$. The components v_R and v_ϕ of the orbital velocity at the point (R, ϕ) are

$$\begin{aligned} v_R &= v_c \varepsilon_R \sin 2\phi, \\ v_\phi &= v_c(1 + \frac{1}{2}\varepsilon_v \cos 2\phi), \end{aligned} \quad (2.14)$$

where v_c is the circular velocity, given by $v_c^2(R) = RV'_0(R)$ and ε_v is the velocity contrast $1 - v_{\min}/v_{\max}$. By means of the standard epicyclic equations we find

$$\begin{aligned} \varepsilon_R &= \frac{6(4-\gamma)r_3r_4}{(R+r_4)^{5-\gamma}} \left[\frac{3R+\gamma r_0}{(R+r_0)^{4-\gamma}} \right. \\ &\quad \left. + \frac{r_1[8R^2 + (7\gamma-12)Rr_2 + \gamma(\gamma-2)r_2^2]^{-1}}{2(R+r_2)^{6-\gamma}} \right], \end{aligned} \quad (2.15)$$

$$\varepsilon_v = \varepsilon_R - \frac{2V_2}{v_c^2},$$

$$v_c^2 = GMR^{2-\gamma} \left\{ \frac{1}{(R+r_0)^{3-\gamma}} + \frac{r_1[2R + (\gamma-2)r_2]}{2(R+r_2)^{5-\gamma}} \right\}.$$

Whereas the closed orbits are circular at large radii, where the potential becomes Keplerian, they become increasingly elliptic at smaller radii, and reach a finite ellipticity in the cusp. In the limit $r/r_0 \rightarrow 0$ we have

$$\begin{aligned} \varepsilon_{R,0} &= \frac{2(3-\gamma)(4-\gamma)(1-p_0^\gamma)}{\gamma[4q_0^\gamma - (2-5\gamma+\gamma^2)(1+p_0^\gamma)]}, \\ \varepsilon_{v,0} &= \frac{4\varepsilon_{R,0}}{4-\gamma}, \end{aligned} \quad (2.16)$$

$$v_c^2 = \frac{3GM}{r_0} \frac{[4q_0^\gamma - (2-5\gamma+\gamma^2)(1+p_0^\gamma)]}{2\gamma(5-\gamma)(1+p_0^\gamma+q_0^\gamma)} \left(\frac{R}{r_0} \right)^{2-\gamma}.$$

For a given shape, the orbits reach larger central ellipticities for shallower density profiles. For models with a constant central density ($\gamma=0$), the central ε_R given above in fact diverges, consistent with the fact that in such models the orbits become y -axis oscillations at a finite radius (e.g. de Zeeuw & Merritt 1983). This provides an a posteriori check on the validity of the first-order epicyclic expressions for any given shape. A comparison with numerical orbit integrations shows that the epicyclic approximation is sufficiently accurate for orbits with values of ε_R up to ~ 0.3 . For considerably flattened models, or for shallow central density profiles, our results can therefore be used only outside a certain radius. Inside this radius one needs to use numerical orbit calculations to describe the closed orbits. Fig. 1 shows contours of constant $\varepsilon_{R,0}$ as functions of p_0 and q_0 for $\gamma=1, 1.5$ and 2 . It illustrates that, in models with axial ratios as small as 0.7 and 0.8 , the simple closed orbits can be described accurately by first-order epicyclic theory.

3 PROJECTED PROPERTIES

3.1 Projected surface density

The simple form (2.4) of the density of the models allows a straightforward calculation of the projected surface density Σ , for any viewing direction. We follow the convention of de Zeeuw & Franx (1989, hereafter ZF), and choose coordinates (x'', y'', z'') with the z'' -axis along the line of sight, and with the x'' -axis in the (x, y) -plane, so that the z -axis projects along the y'' -direction and (ϑ, ϕ) are the standard spherical

Figure 1. Limiting ellipticity of simple closed orbits as a function of the central axial ratios p_0 and q_0 of the density. Curves are drawn for three values of ε : 0.1, 0.2 and 0.3, for models with $\gamma = 1, 1.5$ and 2. Models with limiting ellipticity less than 0.1 lie in the densely hatched region; those with $\varepsilon \leq 0.2$ and 0.3, respectively, lie in the less densely hatched areas.

coordinates of the line of sight. We define (R'', Θ) as polar coordinates in the (x'', y'') -plane. It follows that Θ is the position angle measured on the sky counted counter-clockwise from the positive y'' -axis (see fig. 1 of ZF).

After transformation to (x'', y'', z'') by means of equation (3.1) of ZF, and integration over z'' , we obtain the simple result

$$\Sigma(R'', \Theta) = \Sigma_0(R'') + \Sigma_2(R'') \cos 2(\Theta - \Theta_*), \quad (3.1)$$

where the two radial functions are given by

$$\begin{aligned} \Sigma_0(R) &= 2F_1 + (1 - 3 \cos^2 \vartheta) [G_1 - \tfrac{3}{2} G_2] \\ &\quad + [6H_1 - 9H_2] \sin^2 \vartheta \cos 2\varphi, \end{aligned} \quad (3.2)$$

$$\begin{aligned} \Sigma_2^2(R) &= [6H_2 \cos \vartheta \sin 2\varphi]^2 \\ &\quad + [\tfrac{3}{2} G_2 \sin^2 \vartheta - 3H_2(1 + \cos^2 \vartheta) \cos 2\varphi]^2, \end{aligned}$$

and we have defined the integrals

$$G_1(R) = \int_R^\infty \frac{rg(r) dr}{\sqrt{r^2 - R^2}}, \quad G_2(R) = R^2 \int_R^\infty \frac{g(r) dr}{r \sqrt{r^2 - R^2}}, \quad (3.3)$$

and similarly for F_1, H_1 and H_2 in terms of the functions $f(r)$ and $h(r)$, respectively. The function $\Sigma_0(R)$ (3.1) is the azimuthally averaged surface density profile. The expressions above show that only one-dimensional integrals which are independent of the viewing angles ϑ and φ are required to find the projected surface density of a model with a density of the general form (2.4) (cf. Palmer 1994). For the specific choice (2.5) and general γ , the radial integrals (3.3) need to be evaluated numerically. They can be expressed in terms of elementary functions for integer γ (see Appendix A).

The projected surface density (3.1) has its major axis at position angle Θ_* given by

$$\tan 2\Theta_* = \frac{Th_3}{h_1 + (1 - T)h_2}, \quad (3.4)$$

where we have written

$$\begin{aligned} h_1 &= \sin^2 \varphi - \cos^2 \varphi \cos^2 \vartheta, \\ h_2 &= \cos^2 \varphi - \sin^2 \varphi \cos^2 \vartheta, \\ h_3 &= \sin 2\varphi \cos \vartheta, \end{aligned} \quad (3.5)$$

which depend only on the two viewing angles. Note that $h_1 + h_2 = \sin^2 \vartheta$. The quantity T in equation (3.4) is the *triaxiality* parameter, which is a function of radius, is independent of the viewing angles, and is given by

$$T = T(R'') = \frac{4H_2(R'')}{G_2(R'') + 2H_2(R'')}. \quad (3.6)$$

For a given model and viewing angles, equation (3.4) has two solutions, Θ_* and $\Theta_* - (\pi/2)$. The former is the position angle of the major axis and the latter of the minor axis when Θ_* satisfies the conditions

$$H_2 h_3 \sin 2\Theta_* \leq 0, \quad -\frac{\pi}{2} < \Theta_* \leq \frac{\pi}{2}. \quad (3.7)$$

When T , i.e. H_2/G_2 , is independent of radius, then Θ_* is constant, so that the projected surface density does not show twisting isophotes. For an arbitrary radial T profile, the axial ratio of the projected surface density is a function of radius, and has to be computed by numerical means.

In the case of our specific set of models (2.4), the triaxiality parameters T_0 and T_∞ at small and, respectively, large radii are related to the axial ratios of the density at these radii by

$$T_0 = \frac{1 - p_0^\gamma}{1 - q_0^\gamma}, \quad T_\infty = \frac{1 - p_\infty^4}{1 - q_\infty^4}. \quad (3.8)$$

As a result, the position angle of the apparent major axis of the isophotes is the same at large and small radii if $T_0 = T_\infty$, which via equation (2.6) translates into the requirement $r_2 = r_4$. In this case the functions $g(r)$ and $h(r)$ of equation

Figure 2. The projected surface density distribution Σ of a triaxial γ -model. The left panel shows Σ for a model with $\gamma=1.5$ and $p_0=0.9$, $q_0=0.7$, $p_\infty=0.88$, $q_\infty=0.72$, seen from the direction $\vartheta=45^\circ$, $\varphi=30^\circ$. The right panel shows the variation of ellipticity ϵ , position angle (PA) of the major axis, and the isophotal shape parameter a_4 of the same model, all as a function of radius. The small negative value of a_4 shows that the isophotes are slightly boxy.

(3.5) differ by just a constant factor, so that H_2/G_2 and hence T are indeed independent of R , and there is no isophote twist. When $\rho=\rho(m^2)$, with $m^2=x^2+y^2/p^2+z^2/q^2$, the triaxiality parameter is defined as $T=(1-p^2)/(1-q^2)$, and is independent of the radius.

Fig. 2 shows an example of the projected surface density distribution of these models, together with the radial variation of the ellipticity and the position angle of the major axis of the isophotes, and the variation of the isophotal shape parameter a_4 , for a moderately flattened model with $\gamma=1.5$. It shows a substantial amount of isophote twisting, and isophotes which are slightly boxy (Lauer 1985).

3.2 Integrated surface density

The integrated surface density inside radius R , which we denote by $S(R)$, can be written as

$$S(R) = S_0(R) + R^2 \{ [G_1(R) - G_2(R)] (\frac{1}{2} - \frac{3}{2} \cos^2 \vartheta) + 3 [H_1(R) - H_2(R)] \sin^2 \vartheta \cos 2\varphi \}, \quad (3.9)$$

where

$$S_0(R) = 4\pi \int_0^R F_1(R) R dR \\ = M - 4\pi \int_R^\infty r f(r) \sqrt{r^2 - R^2} dr \quad (3.10)$$

is the integrated surface density of the spherical limit. Whereas evaluation of the function S_0 requires integration of $RF_1(R)$ with respect to R , as expected, the similar radial integrals of the combinations $2G_1 - 3G_2$ and $2H_1 - 3H_2$ which appear in expression (3.2) for $\Sigma_0(R)$ are equal to $R^2(G_1 - G_2)$ and $R^2(H_1 - H_2)$, so that in fact no extra integration is necessary for the triaxial models. This result holds for an arbitrary density of the form (2.4). The integral for S_0

must be evaluated numerically for the choice (2.5) and arbitrary γ . The elementary results for integer γ are given in Appendix A.

For chosen viewing angles (ϑ, φ), the function Σ_0 depends only on R/r_0 . This is because the ratios of $r_1, \dots, r_4$ to r_0 can be set by choosing the axial ratios of the triaxial mass model at large and small radii (Section 2.1). Thus we can set the scalelength r_0 of the model by relating it to the effective radius R_e , defined as the projected radius containing half the total mass. This follows by solving $S(R_e) = M/2$, and can be compared with the observed effective radius of the projected surface brightness of an elliptical galaxy, which is defined in the same way. Alternatively, one can fit the profile $\Sigma_0(R)$ directly to the observed azimuthally averaged light profile. In the spherical limit, the ratio R_e/r_0 is a nearly linear function of γ [see equations (4) and (17) of Dehnen 1993]. For triaxial models, the ratio of R_e and r_0 depends on both the axial ratios and the viewing angles. At fixed γ , the variation around the spherical value is less than 5 per cent, for $0.7 \leq p, q \leq 1.3$.

3.3 Observed axial ratios at small and large radii

For $\gamma > 1$, the projected surface density diverges as $(1/R)^\gamma$ at small radii, and therefore has a cusp. In this case the observed central axial ratio $(b'/a')_0^{\gamma-1}$, defined by $\Sigma(b', \Theta_* - \frac{\pi}{2}) = \Sigma(a', \Theta_*)$, is given by

$$\left(\frac{b'}{a'} \right)_0^{\gamma-1} = \frac{d_0 - \Delta_0}{d_0 + \Delta_0}, \quad (3.11)$$

with

$$d_0 = 1 + t \sin^2 \vartheta \cos^2 \varphi + p_0^\gamma (1 + t \sin^2 \vartheta \sin^2 \varphi) + q_0^\gamma (1 + t \cos^2 \vartheta), \quad (3.12) \\ \Delta_0^2 = [(1 - q_0^\gamma)h_1 + (p_0^\gamma - q_0^\gamma)h_2]^2 + (1 - p_0^\gamma)^2 h_3^2,$$

where $t = (3 - \gamma)/(\gamma - 1)$ and h_1, h_2, h_3 are the functions of ϑ and φ defined in equation (3.5). Thus, the observed central axial ratio depends only on the axial ratios p_0 and q_0 of the three-dimensional density at small radii, which is not surprising since it dominates the observed central projected surface density.

For $\gamma = 1$ the functions F_1, G_1 and H_1 diverge logarithmically at small radii, but the functions G_2 and H_2 remain finite. In this case the projected surface density has a logarithmic cusp, with isophotes that become more nearly circular at smaller radii. This happens even when the cusp in the density ρ is flattened, and is caused by the fact that the density at larger radii now provides a non-negligible contribution to the central surface density.

When $\gamma < 1$, the central cusp in ρ is not sufficiently steep to produce a cusp in Σ , and the central surface density is finite. The axial ratio of the isophotes now depends on contributions from light at all radii, and no simple relation with the intrinsic axial ratios can be given.

At large radii the projected surface density declines as $(1/R'')^3$ for all our models. The axial ratio $(b'/a')_\infty$ is given by an expression that is identical to equation (3.12), but with $\gamma = 4$, i.e. $t = -1/3$, and p_0^γ and q_0^γ replaced by p_∞^4 and q_∞^4 , respectively.

Finally, we remark that, whereas the requirement that $\Theta_{*,0} = \Theta_{*,\infty}$ results in no isophote twisting for our specific set of models (Section 3.1), the requirement that $(b'/a')_0 = (b'/a')_\infty$ does not result in a constant ellipticity at all radii, even though models of this kind have nearly flat ellipticity profiles. The largest deviations occur around $3r_0$.

3.4 Deprojection

When $\gamma > 1$, we can derive the intrinsic axial ratios of the models for given viewing angles ϑ and φ from the observed values of the apparent axial ratios and the major axis position angles at small and large radii. Given Θ_* at a radius R'' , equation (3.4) can be solved to give T at that radius as

$$T = \frac{\sin^2 \vartheta \tan 2\Theta_*}{h_3 + h_2 \tan 2\Theta_*}. \quad (3.13)$$

So, if we measure Θ_0 and Θ_∞ , we can calculate T_0 and T_∞ . At small radii we can then solve equation (3.11) to give

$$1 - p_0^\gamma = -\frac{2\gamma\delta' \sin^2 \vartheta \sin 2\Theta_*}{2(\gamma - 1)h_3 \sin^2 \vartheta - P\delta'}, \quad (3.14)$$

where we have defined

$$\delta' = 1 - \left(\frac{b'}{a'}\right)^{\gamma-1}, \quad (3.15)$$

and

$$P = (\gamma - 1)h_3 \sin^2 \vartheta + [2 - (3 - \gamma) \sin^2 \vartheta]h_3 \cos 2\Theta_* + 2\{\cos 2\varphi + [2 \sin^2 \varphi + (\gamma - 2) \cos^2 \varphi] \sin^2 \vartheta\} \sin 2\Theta_*. \quad (3.16)$$

The value of $1 - q_0^\gamma$ then follows from the definition of T_0 (see equation 3.8). At large radii the same equations can be

used, but with $\gamma = 4$, and (p_0^γ, q_0^γ) replaced by (p_∞^4, q_∞^4) , to deduce p_∞ and q_∞ from T_∞ and $(b'/a')_\infty$.

If we have only $\Sigma(R'', \Theta)$ for a galaxy, then we will not be able in practice to deduce both Θ_0 and Θ_∞ independently, because we do not know the orientation of the galaxy with respect to north. We can then only measure $\Theta_0 - \Theta_\infty$. Additional information is needed to determine fully the intrinsic axial ratios at large and small radii. This might, for example, be provided by the kinematics of the cold/warm interstellar medium (e.g. ZF; Franx et al. 1994).

3.5 PSF convolution

For a comparison of our triaxial mass models with the observed surface brightness distribution of elliptical galaxies, the projected surface density $\Sigma(R'', \Theta)$ generally needs to be convolved with a two-dimensional PSF. This can describe either the atmospheric blurring, i.e. the seeing, or just the instrumental broadening, as for the case of *Hubble Space Telescope* data. If we denote the normalized PSF by S , then the PSF convolved surface density $\Sigma^{\text{obs}}(x'', y'') = \iint S(x'' - x, y'' - y) \Sigma(x, y) dx dy$. We consider the simple case where $S(x', y')$ is a single circular Gaussian with dispersion σ , i.e. $S(x', y') = (2\pi\sigma^2)^{-1} \exp(-R'^2/2\sigma^2)$, with $R'^2 = x'^2 + y'^2$. The results for the sum of multiple Gaussians (e.g. Tonry 1987) then follow by linear combination. Transformation to the polar sky coordinates (R'', Θ) then gives

$$\Sigma^{\text{obs}}(R'', \Theta) = \frac{1}{2\pi\sigma^2} \int_0^\infty \int_0^{2\pi} \Sigma(R, \theta) \exp\left(-\frac{R''^2 + R^2}{2\sigma^2}\right) \exp\left(\frac{R''R \cos(\Theta - \theta)}{\sigma^2}\right) R dR d\theta, \quad (3.17)$$

where we still have to substitute expression (3.1) for $\Sigma(R, \theta)$.

In the special case where Θ_* is independent of projected radius, so that there is no isophote twist (Section 3.1), the θ -integration can be carried out analytically. The result can be written as

$$\Sigma^{\text{obs}}(R'', \Theta) = \Sigma_0^{\text{obs}}(R'') + \Sigma_2^{\text{obs}}(R'') \cos 2(\Theta - \Theta_*), \quad (3.18)$$

where now the two radial functions are given by

$$\Sigma_0^{\text{obs}}(R) = \frac{1}{\sigma^2} \int_0^\infty dR R \Sigma_0(R) \exp\left(-\frac{R''^2 + R^2}{2\sigma^2}\right) I_0\left(\frac{R''R}{\sigma^2}\right),$$

$$\Sigma_2^{\text{obs}}(R) = \frac{1}{\sigma^2} \int_0^\infty dR R \Sigma_2(R) \exp\left(-\frac{R''^2 + R^2}{2\sigma^2}\right) \left[I_0\left(\frac{R''R}{\sigma^2}\right) - \frac{\sigma^2}{R''R} I_1\left(\frac{R''R}{\sigma^2}\right) \right], \quad (3.19)$$

and I_0 and I_1 are modified Bessel functions of order 0 and 1, respectively, available in the standard numerical libraries. It follows that $\Sigma^{\text{obs}}(R'', \Theta)$ has the same form as $\Sigma(R'', \Theta)$, but the radial functions $\Sigma_0(R'')$ and $\Sigma_2(R'')$ are replaced by the convolutions (3.19). The constant position angle of the major axis of the isophotes is not influenced by a circular

PSF, but the axial ratio is, as is physically evident. In the general case, where $\Theta_* = \Theta_*(R'')$, the double integral (3.17) must be evaluated numerically, and now both the position angle of the major axis and the axial ratio of the isophotes are modified by the PSF.

For a given model of the form (2.4), the radial integrals F_1 , G_1 , G_2 , H_1 and H_2 required for the evaluation of $\Sigma_0(R)$ and $\Sigma_2(R)$ are independent of the chosen viewing angles ϑ and φ (Section 3.1). Thus, each of these radial functions (as well as the products required for evaluation of Σ_2) can be convolved as in equations (3.17) or (3.19), and then $\Sigma_0^{\text{obs}}(R'', \Theta)$ follows by linear combination, with the coefficients a function of ϑ and φ . For a given density profile, the integrals (3.17) or (3.19) therefore need to be done only once for every radius.

Bailey & Sparks (1983) derived the relation between $\Sigma_0(R)$ and $\Sigma_0^{\text{obs}}(R)$ for a spherical galaxy and a Gaussian PSF, and they showed that the resulting integral can be evaluated explicitly for pure power-law density profiles, although this leads to hypergeometric functions. The above shows that a similar reduction to one-dimensional integrals is possible also for our flattened models with no isophote twist. We have shown this explicitly for the lowest order harmonic terms, but similar relations can be derived for the higher harmonics. The power-law case is again easy to evaluate. When the PSF $\sigma \ll r_0$, this may be adequate for a description of flattened, PSF-convolved cusps.

When fitting a PSF-convolved projected triaxial γ -model to the observed surface brightness of an elliptical galaxy, one can choose the two viewing angles θ and ϕ , and then use the four intrinsic axial ratios as parameters to optimize the fit. Initial values of these parameters can be found from the unconvolved deprojection discussed in Section 3.4.

4 SUMMARY AND CONCLUSIONS

We have shown that a simple deformation of the potential of the spherical γ -models, by means of the addition of the two lowest-order spherical harmonic terms, leads to a useful family of triaxial mass models with central density cusps. The axial ratios of the models may be chosen freely at small and large radii, but, once this is done, the ellipticity profile and the variation of the major axis position angle with radius, i.e. the isophote twisting, are fixed. The models provide a simple analytical representation of the observed surface brightness distributions of moderately triaxial elliptical galaxies.

For a large range of flattenings and cusp slopes, the properties of the simple closed orbits in the triaxial γ -models can be described by first-order epicyclic theory. This formalism is well suited to investigating, for example, the velocity fields of the nuclear discs of ionized gas seen in many elliptical galaxies.

It is possible to refine these models further, by introducing additional terms in the expansion (2.1). This will not significantly complicate the calculation of the projected surface density, but will produce more nearly elliptical isophotes, even for substantially flattened models. At the same time, it will provide more freedom in the variation with radius of the ellipticity and position angle of the major axis.

ACKNOWLEDGMENTS

The authors thank Joanna Lees and Paola Amico for their contributions in the early phases of this work and the referee, Wyn Evans, for constructive comments. CMC was supported by HCM grant ERBCHICT940967 of the European Community.

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APPENDIX A: AUXILIARY FUNCTIONS

The integrals that define the functions F_1 , G_1 , G_2 and S_0 , which occur in the expressions for the projected surface brightness $\Sigma(R'', \Theta)$ and its integrated profile $S(R'')$ given in Sections 3.1 and 3.2, reduce to elementary functions when γ is an integer. They are given in Table A1, in terms of auxiliary functions $W_n(x)$, which are defined as

$$W_n(x) = \int_x^\infty \frac{dt}{(t+1)^n \sqrt{t^2 - x^2}}. \quad (\text{A1})$$

Table A1. The functions F_1 , G_1 , G_2 and S_0 for special values of γ , expressed in terms of the elementary functions $W_n(x)$.

γ	Function
0	$F_1(R) = \frac{3M}{4\pi r_0^2} [W_3(R/r_0) - W_4(R/r_0)]$ $G_1(R) = \frac{Mr_1}{\pi r_2^2} [W_3(R/r_2) + 3W_4(R/r_2) - 9W_5(R/r_2) + 5W_6(R/r_2)]$ $G_2(R) = \frac{Mr_1 R^2}{\pi r_2^2} [W_5(R/r_2) + 5W_6(R/r_2)]$ $S_0(R) = M \left[1 - \frac{3}{2} W_1(R/r_0) + \frac{5}{2} W_2(R/r_0) - W_3(R/r_0) \right]$
1	$F_1(R) = \frac{M}{2\pi r_0^2} W_3(R/r_0)$ $G_1(R) = \frac{Mr_1}{\pi r_2^2} [W_3(R/r_2) + 3W_4(R/r_2) - 3W_5(R/r_2)]$ $G_2(R) = \frac{Mr_1 R^2}{\pi r_2^2} \left[\frac{r_2^2}{R^2} - W_2(R/r_2) - 2W_3(R/r_2) - 3W_4(R/r_2) - 3W_5(R/r_2) \right]$ $S_0(R) = M [1 - W_1(R/r_0) + W_2(R/r_0)]$
2	$F_1(R) = \frac{M}{4\pi r_0^2} \left[\frac{\pi r_0}{2R} - W_1(R/r_0) - W_2(R/r_0) \right]$ $G_1(R) = \frac{Mr_1}{2\pi r_2^2} \left[\frac{3\pi r_2}{2R} - 3W_1(R/r_2) - 3W_2(R/r_2) - W_3(R/r_2) + 3W_4(R/r_2) \right]$ $G_2(R) = \frac{Mr_1 R^2}{2\pi r_2^2} \left[\frac{3\pi r_2^3}{4R^3} - \frac{4r_2^2}{R^2} + 4W_2(R/r_2) + 5W_3(R/r_2) + 3W_4(R/r_2) \right]$ $S_0(R) = \frac{MR^2}{r_0^2} \left[\frac{\pi r_0}{2R} - W_1(R/r_0) \right]$

For integer n these can all be expressed in terms of elementary functions. We find

$$W_1(x) = \begin{cases} \frac{1}{\sqrt{1-x^2}} \ln \frac{1+\sqrt{1-x^2}}{x} & (0 \leq x < 1) \\ 1 & (x = 1), \\ \frac{1}{\sqrt{x^2-1}} \arccos \frac{1}{x} & (x > 1), \end{cases} \quad (\text{A2})$$

and

$$W_2(x) = \frac{W_1(x) - 1}{1 - x^2}. \quad (\text{A3})$$

The expressions for W_n with $n > 2$ can be evaluated most easily by means of the recursion relation

$$W_n(x) = \frac{(2n-3)W_{n-1}(x) - (n-2)W_{n-2}(x)}{(n-1)(1-x^2)}. \quad (\text{A4})$$

By direct evaluation of the integral (A1) it follows that

$$W_n(1) = 2^{-n} B(n, 1/2) = \frac{(n-1)!}{(2n-1)!!}, \quad (\text{A5})$$

where B is the beta function. For small values of x the functions $W_n(x)$ diverge logarithmically:

$$W_n(x) \sim \ln \frac{2}{x} + \dots \quad (x \ll 1), \quad (\text{A6})$$

while for very large x

$$W_n(x) \sim \frac{B(n/2, 1/2)}{2x^n} + \dots \quad (x \gg 1). \quad (\text{A7})$$