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QUESTIONS OF PROOF*.

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Mostra-se que o conceito de prova é sobre-determinado no sentido que nem todas as asserções comumente feitas sobre provas são compatíveis. Mostra-se como estas asserções diferentes podem ser reconciliadas por uma série de distinções, em particular aquela entre ato-de-prova, objeto-de-prova e traço-de-prova.

It is shown that the concept of proof is over-determined, in the sense that not all the claims commonly made about proofs are compatible. It is shown how these diverse claims can be reconciled by making a series of distinctions, in particular that between proof-act, proof-object and proof-trace.

There are many reasons for having an interest in questions of proof. My foremost personal reason is that I have strong constructivist tendencies within the philosophy of mathematics and proofs are of crucial importance to mathematical constructivism, even at the level of meaning. The questions I shall pose concerning proofs will sometimes also concern 'proof'; in particular, I shall be concerned to draw some distinctions that can be used to resolve a number of 'proof' related ambiguities. In this

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connection, it should be pointed out that English is perhaps an unsuitable language in which to carry out my discussion. 'Proof' derives from the Latin *probare*, so, at least from an etymological point of view, there is little or no difference between the provable and the probable. Other European languages are more fortunate: Swedish *bevis*, German *Beweis*, French *démonstration*, Dutch *bewijs*, and so on, all derive from the same stem as the English *demonstration*.

The obvious first question to ask concerning proofs is undoubtedly:

What is a proof?

A straightforward answer to this question would be simply to hold that if one wants to know that proofs are, one has to look in places where proofs can be found. The right place to look for proofs clearly is a well-equipped mathematics library, where the monographs and the bound volumes of journals contain as many proofs as one could possibly wish for. Consider for a typical example:

> *Theorem* (Arzela-Ascoli): Every uniformly bounded, equicontinuous sequence **F** of functions on a compact interval contains a uniformly convergent subsequence.
>
> *Proof:* The rational points in the compact interval form an enumerable dense set of points. Let $q_1, q_2, q_3, \ldots,$ be such an enumeration ..., and so on.

Clearly, whatever proofs may be, this must be one of the most central notions: proofs are what mathematicians write down and the we find in mathematics journals. Many other aspects of proofs, however, are dealt with rather by philosophers than by mathematicians. In the style of a literal-minded Oxford Ordinary Language Philosopher, I can report that, upon checking a standard (Dutch) dictionary definition (translated into English by me) turns out to run in the following way:

Proof

1. the act of proving; that through which is shown irrefutably that something is as one claims or presupposed, as well in the sense of argumentation as in the sense of justification.

3. token from which one can make out the existence or the correctness of something. *syn* sign[1].

Perhaps anti-anti-realist opponents will now object that the emphasis on acts of proof given here just shows how pernicious Brouwer's influence has been in Holland: it has even entered into the world of lexicography[2]. However, the mathematical anti-realists, among whom I count myself, can afford to remain unimpressed by such a charge, since it can be adequately countered by an admissible *tu quoque* argumentation. The place to look in the *Oxford English Dictionary* is not under 'proof', but under 'demonstration', where

3. the act or process for making evident or proving

is offered as a central meaning. Thus the importance given to acts in connection with 'proofs', or better still, 'demonstrations', is certainly not a Dutch idiosyncrasy, be it intuitionistic or not. The putative critic would certainly be right, though, in his claim that Brouwer did hold just that:

Proofs are (languageless) mental acts/processes[3].

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[1] Geerts & Heestermans (1992), entry 'Bewijs'.

[2] Indeed, the difference between Van Dale (lexicographer) and (Dirk) van Dalen (logician and biographer of Brouwer) is only an 'n'.

[3] See Brouwer (1947), p.197 and also the so-called *First Act of Intuitionism* at p.140-1, in Brouwer (1952).

This dictum has, of course, been a source of heated controversy among philosophers of mathematics, since the term 'mental' provokes a charge of 'psychologism'. Michael Dummett, in particular, has been much concerned to offer justifications for the intuitionistic critique of classical reasoning, freed from this alleged Brouwerian psychologism[4]. Brouwer's pupil, and successor in the Amsterdam chair, Arend Heyting provided the first impetus to such work with his meaning-theoretical explanations of intuitionistic propositions and the logical constants. According to Heyting, proofs of propositions are mathematical objects, 'constructions', that satisfy certain conditions, '*die gewisse Bedingungen genügen*'[5]. In particular, a mathematical proposition is explained by laying down what condition has to be fulfilled by a construction-object that serves as a proof for the proposition in question. To prove that the proposition is true is to construct an object with the required properties. One further has to distinguish between a proposition and a theorem. A theorem is the affirmation of a proposition. Through *Mathematische Existenz,* the book by Oskar Becker, Heyting was explicitly influenced by Husserl's phenomenology and, in particular, by the two last Logical Investigations: a proposition expresses an Intention towards a construction that satisfies certain conditions and one proves the truth of the proposition in question through the fulfilment of the relevant Intention. Thus a theorem is the result of an affirmation that a certain construction fulfils a given Intention.

In the Thirties the topologist Hans Freudenthal, who had an interest in matters intuitionistic, polemicized with Heyting over the right notion of proposition to use in intuitionistic mathematics. Freudenthal objects to the use of 'bedingte Konstruktionen', that is, the hypothetical constructions that Heyting needed for his explanation of the implicational proposition from an intu-

[4]See a number of the essays collected in his Dummett (1978).
[5]See especially Heyting (1931), and (1934).

itionistic point of view. In particular, Freudenthal claimed that:

> [J]eder Satz, wenn man ihn erst intuitionistisch einwandfrei formuliert, [enthält] automatisch seinen ganzen Beweis (Freudenthal (1937), p.112)[6].

From a meaning-theoretical point of view this claim makes sense only if we interpret *Satz* as 'theorem', since it would be clearly too much to demand that a proposition should contain its proof. Indeed, the necessary gap between meaningfulness and truth, commented upon already by Wittgenstein in the *Tractatus*, would then be closed and only true propositions would be meaningful.

Brouwer and Freudenthal were famous mathematicians and it was an agreeable feature of the foundational debate in the first third of the century that mathematicians of the first rank took a very active part. G.H. Hardy's very stimulating Rouse Ball Lecture 'Mathematical Proof' provides a further example of first-rate participation by a top mathematician. According to Hardy (and his long-term collaborator Littlewood) proofs are "gas", rhetorical flourishes intended to stimulate the reader, or learner, to "see", to discover for himself (Hardy (1929), p.18).

I want to supplement the above fairly general points about proofs with a couple of recent remarks made in the anti-realist literature. In his famous paper 'The Philosophical Basis of Intuitionistic Logic', Michael Dummett observed that:

> [A] mathematical [proposition] is intuitionistically true if there exists an (intuitionistic) proof of it (Dummett (1975), p.239).

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[6] English translation:

> Every Theorem (Proposition), when formulated in an intuitionistically unobjectionable way, automatically contains its entire proof.

This, of course, is a point that does not originate with Dummett. The first to make it, as far as I have been able to find out, was Paul Lévy, one of the more acute participants on the realist side of the debate referred to above, and to which Heyting contributed his explanation of the (intuitionistic) notion of a proposition (Lévy (1926), pp.545-51). Dummett continues his remarks with a number of pertinent observations:

> we have to distinguish between a proof proper, a proof in the sense of 'proof' used in the explanations of the logical constants, and a cogent argument. ...
>
> We thus appear to require a distinction between a proof proper – a canonical proof – and the sort of argument which will normally appear in a mathematical article or textbook, an argument which we may call "demonstration". A demonstration is just a cogent a ground for the assertion of its conclusion as is a canonical proof, and it is related to it in this way: that a demonstration of a proposition provides an effective means for finding a canonical proof. But it is in terms of the notion of a canonical proof that the meanings of the logical constants are given (Dummett (1975), p.240).

The computer-assisted proof of the Four Colour Theorem in 1977 gave rise to a lively discussion concerning the notion of proof. In one of the seminal contributions a succinct list of requirements on proofs was given (Tymoczko (1979), p.59)[7]:

(a) Proofs are convincing
(b) Proofs are surveyable

[7]The present section, and later passages arising out of it, were not part of the lecture as originally delivered, but was added in response to a question from Michael Wrigley.

(c) Proofs are formalizable.

The first claim is, of course, nothing but the central ingredient in the dictionary definitions that were commented upon above. It is a point that can be found in Wittgenstein's later philosophy of mathematics:

> What convinces us – *that* is the proof: a configuration that does not convince us is not the proof, even when it can be shewn to exemplify the proved proposition (Wittgenstein (1978), p.171).

The second point (b) can also be found in Wittgenstein, where it is firmly stressed:

> 'A mathematical proof must be perspicuous'. Only a structure whose reproduction is an easy task is called a "proof". It must be possible to decide with certainty whether we really have the same proof twice over, or not. The proof must be a configuration whose exact reproduction can be certain. Or again: we must be sure we can exactly reproduce what is essential of the proof (Wittgenstein (1978), p.143)[8].

In a certain sense, this constitutes a further restriction of the first point: if a proof has to be convincing, it is reasonable to demand that it be surveyable, in order that we may become convinced. The third point (c) was first brought out clearly by

[8] There are many other congenial passages in this section, e.g.

> The proof (the pattern of the proof) shews us the result of a procedure (the construction); and we are convinced that a procedure regulated in *this* way always leads to this configuration.
>
> (The proof exhibits a fact of synthesis to us). (p.159)

The interest of this passage, and especially of its rider, is manifest against the background of Martin-Löf's paper, cited in note 12 below.

Leibniz, who stressed the *formal* nature of proof. Proofs proceed according to formal rules, that is, rules that do not depend on the particular contents under discussion, but only on the form in which they are presented.

A sufficient number of claims have now been listed from which to cull a short list of theses that, one could hope, would serve to delineate the concept of proof.

(1) Proofs are what we find in mathematics journals, etc. (folklore)
(2) Proofs are acts of getting to know (dictionary entry).
(3) Proofs are mental (languageless) constructions (Brouwer)
(4) Proof-constructions are mathematical objects (Heyting)
(5) Meaning is explained in terms of proofs (Heyting)
(6) The theorem, when fully explicit, contains its proof (Freudenthal)
(7) Proofs are (rhetorical) "gas" (Hardy)
(8) The proposition *A* is true if and only if there exists a proof of A (Dummett)
(9) Meaning is explained in terms of canonical proofs; "demonstrations", found in journals and textbooks, are means for finding canonical proof (Dummett)
(10) Proofs are convincing (aspect of (2) above)
(11) Proofs are surveyable (Wittgenstein)
(12) Proofs are formalizable (Leibniz)

The points on our list are fairly general and rather well-known. The question is now whether they jointly serve, not just to determine the concept of proof, but in fact to overdetermine it:

Proofs are *mental* acts, and yet they occur *on paper* in libraries. They are mathematical *objects*, but also "gas". A theorem *contains* its proof, but meaning is *explained* in terms of proofs, which, on the other hand, have to be '*languageless*' mental acts. A thorny thicket indeed!

A first step towards a way out of this conundrum lies in a distinction that I first applied ten years ago in my paper 'Constructions, Proofs and the Meaning of the Logical Constants' (Sundholm (1983), p.164). The term 'construction' is sensitive to a process/product ambiguity. It can be used to refer to, at least, the following three types of entity:

(a)(Construction-)object, that is the object constructed in a
(b) (Construction-)process/act.
(c) Act/process of construction regarded as an entity.

I then remarked that Heyting's notion seemed to be that of (a), whereas Brouwer's notion of construction seemed to encompass elements of all three. Today I would still uphold my position on Brouwer's view, but as to Heyting's notion I think one does more justice to his choice of word by also including features of (b) and of yet a further component:

(d) Construction-"blueprint",

that is, a general description for how to carry out a certain type of construction-act/process in order to obtain a construction-object as result.

In his lecture 'On the Relation between Mathematics, Logic, and the Theory of Knowledge', delivered at a conference in Paris, in April 1992, Per Martin-Löf drew attention to an analogous tripartite distinction with respect to the term 'proof':

(a) Proof-act/process
(b) Proof-object/construction
(c) Proof-trace/track

The sense (a) is fairly uncontroversial and is the one taken from the dictionary entry for 'demonstration', that is, proofs are acts of getting to know. The sense (b), on the other hand, is that of Heyting's proof-objects, that is, the mathematical construction-objects that one has to be given in order to have the right to make mathematical assertions. The sense (c), finally, as far as I know, has not been given much attention prior to Martin-Löf's Paris discussion[9].

In order to put this distinction into proper perspective it is convenient to recall the scholastic logic that was, more or less, current until Kant. Logic, according to St.Thomas Aquinas, is the study of terms of the 'second intention', that is (mental) terms that have themselves mental terms as their intention. The most important examples here are the terms Term, Judgement and Inference, that is, those terms that serve to indicate the (mental) products of the three Operations of Mind. First, act of Simple Apprehension have (mental) Terms (notions, concepts, ideas) as their products. The second Operation of the Mind is that of Judgement (or, perhaps better, of Judging) and acts of judgement have (mental) propositions (judgements) as their products. In the scholastic tradition, an act of judgement is an act of composition, or division, with respect to two mental terms that have been previously obtained as products of acts of Simple Apprehension. The judgement made is thus of the form:

$$S : P$$

where the colon indicates the copula. Finally, acts of Inference, that is, acts of the third Operation of the Mind, presuppose that one or more acts of the second Operation have been completed. Typically, in a syllogism, two judgements, that is, two mental products of the second operation are used as premisses of the act of inference, and here the product is a (mental) inference.

For all three Operations, supplementation of the mental, inner, product, by means of outward tokens, or signs, is possible. The outward sign of a (mental) terms is a spoken, or written, term, whereas that of the (mental) proposition/judgement made

[9] I am most indebted to Per Martin-Löf for telling me about the notion of a proof-trace and allowing me to see his unpublished notes for the Paris talk.

is an assertion. Finally, the sign/token for the Inferences are written, or spoken, pieces of argumentation.

An important distinction with respect to acts of judgements is that between immediate and mediate acts. In an immediate act the division, or composition, takes place in one fell swoop, with no intermediate steps, whereas in a mediate act, one, or more, acts of inference will be carried out prior to obtaining the final proposition as the conclusion of the whole chain of intertwined inferences. The immediate/mediate distinction can be transposed, from the acts of making the judgements, to the knowledge obtained, that is, to the judgements made. Hence one can speak about *mediate* knowledge, etc., according to how the act of obtaining the knowledge fares with respect to the distinction in question.

Naturally, a number of changes can be found in this scholastic, traditional, picture. Thus, according to Kant terms are not conceptually prior ('prior in formula', or 'in definition', as the scholastic terminology has it) to judgements. On the contrary, we can reduce '*alle Handlungen des Verstandes auf Urteile*' (Kant (1787), A 69). Furthermore, the conceptual priorities between the exterior and interior aspects of acts were reversed in our century. The traditional view of acts in certainly psychologistic; for instance, even for the arch-antipsychologist Frege, assertion is the exteriorization of an interior judgement. Towards the end of the Twenties, these priorities were reversed by Wittgenstein, and, or so I have been told, by Heidegger[10].

For the present discussion, the major change was effected by Bolzano, who, in 1837, disposed of the old form of judgement. In place of this subject/copula/predicate form, we now find:

$$A \text{ is true}$$

where A is a *Satz an Sich*, that is, a Fregean *Gedanke*, or propo-

[10] Wittgenstein (1958), and the early sections of Heidegger (1927).

sition *in the modern sense.* Such a proposition is no longer the product of an act of judgement. Rather, it is the *content* of a judgement made. The post-Bolzano (Frege, Russell, Wittgenstein) situation with respect to an act of judgement can thus be drawn as follows:

Judgement
-act

content

A is true

judgement made.

When we apply the act/object terminology here, the product/object of the act of judgement is the judgement made, that truth adheres to the content in question. It is also natural to speak of the act of judgement as a(n act of) *proof.* The object of a proof-act is not a proof-object, but the judgement made in the act, that is, the theorem one gets to know through the proof-act in question.

The above scheme of the judgemental act, and its correlates, thus raises a number of issues related to each of the three items involved: the act, the product, and the content of the product. In particular, the notion of truth is of central importance, since it serves to fix the form of judgement. Frege, and Bolzano, had little to say about the notion of truth. This, however, is hardly surprising. It is the central, absolute, notion on, and around which both their respective systems are developed. Consequently, concerning this core notion, in terms of which other, less central notions are given, one can hardly expect to be told very much. It fell to others, in particular Brentano, Russell, Moore, and the

Wittgenstein of the *Tractatus*, to offer a substantial analysis of the notion of truth. Even though their respective efforts differ in the details of their execution, it is not inappropriate to speak of *an* analysis, since they all share a common pattern, namely that of the truth-maker view of truth. Such an analysis holds that

proposition A is true = there exists a truth-maker for A

or, if we prefer a more scholastic formulation

= the concept *truth-maker for A* has existence/EXISTS.

Such a truth-maker analysis clearly depends on the category of truth-makers for A, as well as on the relevant notion of existence. Owing to this one cannot expect to make too many general claims concerning truth-maker analyses in general, since many of their properties will depend on the specific choices made. A negative general point that can be made, though, is that the notion of existence involved in a truth-maker analysis cannot be that of the existential quantifier. The meaning of the latter is explained via the truth-condition for an existential proposition, that is, through consideration of the form of judgement:

$$(\exists x \in D)P(x) \text{ is true}$$

But, on the truth-maker analysis of truth, this judgement then has to be analyzes as:

there exists a truth maker for $(\exists x \in D)P(x)$

so a vicious regress would result if the notion of existence were that of $(\exists x \in\ )$. By the same token, we also see that the relation of truth-making will have to be non-propositional.

On my preferred constructivistic reading, the relevant truth-maker is, of course, the one introduced by Heyting, namely, that of a proof(-object) of the proposition in question. Here the link with meaning is also clear: you explain the proposition by explaining what is required of a proof-object.

The notion of existence, on the other hand, is also tied to constructivist notions, and was, as far as I know, adumbrated by Hermann Weyl[11] When α is a general concept, that is, a notion such that

$$a : \alpha$$

is a logically possible judgement for certain terms a, then

$$\alpha \text{ EXISTS}$$

is a form of judgement. In order to explain a judgement one has to explain what one has to know in order to have the right to make the judgement in question. For an existential judgement, what one has to know in order to have the right to make the judgement is that the concept α is instantiated, that is, has to know an:

[11] Weyl writes:

> *Ein Existentialsatz* – etwa "es gibt eine gerade Zahl" – *ist überhaupt kein Urteil im eigentlichen Sinne, das einen Sachverhalt behauptet*; Existential-Sachverhalte sind eine leere Erfindung der Logiker. "2 ist eine gerade Zahl": das ist ein wirkliches, einem Sachverhalt Ausdruck gebendes Urteil; "es gibt eine gerade Zahl" ist nur ein aus diesem Urteil gewonnenes *Urteilsabstrakt* (1921, p.54).

(English translation

> *An existential Proposition* – for instance, "there is an even number" – *is not at all a proper judgement that expresses a state of affairs*; existential states of affairs are an empty invention of logicians. "2 is an even number": that is a real judgement that expresses a state of affairs; "there is an even number" is only a judgement-abstract that has been obtained from this judgement.)

$$a : \alpha.$$[12]

Enough preparatory work has now been carried out so as to illustrate the distinctions made above. When I prove a mathematical theorem, I carry out a mental act of proof. This act of proof encompasses an act of construction that produces a construction-object c that 'satisfies certain conditions', depending on the proposition A that serves to give content to the theorem in question, and required by the meaning-explanation of A. On the basis of this construction, with the ensuing construction object, I am entitled to affirm the theorem:

A is true

Owing to the truth-maker analysis, this theorem is of the form:

the concept proof-object of A has existence.

Thus, in the explicit version:

construction c is a proof-object of the proposition A

it does contain its proof(-object), just as Freudenthal claimed. It should be noted here that on this proof-theoretical version of the truth-maker analysis, the novel, Bolzanian, form of judgement:

A is true

has been brought back to the traditional:

$$c : A$$

that is:

[12]The explicit formulation of the form α EXISTS was given by Martin-Löf in his lecture "Analytic and Synthetic Judgements in Type Theory", forthcoming in the preceedings of the Workshop on Kant and Contemporary Philosophy, Florence 27-30 May 1992.

construction c is a proof of proposition A

in the sense that the former is regarded as an ellipsis for the latter.

The proof-act and the proof-object have now been accounted for. The proof-trace still remains to be treated of. When the proof-act has been completed, what remains is its object, that is, the theorem proved/judgement made, part of which is a proof-object, being the construction-object of a construction-process. The theorem is the product/object of the act, but other traces might still be left. Martin-Löf considers the case of a tour of cross-country skiing, performed by a sportsman. The act, or action, is that of completing a certain tour. Its object, naturally enough, is the goal reached. The trace, or track, left by this act of skiing, is, in this particular case, owing to the nature of the activity, nothing but a pair of tracks, perhaps also supplemented with signs or flags. This trace of the act of skiing has the property of enabling another sportsman to carry out an act that will produce the same object, namely, the goal in question, simply by following the pair of tracks and attending to the flags and other signs. The act of skiing might leave yet a further trace in the form of a written record or description of the path taken. Also this form of trace might serve to enable someone to carry out an act of skiing that will complete the same tour as the original one. Indeed, this seems to be the exact situation with respect to travel-guides (provided, of course, that they have not been written on the basis of "armchair travel", that is, the copying other travel-guides).

In addition to Martin-Löf's example we might add two more. One drawn from culinary art and the other from the game of chess. For the first, consider the act of preparing a Sauce Béarnaise: one prepares a reduction of pepper, vinegar, tarragon, and an egg-yolk, into which lukewarm melted butter is whisked in drop by drop. The object of the act is, of course, the sauce itself. The trace of the act of preparation can be found in the kitchen:

a half-empty bottle of finest vinegar, a mortar and pestle with traces of white and black pepper-corns, tiny twigs of tarragon, tiny specks of melted butter, a buttery *au bain marie* pan, a set of used knives, forks and spoons, etc. If the utensils have been stowed away in the order they were used, a gifted culinary expert could possibly abduct the making of the sauce in question. Whether the act could be copied with the same ease as in the case of the skiing example given by Martin-Löf is, however, doubtful. Indeed, the written trace, or description, that is, the *recipe*, for (the act of) preparing the sauce, here seems essential for enabling someone else to produce the same object[13].

One should note here that a description of an act need not be obtained as a trace of the act. An act clearly can be described upon its completion – first I did this, and then I did so, etc. – but, in principle a set of instruction can be written down irrespectively of whether they haven been carried out or not. Someone could create a delicious dish mentally, write out a recipe and leave the actual realization to a chef. Similarly, one could draw up a skiing-tour simply by looking at a map and writing out a set of suitable directions, without the tour's ever having been made. A further parallel is that of a composer working with, or without, a piano while composing. In the former case, but not in the latter, the score will be a trace of the act of playing various pieces of music.

Consider now a game of chess being played. Here the act is a joint act of two (or possibly more, '*Beratende*', as they are called in old-fashioned German chess books) antagonists. One way of taking the product/object of this act is, of course, the completed game itself. I prefer, however, to view the *result* of the game,

[13] Per Martin-Löf pointed out in conversation that the recipe might best be seen as a *programme* and the raw-materials used as the input. The execution of the programme/procedure/recipe has the prepared dish as its result. The recipe together with the *mise en place* of raw-materials is hence the analogue of a non-canonical proof-object. The finished dish obtained through the execution of the recipe, on the other hand, is in canonical form.

that is, the win/loss, or draw, as the object/product of the act of playing. In this case, the trace of the act of playing that produced the result in question, will be the signed score-sheets of the players. Other players, or they themselves, for all that matters, could use the score to play the game all over again[14]. If the game is of a very high quality, and/or, the players involved very eminent, say of top Grandmaster level, the score might be published in chess journals all over the world, in which case the traces of the (act of playing the original) game have become quite substantial. Many these journals will carry, not only the moves played, but also more or less thorough annotations, often by one of the opponents – most commonly, the winner. These annotations do not constitute a direct trace of the act of playing the game, but only in an indirect sense are they called forth by the original act of playing. The winner, for instance, would not have written these annotations had he not won the game, of course, but further inducements might be needed, such a sizeable fee. They are also not necessary in any way for a replaying of the game; rather they serve as hints for how to play the game of chess in a skilful way. They belong to chess methodology, rather than to chess itself. Also in the other case something similar is not unknown. Good guides to culinary art often provide analogous advice, such as that of adding some very hot water to a butter sauce that has curdled, or that of recommending the making of a mayonnaise with eggs at room temperature, but not colder, etc.

If we now return to the mathematical case, we see that the proof-traces here commonly correspond to the recipes in works in culinary art, or to the score-sheets of music, etc. They are the proofs we find as 'proofs' in mathematical journals and mono-

[14] Although it is questionable whether they would then be playing a game of chess; the Laws of Chess (Article 15.19(a)) do expressly forbid the use of written material as an aid during the playing of a game of *chess*. This rather subtle point concerning the identity of games relative to their rules is not of importance for my example and need not detain us further.

graphs. The annotations of chess, and the hints from culinary art, correspond to pointers within the methodology of mathematics. Thus, here one would not be concerned with a description of an act of construction for a particular proof-object, but more with general, "strategic", principles for how to find proofs-objects with certain properties. Polya's work on 'plausible reasoning' here springs to mind.

It is now convenient to return to the list of points (1)-(12) above, that served to overdetermine the concept of proof, with a view to disentangling the issues through the use of the distinctions just drawn. Point (1) concerning proofs in books and journals clearly refers to proof-*traces*, whereas the dictionary/Brouwer points (2) and (3) concern proof-*acts*. The Heyting's claims (4) and (5), as well as the Freudenthal point (6), hold for proof-*objects*. Hardy's (7), on proofs as 'gas', deals with the *traces*, or descriptions, of acts of proof.

The three points culled from Tymoczko are also easily dealt with. The proofs that have to be convincing (10) are, of course, in the first instance, the proof-*acts*. However, in a derivative sense, also a proof-trace can be "convincing", namely when the proof-acts it describes are convincing. The surveyability (11), on the other hand, refers to the proof-*traces*. The proof-*object* obtained from a proof-act carried out according to a surveyable proof-trace will also be surveyable. Finally, Leibniz' point concerning the formalizability of proofs (12) concerns the proof-objects, traces and acts. Mediate proof-acts have to be analyzable in terms of immediate steps according to certain formal rules. Of course, such rules need not be explicitly formulated prior to the act, since it is possible to discover new axioms and formal rules.

The two Dummett points (8) and (9), finally, demand a more careful analysis. As soon as the meaning of a sentence has been explained, that is, as soon as the proposition has been grasped, the proof-object will be a logically possible object in the sense that it has been laid down what properties it must have. The notion of a proof-object of a proposition is not a contradiction in terms. Since no explanation has been given of what proof-objects for numbers would be, a proof-object for a natural number, on the other hand, is not a logically possible object; one would get a category mistake. For any proposition, also for the absurd proposition $\perp$, a proof-object for the proposition in question is a logically possible object. Only for some propositions, though, is a proof-object *really possible*, in order to speak with Kant. For instance, a proof-object for $\perp$ is only a logical, but not a real possibility. Finally, for a proved proposition, the proof-object is even actual[15].

Thus, the truth of a proposition will also be further analyzed according to the modal status of the proof-object in question. As soon as the proposition has been explained we can use the notion of its truth *simpliciter*, that corresponds precisely to the proof-objects' being logically possible. A proved proposition is actually true. The notion of a potential truth, that is, a proposition for which a proof-object is really possible, finally is used to make sense of such clauses as:

$A \vee B$ is true

iff

A is true or B is true.

Here, $A \vee B$ can be actually true, that is, known to be true, without it either being known that A is true or that B is true. On the other hand, given that a proof-object can be found for $A \vee B$, one can be found for A or one can be found for B (but need not yet have been found, of course). Thus, potential truth is preserved.

Meaning is explained in terms of canonical proof-*objects*. Just as natural numbers can be given in direct or indirect ways, so can

[15] See Per Martin-Löf (1991), pp.141-149.

proof-objects. In formal systems

$0, s(0), s(s(0)), \ldots,$ and so on,

are the canonical terms used for the natural numbers whereas in the standard informal language of mathematics the Arabic numerals serve as canonical terms for the natural numbers. In the appropriate terminology of Wolfgang Künne, they are 'presenting terms'[16]. Natural numbers can, however, be given in countless other ways, using, for instance, various primitive recursive functions or what have you, to produce highly complex number terms that must yet be evaluated to canonical form. The same holds also for proof-objects. The proof-objects provided by the above Heyting(-like) meaning explanations are all in (Gentzen) introduction form, but as we all know, it is certainly possible to prove the truth of a proposition by means of non-canonical inferences, for instance, applications of elimination rules, and the proof-object provided in such an act of proof will be non-canonical, but must, of course, be evaluable to canonical form. Thus, the canonical proof-objects stand opposed to non-canonical proof-objects. Properly speaking, Dummett's "demonstrations", on the other hand, do not correspond to non-canonical proof-objects. Just as the canonical proofs, the latter are mathematical objects, whereas the Dummett "demonstrations" are written proofs in journals, intended to convince. Within the present framework, they are proof-traces, rather than non-canonical proof-objects, and so, it appears, that Dummett has conflated two distinctions, owing to his neglect of the fundamental distinction between proof(-acts) of theorems and proof(-objects) of propositions.

Wittgenstein wrote:

> I should like to say: mathematics is a MOTLEY of techniques of proof. – And upon this is based its

[16] Künne (1983), §4.7.

> manifold applicability and its importance (Wittgenstein (1978), p.176).

Indeed, it is not possible to list in advance all the ways of proving a proposition. What can be done, and what has to be done, is to lay down in advance what properties a canonical proof-objects has to have. In order to explain the proposition, 'gewisse Bedingungen', to be satisfied by the mathematical constructions that serve to prove to proposition in question, have to be laid down. But for these strictures on canonical proofs, nothing further is imposed. Thus, there will, in general, be no end of non-canonical means for presenting objects of the right type. The only requirement is that the presentation chosen be effectively evaluable to canonical form. In the same fashion, the creative acts of mathematicians are in no way circumscribed or hampered, save as required by the meaning-explanation for the proposition serving as content of the theorem (to be) proved. All and any means are admissible for obtaining the sought-after proof-objects, always provided that the latter admit effective evaluation to canonical form as specified by the relevant meanings.

This, then, is the most liberal resolution I have to offer for (at least some of) the tensions present among the theses (1)-(12). Similarly, I also hope to have made clear how the threefold distinction between proof-acts, -objects, and -traces, serves to provide answers to a fair number of difficult questions of proof implicit therein.

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