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THE COMPONENT OF THE GALACTIC GRAVITATIONAL FIELD PERPENDICULAR TO THE GALACTIC PLANE, K_z ¹⁾

BY E. R. HILL

A critical review is made of all aspects of K_z computations, and likely avenues for future researches, both observational and theoretical, are advanced. The main objective has however been to derive a new K_z based on a fresh study of the distribution above and below the galactic plane of K giants and on an improved analysis of their Z -velocity distribution.

The Z -velocity distribution in the galactic plane has been represented as the sum of three components, a strong-line, weak-line, and high-velocity component. The proportion in which the latter are present, and their mean Z -speed have been deduced from radial velocities of high-latitude K giants — and particularly from those with m_{pv} about 11. With this analysis, the dispersion of Z as a function of distance from the plane z could be estimated. For all this work a tentative K_z had to be adopted. With the aid of further studies of the peculiar and solar motions of the three groups of stars parallel to the galactic plane, distributions of transverse velocity of high-latitude K stars were deduced at various z -distances.

A z -density distribution of K giants has been derived from high-latitude K-star counts to $m_{pg} = 13$ based on the *Henry Draper Catalogue* and *Bergerdorfer Spektral-Durchmusterung* and using observed total proper-motion distributions of K stars to assess the proportions of giants at various apparent magnitudes. This necessitated using the VAN RHIJN-SCHWASSMANN luminosity function for K stars. Total proper-motion distributions were compiled from Boss's *General Catalogue* and the Yale, Radcliffe-Pulkovo, and Groningen catalogues. Galactic-rotation effects were removed from observed motions, which were reduced to the FK₃ system. Observed distributions were corrected for accidental errors. Computation of the proportion of giants involved using the above-mentioned transverse-velocity distributions to compute proper-motion distributions of K stars in several luminosity groups at various apparent magnitudes. The combination of these computed distributions which best fitted those based on observation gave the required proportions.

With the dispersion of Z and the density $\Delta(z)$ so deduced for K giants, a new K_z was derived which should be reliable to about $z = 500$ pc. Beyond this distance it appears to violate Poisson's equation; this has been attributed to uncertainties in the number of K stars at fainter photographic magnitudes. The new K_z yields a local density of 0.13 solar masses per cubic parsec, which is larger by a factor 1.5 than that deduced by OORT in 1932.

1. INTRODUCTION

1. *The early history, and the basic assumptions of K_z computations*

In 1920, KAPTEYN and VAN RHIJN (1920a) published results of a research aimed at providing the distribution of stars throughout the Galaxy. This analysis rested on very extensive general star counts and a luminosity function which, although derived from studies of relatively nearby stars, was assumed to be applicable to all parts of the Galactic System. The results indicated that the Galaxy was a flattened system in which surfaces of constant star density took the form of oblate spheroids; the star density decreased

outwards from the centre, which was thought to be quite close to the sun.

In a subsequent paper, KAPTEYN (1922) made a tentative attempt to compute the gravitational field in this system with the aid of relations provided by the kinetic theory of gases. It was assumed in the first place that the system had settled down into a stable spatial configuration, now commonly referred to as a steady state²⁾. This assumption was vital to the application of

¹⁾ This article was written in 1956. Through various circumstances its publication has been delayed.

²⁾ More specifically, the steady-state condition is one in which the number of stars entering each volume element within the Galaxy with certain velocity components equals the number leaving with the same components.

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the kinetic theory, but it could be neither supported nor denied by observational material available at the time. Still, the employment of conclusions reached from the kinetic theory as applied to a gas at rest under its own gravitational attractions was considered with caution, principally because of the non-isotropic distribution of motions of stars in the sun's neighbourhood. Nevertheless it was hoped that the kinetic theory results relating to the density distribution in an isothermal gaseous atmosphere stratified in plane-parallel layers might be applied to the study of the density distribution of stars along a line passing through the sun¹⁾ and crossing perpendicularly to the symmetry plane of the Galaxy. This line will be referred to hereafter as the z -direction; z being measured away from the plane of symmetry of the Galaxy. The stellar density distribution along this line will be denoted by $\Delta(z)$.

Stars distributed along the z -line would be quite close to the axis of rotation of the KAPTEYN-VAN RHIJN Galaxy, so that their motions should not be greatly influenced by rotational motion. Therefore the density distribution of such stars, $\Delta(z)$, will be fixed by the distribution of their Z -speeds and the z -component of the gravitational field prevailing along the z -line. KAPTEYN assumed the Z -speed distribution of the stars as they pass through the galactic plane near the sun to be gaussian with mean square speed $\overline{Z^2}$, so that $\Delta(z)$ and the acceleration produced by the gravitational field perpendicular to the plane of the Galaxy at distance z , K_z , are related by

$$K_z = \frac{\overline{Z^2}}{\text{Mod}} \cdot \frac{d \log \Delta(z)}{dz}, \quad (\text{Mod} \equiv \log_{10} e)$$

according to the kinetic theory²⁾. Thus once $\overline{Z^2}$ and $\Delta(z)$ were known, K_z could be readily computed, and with this information KAPTEYN was able to compute the field throughout the Galaxy.

It is easy to obtain a qualitative impression of the

¹⁾ In practice, position is usually measured with respect to the sun as origin and along a line perpendicular to the conventional galactic plane passing through the sun. There are however clear indications based, amongst other evidence, on faint-star counts and the latitude distribution of spiral arms from 21-cm radio observations that the sun is probably 10 to 20 pc north of a more appropriate galactic plane of symmetry. In general this difference is not important and distances z may, without loss of precision, be measured with respect to the sun. The only exceptions arise when the z -density distribution of stars, or any other objects very concentrated to the plane are considered—such as A stars, interstellar gas, etc.

²⁾ This equation is closely related to the barometric equation derivable for a plane-parallel isothermal atmosphere, which for many purposes bears a sufficiently close approximation to the planetary atmosphere. The only real difference is that in the atmosphere problem the gravitational field is fairly uniform over the regions so far investigated directly, whereas in the present case it is definitely non-uniform.

way in which K_z should vary with z . At a point in the plane of symmetry of the Galaxy ($z = 0$), the acceleration is clearly zero; the attraction by material above the plane will cancel that arising from material below the plane. However at any position away from the plane the attractions tending to pull stars further from the plane (due to stars on the same side of the plane, but beyond the position considered) will always be more than counterbalanced by those from the greater remaining part of the mass of the system which tend to draw the star back into the plane. The result is that K_z will always be directed towards the galactic plane and will become larger and larger in magnitude as z increases until the distance becomes sufficiently large to weaken the effects of the main gravitational forces. At z -distances large compared with the dimensions of the Galaxy, K_z will fall off as z^{-2} .

The relationship between K_z and the distributions of density and velocity in the z -direction for the stellar system envisaged by KAPTEYN was reconsidered in more detail by JEANS. JEANS pointed out that even in times of the order of the age of the Galaxy the effects of encounters between stars are of no importance. Thus the Galaxy cannot have reached an equilibrium condition in which stellar encounters play the significant role that atomic and molecular collisions often play in a gas. He therefore considered the properties of a stellar system having a steady spatial configuration in which the motions of stars are governed solely by the gravitational field of the system as a whole.

Introducing the cylindrical co-ordinate system ϖ , θ , z , with the galactic centre as origin, and the galactic plane as the plane $z = 0$, JEANS (1919) was able to show that for a steady-state system with cylindrical symmetry about the axis $\varpi = 0$, the distribution of velocity components, Π , Θ , Z , at a point ϖ , θ , z , takes the form

$$f\left\{\frac{1}{2}(\Pi^2 + \Theta^2 + Z^2) - V, \varpi\Theta\right\} d\Pi d\Theta dZ d\varpi d\theta dz, \quad (1)$$

where V is the gravitational potential at the point. The precise character of the function f cannot be deduced from the statistical theory used in the derivation of (1). Later JEANS (1922) derived the following relationships, valid anywhere in the system,

$$K_{\varpi} = \frac{1}{\Delta} \frac{\partial(\Delta \Pi^2)}{\partial \varpi} + \frac{\Pi^2 - \Theta^2}{\varpi}, \quad (2a)$$

$$K_z = \frac{1}{\Delta} \frac{\partial}{\partial z} (\Delta \overline{Z^2}), \quad (2b)$$

where Δ here denotes the density at the point ϖ , z .

The results summarized in (1), (2a) and (2b) are of importance for several reasons. Firstly they show how

both gravitational field components, K_{ω} and K_z , in the KAPTEYN-VAN RHIJN Galaxy are related to distributions of density and velocity, irrespective of the presence of galactic rotation and the form of the velocity distribution¹⁾. In a sense this was an advantage since shortly after the work of KAPTEYN and VAN RHIJN was published, it became clear that the sun was very far from the galactic centre and in a region of considerable rotational velocity. Moreover it was definitely established by OORT that the Z -velocity distribution was not purely gaussian as supposed by KAPTEYN. And secondly, expression (1) provided an additional criterion by which the validity of the steady-state hypothesis might be judged. For since the Π - and Z -motions appear in a similar manner in the velocity distribution (1), the dispersions of Π - and Z -motions should be identical (but not generally the same as the dispersion of Θ -motions). Numerous studies of various observational data have since shown that this condition is not satisfied by stars in our Galaxy — they show that generally $\overline{\Pi^2}$ is unmistakably larger than $\overline{Z^2}$.

Thus it became clear that even the steady-state condition considered by JEANS probably had not been reached by the Galaxy, and that equations (2a) and (2b) have to be used with caution. Apparently insufficient time has elapsed for the gravitational field of the Galaxy to shuffle up the Π - and Z -motions of the stars²⁾ to the extent required by JEANS's condition.

In 1932, OORT made a direct study of the average Z -velocity of stars at various distances from the galactic plane. This work was based on peculiar radial velocities (converted statistically into Z -velocities) of high-latitude stars. It was found that in all probability the common stars show no sign of possessing systematic Z -motions within several hundred parsecs of the galactic plane³⁾. This result led to a new foundation for a K_z -calculation; OORT introduced the supposition that as far as their oscillations to and fro

across the galactic plane are concerned, i.e., their Z -motions, the stars are "well-mixed". Or more explicitly, of stars in a group comprised only of those which cross the galactic plane within a very narrow range of Z -velocity (but not necessarily at nearly the same times), as many are observed to enter as leave any layer above and below the galactic plane. Thus the density distribution, $\Delta(z)$, of stars above and below the galactic plane as observed at the present time is considered permanent. To obtain a true picture of the density distribution it is then not necessary to average density distributions over times large compared with the oscillation times (which are of the order of 10^8 years) in order to compensate for large fluctuations in density due to some orderliness which may still prevail in the Z -motions. In essence the well-mixed condition adopts, on the basis of observation, those features of JEANS's theory insofar as the Z -velocities alone are concerned. As nothing is postulated about the Π - and Θ -motions, inconsistencies arising from the more general steady-configuration hypothesis are bypassed. Thus equation (2b) is supposed to be valid. The implication is then that the problem of computing K_z can be solved in terms of only the Z -motions of the stars, for which $\Delta(z)$ is known, as they cross the galactic plane near the sun, or alternatively, in terms of the mean square velocity which they possess, as a group, at various distances from the galactic plane⁴⁾.

Before turning to the discussion of results of various K_z -calculations, some frequently-used relations, derived by OORT from the well-mixed condition, might be noted without proof.

If the Z -speed distribution of a group of stars at the galactic plane is gaussian, i.e. if

$$\varphi(z=0, |Z|) = \frac{1}{\pi|\overline{Z}|} e^{-Z^2/2\overline{Z}^2}, \quad (3a)$$

¹⁾ Provided of course that velocity components are distributed in accord with expression (1).

²⁾ Whether this explanation of the divergence between theory and observation is reasonable might be made clear by calculations, which are being performed by Mr OLLONGREN, of the orbits traversed by stars during 10^9 years or so in the prevailing gravitational field of the Galaxy. There does exist another possible reason for this disagreement. It stems from the possibility that the velocity distribution (1), which is a function of the constants of the motions in the system, might contain another constant apart from the total energy and angular momentum which are only two out of the *five* that could conceivably exist. If another constant could be found, it is possible that its presence in expression (1) could modify the relation already existing there between Π - and Z -motions and thus diminish the gap between theory and observation.

³⁾ It might be added that this calculation has been repeated by the author and its conclusions confirmed, using the more extensive and precise radial-velocity data now available.

⁴⁾ Although there are no objections to these ideas, so long as they concern stars with low or moderate Z -velocities, they do appear questionable as regards high Z -velocity stars and consequently the computation of K_z for large z -distances. The objection which might be raised is the following. Stars which are observed at great distances from the galactic plane must, very probably, have crossed the galactic plane far from the sun with a large Z -speed, since such speeds are generally accompanied by large Π - and Θ -motions. This would mean that the Z -motions exhibited by stars at large z -distances are not directly related in a one-dimensional fashion to the Z -motions of stars with the same spectrum near the sun as supposed, but are related in a much more complex three-dimensional fashion involving the Π - and Θ -motions. Although this criticism may possibly be serious, the extent to which it affects K_z -computations is unknown; the pursuance of issues raised by this question probably constitutes a complete investigation in itself and so cannot be considered here.

then the z -density distribution of members of the group may be shown to be

$$\Delta(z) = \Delta(0) e^{\frac{1}{\bar{Z}^2} \int_0^z K_z dz}, \quad (3b)$$

from which it readily follows that

$$K_z = \frac{\bar{Z}^2}{\text{Mod}} \cdot \frac{d \log \Delta(z)}{dz}. \quad (3c)$$

In this case $\bar{Z}^2$ is independent of distance from the galactic plane. This comes about in the following manner. Stars crossing the plane with small Z -speeds never reach beyond small distances from the plane and, other things being equal, tend to make $\bar{Z}^2$ at greater distances larger than in the plane, however, those which do reach large z -distances have had their velocities diminished in overcoming the field which is directed towards the plane. These two effects compensate one another precisely, and the same gaussian distribution of Z -motions occurs at all z . Thus $\Delta(z)$ is exactly the same density distribution as would be assumed by an isothermal gas at temperature $T = m\bar{Z}^2/k$ in the same gravitational field¹⁾.

More generally, if $\varphi(z=0, |Z|)$ can be considered to be made up of several gaussian components with differing $|Z|$, i.e., if

$$\varphi(z=0, |Z|) = \frac{1}{\pi} \sum_i \frac{\theta_i^0}{|Z|_i} e^{-Z^2/2\bar{Z}_i^2} \quad (4a)$$

where the θ_i^0 's are the proportions of the components found in the galactic plane, ($\sum_i \theta_i^0 = 1$), then it follows from (3b) that

$$\Delta(z) = \Delta(0) \sum_i \theta_i^0 e^{\frac{1}{\bar{Z}_i^2} \int_0^z K_z dz}. \quad (4b)$$

In this case, K_z is given by

$$K_z = \frac{1}{\Delta(z)} \frac{d}{dz} (\Delta(z) \bar{Z}^2). \quad (4c)$$

From equation (4b) it is clear that if several gaussian components make up the Z -speed distribution near the galactic plane, then as z increases the group with the largest Z -speeds tends to predominate more and more. Under these circumstances $\bar{Z}^2$ increases with z and there is no longer an analogy with an isothermal gas, but rather with one in which the temperature increases with z . This differentiation between velocity groups may manifest itself in several ways. For instance, variations will occur, with z , of the distributions of motions parallel to the

¹⁾ Here m is the mass of the gas particles, and k the Boltzmann constant.

galactic plane, and in the luminosity function, if these characteristics vary from one component to another. Such effects may become important when analysing proper-motion distributions, or apparent-magnitude distributions, in order to obtain the z -density distributions.

In principle, if $\Delta(z)$ is known, and if (as will usually be the case), $\varphi(z=0, |Z|)$ is not purely gaussian, there are two distinct methods by which K_z may be computed depending on the way in which the Z -motions are studied. Either $\bar{Z}^2$ may be derived as a function of z directly from the observational data, and K_z computed using (4c), or, the Z -motions exhibited by members of a group of stars, while in the galactic plane, may be analysed into gaussian components and equation (4b) solved for K_z when $\Delta(z)$ is known²⁾.

The first alternative is of no practical importance at the present time; the existing observational material is inadequate in both accuracy and quantity to be of much use. This point will be discussed in more detail later. However, the second method is not without its own difficulties — these will become clear later when the present calculations are described.

The principles upon which OORT has established his calculations have not passed without criticism of a definitely fundamental character. SCHILT (1950) has directed criticism at the combination of Z -motions of stars spread out over great distances in computing the Z -speed distribution at the galactic plane. It is suggested that the Z -speed distribution in the galactic plane is not continuous with a maximum frequency at the smallest speeds, but that it is a segmented distribution, i.e., one in which speeds are concentrated around certain discrete values. Groups of stars defined by these "allowed velocities" do not resemble open clusters since at any time members of any group are to be seen crossing the galactic plane with about the same speed but in opposite directions.

From SCHILT's viewpoint, the continuous Z -speed distributions arise from the filling up of gaps between the allowed velocities at the galactic plane by stars in differently placed segments away from the galactic plane. (The Z -speed distribution for $z \neq 0$ will also be segmented according to SCHILT's viewpoint, but the allowed speeds will be lower than those near the

²⁾ This solution may be performed in a direct way which is preferable to the iteration method used in the past. For, once $\varphi(z=0, |Z|)$ is given in the form (4a), $\Delta(z)$ may be computed as a function of $\int_0^z K_z dz$. Use of this relationship in

conjunction with the observed $\Delta(z)$ allows $\int_0^z K_z dz$ to be evaluated as a function of z . Differentiation with respect to z yields K_z as a function of z .

galactic plane because of kinetic energy expended in moving against the gravitational field).

If this new viewpoint is correct, then the z -distribution of stars with zero peculiar Z -speed should possess pairs of maxima symmetrically situated on either side of the plane of symmetry of the Galaxy. At the plane of symmetry the frequency of zero peculiar Z -speeds is a minimum — a maximum would be expected from continuous Z -speed distributions hitherto used. SCHILT has found several such pairs of maxima, which indicate that the plane of symmetry lies at a distance of about 6 pc north of the galactic plane through the sun. The amplitudes of the z -oscillations of the velocity groups discovered are quite small—of the order of 10 pc as measured from the plane of symmetry.

Comments upon SCHILT's criticism and theory have been reserved for the following section, since it is mainly in relation to the results he obtains for $\frac{\partial K_z}{\partial z}$ near the galactic plane, and the local density of material computed from it, that weaknesses in his argument are most apparent.

2. Previous computations of K_z

The first computation of this kind, that of KAPTEYN, was primarily aimed at reducing the relative values of K_z derivable from the star distribution given by KAPTEYN and VAN RHIJN to absolute values by providing the average masses of stars for various z -distances. These having been found, the gravitational field throughout the Galaxy was estimated assuming surfaces of constant star density to correspond to constant average stellar mass.

Unfortunately, when this work was completed, evidence indicating the presence of interstellar material was accumulating. It was found that this material produced considerable obscuration of star light — especially to the light which has to reach us after passing through large stretches of the Galaxy near the galactic plane, where the gas is strongly concentrated. Moreover it became clear that the luminosity function was by no means the same throughout the Galaxy—the distribution of stars in the Galaxy varies rather significantly with absolute magnitude. Both of these factors have necessitated an interpretation of the star counts which is completely different from that given by KAPTEYN and VAN RHIJN. The available star counts provided then and even now the basis of an adequate description of the stellar distribution only in regions relatively near the sun.

It now appears that the Galactic System is much larger than the KAPTEYN-VAN RHIJN model, although

similar in general outline. Furthermore, through the investigations by SHAPLEY of the globular clusters, and OORT's work on the differential galactic rotation, the sun was shown to be quite far from the galactic centre and in a region of large rotational velocity.

A detailed discussion of the quantitative aspects of KAPTEYN's K_z -calculations is, for the reasons just given, no longer worthwhile. The presence of interstellar absorption has rendered the objective of KAPTEYN's work impossible to attain at present by his methods and in its stead has placed emphasis on two points of considerable interest. In the first place the rate of increase in K_z for small z , (say, less than 100 to 150 pc) gives an estimate of the local density of material, both stellar and interstellar; this was recognized by KAPTEYN. And secondly, by studying K_z at large distances from the galactic plane (at z -distances of 1 kpc or more), general information is provided about those more central parts of the Galaxy¹⁾ of which direct observation is generally denied by the considerable absorption.

The first reliable K_z -computation was made by OORT (1932)—this provided data up to distances of 600 pc from the galactic plane. In this study the z -density distributions of stars of all spectral types in four ranges of absolute magnitude covering the interval between $M_{vis} - 1.5$ to $+5.5$ were used—these distributions were obtained from the work of VAN RHIJN (1925b). The Z -velocity distributions appropriate to each absolute-magnitude range for the galactic plane were derived by combining those of the individual spectral classes in proportion to their contribution to the stars in the galactic plane in the respective absolute-magnitude ranges. Z -velocity distributions representative of each spectral class and absolute magnitude were obtained from radial velocities of high-latitude stars. Since only radial-velocity material was used, conversion of each observation to a Z -velocity had to be performed by applying statistically determined factors to each radial velocity. These factors depended on the latitude of the star and the distribution of space motions of the

TABLE I
The smoothed K_z deduced by OORT (1932)

z (pc)	K_z (cm sec ⁻²)	z (pc)	K_z (cm sec ⁻²)
50	-0.88×10^{-9}	250	-3.79×10^{-9}
100	-1.74×10^{-9}	300	-3.81×10^{-9}
150	-2.57×10^{-9}	400	-3.87×10^{-9}
200	-3.43×10^{-9}	600	-3.96×10^{-9}

¹⁾ This has been done by construction of Galaxy models fitting K_z -data; see for example OORT and VAN WOERKOM (1941).

group (as defined by spectral class and M_{vis}) of stars concerned. The four Z -velocity distributions resulting from the combination of the contributions of each spectral class to the absolute-magnitude intervals were each represented in terms of several gaussian distributions.

The values of K_z reached by OORT in these computations are shown for several z -distances in Table 1; they are also illustrated later in Figure 14.

With the estimate of $-1.7 \times 10^{-11} \text{ cm sec}^{-2} \text{ pc}^{-1}$ for $\frac{\partial K_z}{\partial z}$ near the galactic plane given by Table 1, the computation of $\rho(0)$, the total density of material in the galactic plane may be carried out as follows. If Poisson's equation is applied to an element of volume in the galactic plane surrounding the sun, then in terms of the cylindrical co-ordinate system previously introduced, we have

$$\frac{\partial K_{\varpi}}{\partial \varpi} + \frac{K_{\varpi}}{\varpi} + \frac{\partial K_z}{\partial z} = -4\pi G \rho(0), \quad (5a)$$

where G is the gravitational constant. Because of the supposed cylindrical symmetry of the Galaxy about the axis $\varpi = 0$, $\frac{\partial K_{\theta}}{\partial \theta} = 0$, and no term involving K_{θ} appears in Poisson's equation. The first two terms of equation (5a) may be expressed in terms of the galactic rotation constants A and B as

$$\frac{\partial K_{\varpi}}{\partial \varpi} = (A - B)(3A + B), \quad \frac{K_{\varpi}}{\varpi} = -(A - B)^2. \quad (5b)$$

Then, if the galactic rotation constants are given the recently adopted values, $A = +6.3 \times 10^{-16} \text{ sec}^{-1}$, and $B = -2.2 \times 10^{-16} \text{ sec}^{-1}$ (VAN DE HULST, MULLER and OORT 1954), we then obtain from the above relations, $\rho(0) = 5.8 \times 10^{-24} \text{ g cm}^{-3}$. The primary cause for the uncertainty in the density so calculated

is due to the $\frac{\partial K_z}{\partial z}$ term in (5a); the first two terms are of minor importance and for regions near the galactic plane they may be considered to be known with ample precision.

This estimate of the total density may be compared with the most recent information of the stellar and interstellar make-up of the sun's neighbourhood.

The radio observations of the 21-cm emission line originating from interstellar hydrogen indicate an average density of interstellar hydrogen of 0.77 atoms cm^{-3} (or $1.3 \times 10^{-24} \text{ g cm}^{-3}$) near the sun—the addition of 10% to allow for ionized hydrogen will then give a total contribution from hydrogen of $1.4 \times 10^{-24} \text{ g cm}^{-3}$ (SCHMIDT 1956). Further, adopting the relative abundances of helium to hydrogen as found from the analysis of stellar atmospheres (UNDERHILL 1953), an

additional 15% should be added to include the contribution of interstellar helium. This brings the estimated density of interstellar material up to $1.6 \times 10^{-24} \text{ g cm}^{-3}$.

As for the stellar contribution, the recent data provided by GLIESE (1956) may be used. From a careful study of the stars within 20 pc of the sun, GLIESE finds that stars contribute at least about $3.9 \times 10^{-24} \text{ g cm}^{-3}$ to the local density. Since half of this density is contributed by intrinsically faint M dwarfs it is quite possible that this estimate of the stellar density is rather low because many of the M dwarfs within 20 pc have probably not been found as yet.

Thus the total density of material yielded by direct analysis is at least $5.5 \times 10^{-24} \text{ g cm}^{-3}$, as compared with the $5.8 \times 10^{-24} \text{ g cm}^{-3}$ found above using Poisson's equation with OORT's $\frac{\partial K_z}{\partial z}$. Such close agreement in

the resulting densities is undoubtedly fortuitous in view of the uncertainties in each.

In 1941, OORT and VAN WOERKOM made an estimate of the average K_z at distances from 2 to 10 kpc from the galactic plane using RR Lyrae variable stars. The radial velocities of 29 high-latitude variables were used to obtain the average Z -velocity of 57 km sec^{-1} . Assuming the Z -velocity distribution to be gaussian and adopting the density gradient $\frac{d \log \Delta(z)}{dz} = -0.22 \text{ kpc}^{-1}$ found by SHAPLEY (1939)

to apply over the above-mentioned distance range, OORT and VAN WOERKOM obtained an average K_z of $-8.4 \times 10^{-9} \text{ cm sec}^{-2}$. This determination is rather uncertain because of the assumption that the Z -velocity distribution is gaussian with $|\bar{Z}| = 57 \text{ km sec}^{-1}$ at all z -distances. For, with the OORT-VAN WOERKOM K_z of Figure 14 it appears that for an RR Lyrae variable to reach up to 2 kpc from the galactic plane it would require a Z -velocity of about 80 km sec^{-1} on crossing the plane. And since, according to Table 1 of OORT and VAN WOERKOM's paper, only 7 out of the 29 stars comprising the Z -velocity distribution have such velocities, the assumption made is not very strongly supported by the observational data used. Further, the addition of more recent radial velocities of RR Lyrae variables does not allow a more definite conclusion to be reached.

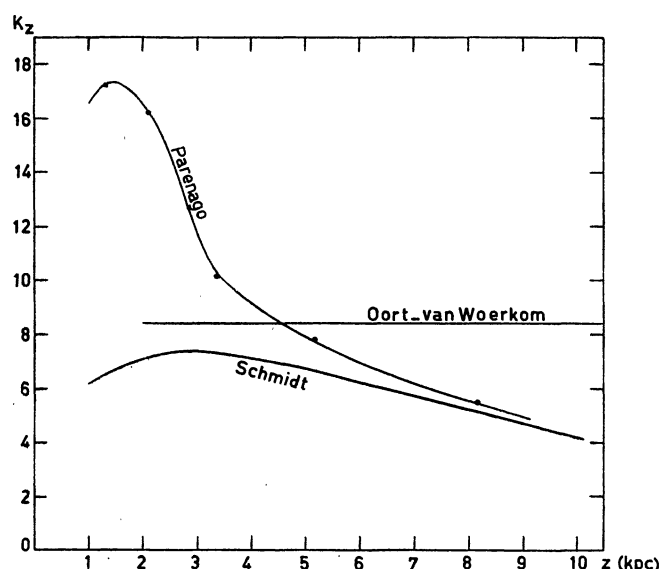
Much more recently, PARENAGO (1952) has reconsidered K_z between 0.8 and 8.1 kpc using RR Lyrae and long-period variables for which KUKARKIN has supplied density distributions. The average value of $\frac{d \log \Delta(z)}{dz}$ found for RR Lyrae stars in KUKARKIN's material is -0.23 kpc^{-1} , or -0.18 kpc^{-1} if the presumably less reliable densities of the nearest and farthest groups are omitted—these are in satisfactory

agreement with SHAPLEY's results. However for RR Lyrae stars, PARENAGO prefers to use $|\overline{Z}| = 70$ km sec⁻¹; again no account is taken of possible changes of $|\overline{Z}|$ with z -distance.

Another estimate of K_z over the large z -distances is given by SCHMIDT (1956) with his model of the distribution of mass in the Galaxy. From this work it is found that the average K_z is about 6.4×10^{-9} cm sec⁻².

Details of K_z as given by the combined results for RR Lyrae stars, and long-period variables as obtained from PARENAGO's data and from the OORT-VAN WOERKOM, and SCHMIDT computations are illustrated in Figure 1.

FIGURE 1



The results of various K_z computations for z -distances greater than 1 kpc. The points were calculated directly from PARENAGO's material and later smoothed to the curve shown.

Some mention must now be made of the results provided by SCHILT's calculations already discussed in section 1. From an estimate of the rate of change in $\overline{Z}^2$ with z for stars comprising one of the velocity groups, SCHILT obtains $\frac{\partial K_z}{\partial z} = -1.5 \times 10^{-9}$ cm² sec⁻² pc⁻¹ for distances less than about 10 pc from the plane of symmetry. No further data regarding K_z at greater distances from the galactic plane could be ascertained, presumably because it was not possible to get sufficiently accurate mean square velocities from the few members of the group observed at the greater distances.

The value of $\frac{\partial K_z}{\partial z}$ reached by SCHILT corresponds to a total density $\rho(0)$ of 5.3×10^{-22} g cm⁻³, which is

about 100 times larger than the density obtained by OORT. As has since been pointed out by OORT (1952), acceptance of this very high density would incur several difficulties. One of the most serious difficulties is with the existence of material which would produce such a high density—the estimates of the local contributions of known stellar and interstellar matter already given are perhaps somewhat low, especially the stellar contribution, but it seems inconceivable that together they fall short of the real density by a factor of 100. Moreover even if it is supposed that the high density only occurs in a very thin layer of thickness about 10 pc, its mode of formation is difficult to envisage. Presumably the material in such a layer must be either gaseous, or of gaseous origin, and in either of these cases the average thickness would be expected to be at least equal to that of the known gas layer, i.e., about 200 pc, or more. Finally, the means whereby the “segmented Z -motions” might originate are difficult to envisage. It seems impossible to suppose them as having been derived from open clusters whose members have in the course of time drifted apart from one another because the Π - and Θ -motions apparently show no segmentation.

Thus the present situation with regard to SCHILT's results would seem to be that there is no known reason for expecting the high density which he obtains (even if it is considered to be very localized), or for explaining the segmented velocity distributions. On the other hand, the calculations carried out with continuous velocity distributions yield densities which are in reasonable accord with independent direct estimates.

3. Some introductory remarks regarding the present work

The impetus for a new calculation of K_z has been provided by the publication of results from extensive proper-motion surveys during the last 15 or 20 years, together with the steady increase in our knowledge of radial velocities and space motions of stars. In the present investigation an attempt has been made to make the fullest possible use of this new material in improving and extending the existing information about K_z .

The description of the work is given in Parts II and III of the paper. Part II is concerned with the collection and reduction of all the observational material preparatory to the K_z -calculation proper, which is described in Part III. In Part II, the description has been divided into division (a) dealing with proper motions and star counts, and division (b) where velocity distributions are considered. To the two divisions of Part II, and to Part III, introductory sections have been inserted in order to explain briefly

the most important details involved in the ensuing discussions. However as the basic methods employed to get distance distributions of stars in the work differ somewhat from those hitherto used, a brief introductory survey of the calculation as a whole seems desirable at this early stage of the description. The remaining introductory sections may be used to obtain some idea of what details are involved in the actual calculations without having to delve into the detailed accounts of the remaining sections.

The decision to make the Henry Draper K stars the subject of the investigation in preference to other spectral classes was reached from the following considerations. In the first place K stars occur in relatively large numbers at high galactic latitudes so that it is possible to form distributions of apparent magnitude, proper motion, etc., which are more reliably defined than those which could be obtained for most other spectral classes. Secondly, there is likely to be a large proportion of giant stars amongst even the 13th and 14th magnitude K stars; this accentuates the advantages accruing from studies of high-latitude stars in obtaining the requisite density distributions out to great distances from the galactic plane.

The initial step in calculating K_z has been to compute the fraction of K-giant stars which are included in counts made of high-latitude K stars. This has been done with the aid of observed total proper-motion distributions (corrected for the influences of accidental errors of observation) compiled for K stars of various apparent magnitudes¹⁾.

The distributions of μ are represented as well as possible by combining distributions computed for K stars in several luminosity groups at various apparent magnitudes for which proper-motion distributions are available. By assessing the proportions in which the computed distributions are combined in order to give the best fit of each observed distribution,

¹⁾ It is advantageous for two reasons, to use distributions of the total proper motion μ , instead of $\mu_\alpha \cos \delta$, μ_δ or some other component. In the first instance the use of distributions of some component of proper motion entails either the laborious computation of the required component from the observed components of each star, or the equally laborious task of computing the transverse-velocity distributions which the proper-motion components reflect, from the space-motion distribution. And secondly, the total proper-motion distribution contains in a concise form practically all the information which can be conveyed by measures of the apparent motion of stars. The only item of information omitted with the one-dimensional total proper-motion distribution is the directional distribution of the total proper motions. Although it is possible that the directional distribution of total proper motions might convey some information regarding the distances of stars, through perhaps the variation of the vertex direction with z -distance, it is not possible to use this approach at present.

and combining the results with general counts of K stars at high latitudes, it is possible to determine the numbers of K stars of each of the luminosity groups. Of these apparent-magnitude distributions, that belonging to the giant stars is likely to be the most reliable.

In computing the total proper-motion distributions just mentioned, due regard has been paid to the variation in the transverse-velocity distribution with distance from the galactic plane. The computed proper-motion distributions are naturally also dependent on a luminosity function, and to a lesser extent on the effects of interstellar absorption, and an estimate of the gradient in the z -density distribution. For simplicity, the change in the transverse-velocity distribution with distance from the galactic plane has been treated in the following way. The Z -velocity distribution of K stars regardless of luminosity, crossing the galactic plane, has been represented by three gaussian distributions having different dispersions, two being in equal proportions whilst the third, the one with the highest $|Z|$ occurs in a much smaller proportion. Details of $|Z|$ for the first two groups have been obtained from earlier work of Miss ROMAN, whilst the proportion and average velocity of the high-velocity group has been deduced from a special investigation in which use is made of the increase in the average Z -velocity with z -distance, as exhibited by the radial velocities of high-latitude K stars. An estimate of K_z has had to be used in this study, so that the outcome is to some extent dependent on this approximation. The increase in the average Z -velocity with z comes about with the seeding out by the gravitational field of the lower Z -velocity groups and results in the occurrence of a larger and larger proportion of the high Z -velocity group as one goes away from the plane. With the proportions of the three groups as seen in the galactic plane, together with their average Z -velocities, and the approximate field, the proportions of the three groups present at various distances from the plane may be computed. Since the characteristic peculiar space motions and solar motion, like the Z -motions, differ from group to group, and may be estimated, the changes in the motions of K stars with z may be computed approximately.

With the aid of the Z -velocity distribution in the galactic plane as found in the manner just described, and a luminosity function for K giants, which is supposed to remain independent of distance because of its small dispersion, and a knowledge of the interstellar absorption as a function of distance, a K_z may be readily obtained which gives the best agreement with observed giant-star counts.

Then in principle the effect of replacing the tentative K_z used in earlier parts of the computations

with the newly derived K_z may be examined and the K_z further improved in an iterative fashion. In practice however, such refinements are unnecessary; existing approximations to K_z appear sufficiently accurate to make the outcome of the preliminary steps wherein a tentative K_z is involved sufficiently insensitive to inaccuracies in the latter.

II. THE COLLECTION AND PRELIMINARY ANALYSIS OF OBSERVATIONAL MATERIAL

(a) *The Total Proper-Motion Distributions and Star Counts*

4. Introduction

In order to conform with the general principles already laid down for the density calculations, distributions of total proper motion, each of which is representative of K stars having a particular apparent magnitude, must be obtained. Ideally, stars comprising these distributions should be drawn from relatively localized regions of the sky lying as near to the galactic poles as is possible. In this way the material becomes very homogeneous, and complicated calculations needed to take into account the effects of variation, from one part of a large sky zone to another, in the total proper-motion distributions, are avoided. Furthermore, the limitation to polar regions ensures that densities are obtained up to the greatest possible z -distances. However in practice a compromise must be made between the desire to meet these needs and the necessity for having sufficient stars to give reliable distributions. Even amongst the numerous faint stars this situation prevails because proper motions are only available for relatively small numbers of stars in sparsely distributed Selected Areas and other standard regions. In the present case this has led to the inclusion of all K stars within 50 degrees of the galactic poles, and for which the necessary data are available.

Distributions characteristic of K stars having a certain apparent magnitude have been obtained by grouping the material according to apparent magnitude and attributing the total proper-motion distribution thus found to K stars with the mean apparent magnitude of the group. In doing this it is advantageous for several reasons, to keep the apparent-magnitude intervals as small as possible. In the first instance the smaller the apparent-magnitude range, the nearer will the proper-motion distribution approximate to that of stars with the mean apparent magnitude. And secondly, in subsequent density calculations, the luminosity function may be used directly without any modifications; otherwise the

luminosity function would have to be spread out further by an amount dependent upon the spread of apparent magnitudes about their mean.

In fulfilling the above requirements a number of features of considerable importance relating to the quality of the observational material must be considered in some detail. The most important factors are brought together briefly at this stage in the interests of clarity although they will be discussed in more detail in ensuing sections.

First and foremost, considerable care must be devoted to the selection of a spectroscopically homogeneous group of stars over the wide range of apparent magnitudes from which the stars are to be drawn. It is absolutely necessary that stars with essentially the same physical and kinematical characteristics enter into the material.

Ideally it would be desirable to select stars on the basis of accurate spectral and, if possible, luminosity classifications from one system which has been constructed in such a way as to be as free as possible from a magnitude equation. Unfortunately, for many practical reasons, such an ideal has hitherto not been achieved. As a consequence there are in existence a number of systems of spectral classification which are by no means identical. In extensive statistical studies the *Henry Draper Catalogue* (subsequently referred to as the HD) has come to be the authority for spectra of the brighter stars, whilst for the faint stars a number of classification systems have been devised and applied to stars in numerous small zones of the sky. Among the more extensive catalogues of faint star spectra one might mention the Bergedorf system in the *Bergedorfer Spektral Durchmusterung* (BSD) associated with KAPTEYN's Selected Areas, and the McCormick Observatory system in McCormick sky zones.

The faint-star systems of classification have diverged somewhat from the HD. This is partly because the spectra are necessarily of different quality requiring different criteria, but also because ideas regarding classification criteria themselves and the methods of applying them have changed from time to time since the HD catalogue was completed. Furthermore, since the aim in recent years has been to extend classifications to very faint stars, conditions are not favourable for classification of the few bright stars on the faint-star fields. Thus the overlap between bright- and faint-star classifications is usually insufficient to obtain an adequate intercomparison between the systems. Several special investigations have provided some inter-relationships between classifications which allow a fairly homogeneous group of stars very similar to HD K stars to be extracted from various spectral catalogues. However, it will become clear later that, despite the care taken in relating different systems of classification, there is still much left to be desired; the

inter-relationships either cannot be, or are not sufficiently well defined¹⁾.

Turning now to the magnitudes, the main requirement here is that they be on a normal system or readily convertible to such a system in order that distance distributions subsequently computed should be reliable. The ideal arrangement would be to specify magnitudes on the carefully constructed international photovisual or photographic scales. This entails, particularly for the brighter stars,

¹⁾ There is some reason to believe that the means for approaching very close to the spectral requirements of extensive statistical investigations are near at hand, although it may be many years before the results of their application to large numbers of stars are available. Descriptions of two new methods whereby both spectral and luminosity classifications can be given for normal stars are given by STRÖMGREN and GYLDENKERNE (1955) and CHALONGE and DIVAN (1952). Of the two techniques the former seems the most readily applicable to faint stars.

corrections to the available magnitudes which are on old and rather poorly defined scales. The more recent photovisual and photographic magnitudes provided for faint stars require no modification on this account.

Regarding the proper motions themselves, a decision should be made at the outset as to which system of proper motions to employ—this must be made primarily on the basis of freedom from systematic errors. Once this has been decided, systematic corrections have to be applied to catalogue motions to bring them into the chosen system. Generally, for absolute motions, these corrections have been published, but many faint-star motions are as yet merely relative. For such motions, estimates of the necessary corrections were made using mean parallaxes of reduction stars derived from studies of their peculiar motions.

Having determined distributions of the observed motions, freed as far as possible from systematic

TABLE 2
The proper-motion material
(See text for detailed explanation)

proper-motion catalogue ²⁾ (system of motions)	sky zones	$ b $	m -interval (mag. system)	m -groups	$\bar{m}$	n	probable errors in $''/a$
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
(a) General Catalogue (GC system)	$ b \geq +40^\circ$	57° 57°	$3.5 \leq m_{vis} \leq 6.4$ (Harvard visual scale)	4.5 to 5.4 5.5 to 6.4	5.1_{vis} 6.1_{vis}	178 508	± 0.002 (approx.)
(b) Yale Catalogue (Yale system)	$ b \geq 50^\circ$, $8 + 10^\circ$ to $+30^\circ$ (Zone A), $8 - 10^\circ$ to $+10^\circ$ (Zone B)	Zone A: 70° Zone B: 60°	$6.5 \leq m_{vis} \leq 9.4$ (various AGC scales)	6.5 to 7.4 7.5 to 8.4 8.5 to 8.9 9.0 to 9.4	6.9_{vis} 8.1_{vis} 8.9_{vis} 9.4_{vis}	A 155 555 760 640 B 192 753 1507 1096	± 0.007 ± 0.007 ± 0.007 ± 0.007
(c) McCormick Catalogues ³⁾ (FK3 system)	266 McC zones $ b \geq 40^\circ$	54° 54°	$9.0 \leq m_{pg} \leq 11.9$ (McCormick)	9.0 to 10.4 10.5 to 11.9	9.8_{pv} 11.3_{pv}	304 462	± 0.008 ± 0.008
(d) Groningen Catalogue (relative motions)	8 Selected Areas $ b \geq 48^\circ$	59° 59°	$10.5 \leq m_{pg} \leq 13.4$ (Bergedorf)	11.5 to 12.4 12.5 to 13.4	12.0_{pg} 12.9_{pg}	253 181	± 0.0032 ± 0.0032
(e) combined Radcliffe- Pulkovo Catalogues (relative motions)	41 Selected Areas $ b \geq 40^\circ$	55° 55° 55°	$10.5 \leq m_{pg} \leq 14.4$ (Radcliffe)	11.5 to 12.4 12.5 to 13.4 13.5 to 14.4	12.0_{pg} 13.0_{pg} 13.8_{pg}	90 112 116	± 0.0030 ± 0.0025 ± 0.0030

²⁾ The following are the references for the catalogues listed above.

(a) B. BOSS, *General Catalogue*, published by Carnegie Institution, Washington, 1937.

(b) *Yale Transactions*, Volumes 10, 16 to 22, and 24.

(c) *Publ. Leander McCormick Observatory*, Volumes VII, 1937 and X, 1948.

(d) P. J. VAN RIJN and L. PLAUT, *Gron. Publ.* No. 56, 1955. This material was kindly supplied to the author before its publication.

(e) The combined motions computed at the Leiden Observatory (see *B.A.N.* 10, 9, 1943; No. 362), have not been published.

References for the individual catalogues are:

Radcliffe: Oxford Univ. Press, London 1934.

Pulkovo: *Publ. Poulkova*, Série 2, IV, 1940.

³⁾ In view of the unreliability of motions for many stars of the first catalogue situated near the edges of plates (see A. N. VYSSORSKY, *Publ. Leander McCormick Obs.* 10, 3, 1948) stars within one centimetre of the plate edges were omitted from this catalogue.

errors, it is necessary to correct them for the effects of accidental errors in the observations. This correction may become particularly important for faint-star distributions because the dispersion of accidental errors sometimes approaches parity with the dispersions in the motions themselves.

5. Proper-motion material, sky zones, magnitude groupings, and spectra

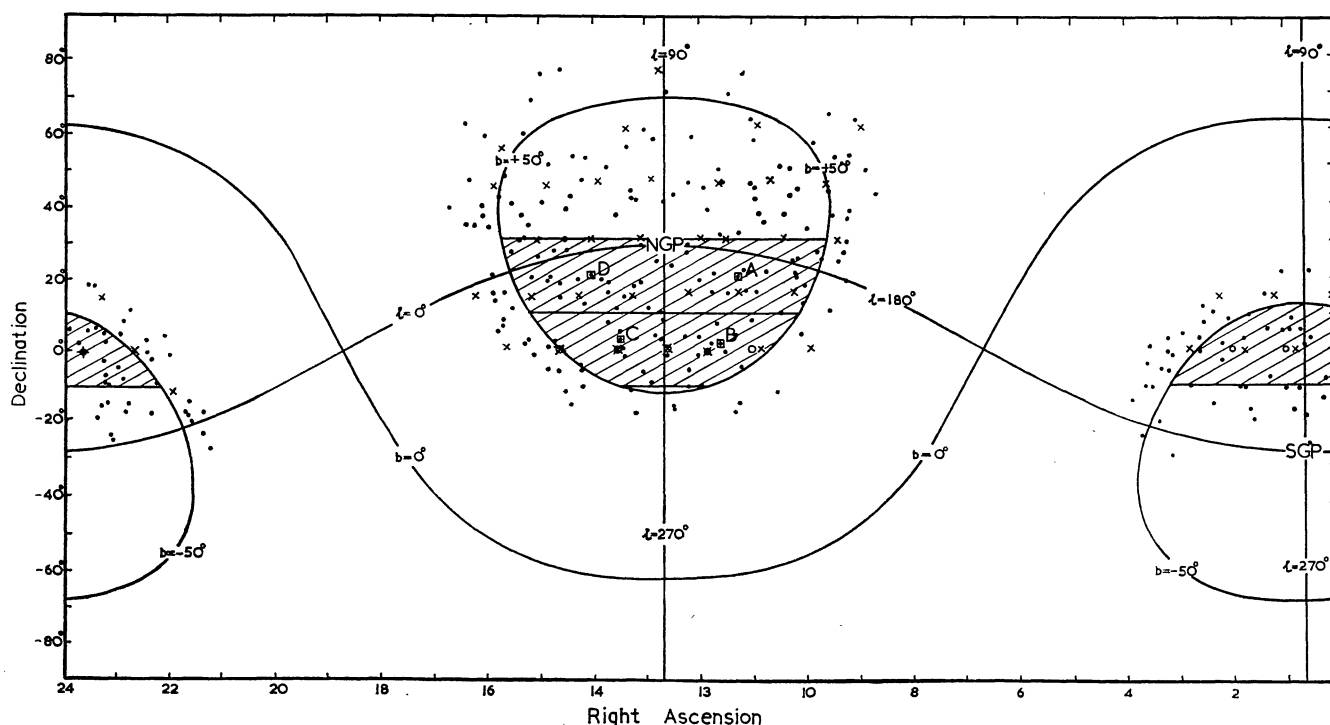
In Table 2, the proper-motion catalogues which have been used, the sky zones and their average galactic latitude, together with the apparent-magnitude intervals (given in terms of the catalogue magnitudes) from which K stars have been selected, are indicated in columns 1 to 4 respectively. Column 4 also contains the catalogue magnitude system, whilst column 5 shows the final magnitude groupings and column 7 the number of stars in the various groups.

The catalogues listed in Table 2 do not exhaust the proper-motion material which might have been used. For example, many more stars might have been obtained from other declination zones of the Yale Catalogues. These motions are generally much less

accurate than those used and their incorporation in the data already collected would not have contributed markedly to the precision of the proper-motion distributions. Other useful lists of proper motions undoubtedly exist, but no thorough-going search was made for them since the material already gathered was as much as could be handled.

Of the 17 proper-motion distributions listed in Table 2, three might conceivably be subject to selection effects. These are the Yale $9^m.0$ to $9^m.4$ distributions and the McCormick $10^m.5$ to $11^m.9$ distribution. The reason is that these distributions may possibly contain an abnormally large percentage of stars with relatively low luminosity. For the Yale stars the magnitude limit of the Yale photographic resurvey of the *Astronomische Gesellschaft Katalog* (AGC) stars has not been sufficiently faint to cover all faint stars in that catalogue. Thus, due to their smaller colour-index relative to giant stars, dwarfs and subgiants will tend to be more prevalent in this visual-magnitude range than in reality. A similar situation prevails in connection with the faintest McCormick group where the photographic limit of the spectral survey has been insufficient to give complete classifications of proper-motion stars. In

FIGURE 2



A sky map showing, for each catalogue used, the regions of the sky from which proper motions have been taken. The shaded areas refer to the Yale catalogues, the 266 dots to the McCormick zones, while the 8 small circles, and 41 crosses indicate Selected Areas from Groningen, and Radcliffe-Pulkovo motions originated. GC motions of the bright stars were taken from the polar caps, $|b| \geq 40^\circ$. The four $\square$ show the directions of the normals to the four planes into which strong- and weak-line star space motions were projected in forming the transverse-velocity distributions of Figure 7.

both cases it is impossible to assess the extent to which the proper-motion distributions are influenced by these effects so that from the present viewpoint they must be considered with caution. Later, in section 9, another reason is given for completely deleting both McCormick proper-motion distributions. The Yale material however has been retained in the final density analysis since no manifestation of the above influences was subsequently revealed in determining the percentages of giants among these stars.

The sky zones, which are shown in Figure 2, are rather extensive; this is unavoidable if proper-motion distributions possessing sufficient precision are to be used. For this purpose it has also been necessary to combine material from both the north and south galactic polar zones. The proximity of the sun to the galactic plane should make such a procedure justified since accurate densities in regions as close as 100 pc are not to be expected.

Finally, the aim has been to choose stars most nearly approximating to Henry Draper K stars from proper-motion catalogues. No difficulties are provided in this respect for GC stars since HD classifications are available for all of them. Concerning spectra to be found in the YC, the principle adopted by compilers of the catalogue was to use HD spectra wherever possible. For the faint stars, $m_{vis} \geq 8.5$, where the HD is far from complete, additional data were used. These were supplied chiefly by the McCormick Observatory; however, some were provided by Miss CANNON and GOEDICKE, whilst a few others were taken from the extension to the HD. For all these classifications, systematic differences with respect to the HD system have already been derived by VYSSOTSKY (1941a), MATEER (1948) and GOEDICKE (1944). These allowed HD K stars to be fairly reliably chosen from the classifications available for faint Yale stars. On the basis of VYSSOTSKY's work (VYSSOTSKY 1941b) it was decided to use all McCormick K stars, whilst from the Bergedorfer spectra for Selected Area stars, G8 to K9 stars were taken to correspond to K in the Henry Draper system.

The term K stars as used throughout this work refers to stars classified as K in the HD and the McCormick catalogues, and as G8-K9 in the BSD.

6. Counts of Henry Draper K stars

In preparation for subsequent density calculations, the numbers of K stars, $A(m)$, per square degree of sky, per unit interval in apparent magnitude are required. These counts should be computed for each individual proper-motion distribution in order to allow for differences in average galactic latitude from group to group. Strict adherence to this precept is unnecessary since the average galactic latitudes do

not differ sufficiently from one another. Thus star counts have been deduced for the entire apparent-magnitude range for galactic latitude 57° which is sufficiently close¹⁾ to the straight mean of the average latitudes of the individual distributions (the two McCormick distributions being ignored).

For stars brighter than $m_{vis} 9.5$, data provided for K stars by VAN RHIJN (1925a, see Table I) have been used. Amongst the brightest stars, where the HD is complete, VAN RHIJN's K-star frequencies are based directly on counts from the HD, whilst at the fainter magnitudes the frequencies were derived by combining the percentage of K stars as given by the HD with total star counts obtained elsewhere. It is quite possible that at about $m_{vis} 9.0$ the K-star frequencies given by VAN RHIJN are faulty. In the first place the influence of overlapping spectra on the percentage of K stars has not been ascertained and furthermore the total star counts contain considerable corrections made to compensate for distortion in the magnitude scale of the old catalogues upon which they are based. A re-investigation of these possible defects was out of the question. As regards the K stars brighter than $m_{vis} 5.0$ the average star counts over the entire sky have been used to diminish the uncertainties occurring if only the few bright high-latitude stars had been used. It is easily shown that the latitude variation is here so small as to be of no importance. For the fainter visual magnitudes, the frequencies adopted are those referring to stars from $|b| 40^\circ$ to 90° ($|b| = 57^\circ$) given by VAN RHIJN. Over the entire visual-magnitude range the K stars amount to about 32% of all stars. The visual-magnitude star counts so obtained are illustrated in Figure 3.

To ascertain the number of K stars at photographic magnitudes it was necessary to use the spectral classifications in the BSD for 43 Selected Areas having $|b| > 40^\circ$ and $|b| = 57^\circ$. But since the BSD begins to be incomplete at about $m_{pg} 12.5$, it was impossible to rely on it for the total numbers of K stars. Consequently the BSD was used to obtain the percentages of K stars (BSD G8 to K9) for various values of m_{pg} , then the required star counts could be reached with the aid of VAN RHIJN's (1929, see Table 6) smoothed total star counts for $b = 57^\circ$.

The percentages of K stars so derived are shown in Table 3 together with K-star counts deduced from these percentages, and total star counts given in the second column. Beyond $m_{pg} 12.5$ the classifications become increasingly uncertain (cf. the last paragraphs of this section). Less weight should therefore

¹⁾ Although the straight mean of the average galactic latitudes without the McCormick data is nearly 60° , much of the computation had been done before McCormick data were rejected and $|b| = 57^\circ$ has been retained — this will produce no significant inaccuracies in the results.

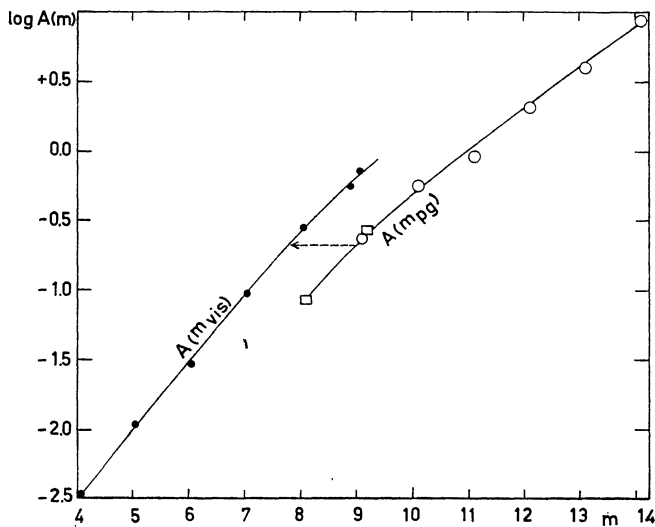
TABLE 3

Counts of K stars for photographic magnitudes at $|b| = 57^\circ$

m_{pg}	$\log A(m)$ all spectra	% K stars (BSD G8 to K9)	$\log A(m)$ K stars
9.1	-0.05	27%	-0.62
10.1	+0.34	26	-0.25
11.1	+0.74	17	-0.03
12.1	+1.07	18	+0.32
13.1	+1.40	16	+0.60
14.1	+1.69	18	+0.94

be given to the percentage and consequent K-star frequency at m_{pg} 13.1, while those at 14^{m.1} have hardly any significance. The photographic counts of K stars are shown in Figure 3 by circles. There are two additional frequencies marked on this figure; the two squares are results for HD K-star frequencies at

FIGURE 3



Observed counts of K stars at $|b| = 57^\circ$. The visual magnitude counts are based on the HD, and the photographic on the BSD with the assumption that BSD G8 to K9 stars are equivalent to HD K0 to K9. Two additional frequencies for $m_{pg} = 8.1$ and 9.1, marked by squares, were computed from the visual star counts. The two curves are taken as the representation of the two sets of observational data — the photographic and visual.

$m_{pg} = 8.1$ and 9.1 computed from the visual-magnitude counts supposing an average colour-index of $+1.2$. This colour-index is based on the conclusion, reached later, that by far the greatest number of bright K stars are giants; the colour-index is taken from a compilation by ALLEN (1955).

For the present calculation it was not necessary to have the frequencies of such photographically bright stars as some just considered. However it is worthwhile to see that there are no indications of important inconsistencies between the visual and photographic counts. The above calculations show no inconsistencies when one considers the colour-index of stars with m_{vis} 8, where the smoothed curves representing

visual and photographic counts indicate an average colour-index of $+1.3$. It is true that at m_{vis} 9 these curves would imply an excessive colour-index of $+1.4$; however the uncertainty in the visual star counts, alluded to earlier, gives little weight to this comparison. Readings from the smoothed star-count curves shown in Figure 3 are given in Table 4.

TABLE 4

Adopted K-star counts at $|b| = 57^\circ$

m_{vis}	$\log A(m)$	m_{pg}	$\log A(m)$
4.0	-2.50	9.0	-0.68
5.0	-2.01	10.0	-0.32
6.0	-1.52	11.0	0.00
7.0	-1.04	12.0	+0.31
8.0	-0.59	13.0	+0.60
9.0	-0.20		

Note: $\log A(m)$ given above for m_{vis} 8 and 9 are compromises between the $\log A(m_{vis})$ inferred from the photographic counts and those obtained from direct counts.

Before concluding this discussion some concrete example of the type of difficulties which are met in the work just described might be of use. In Table 5 are set out the percentages of BSD G7 to K9 stars for several intervals of photographic magnitude as obtained independently by the BSD observers, SCHWASSMANN and WACHMANN¹⁾. The comparison

TABLE 5

The percentage of faint BSD G7 to K9 stars according to SCHWASSMANN and WACHMANN

m_{pg}	SCHWASSMANN	WACHMANN	Mean
10.5 to 11.4	18%	22%	20%
11.5 to 12.4	18	27	22
12.5 to 13.4	18	32	25

between the results of the two observers shows clearly the large discrepancies that may arise between two observers attempting to apply the same criteria to the classification of faint stars. Furthermore the average percentages shown in the last column of Table 5 (which would probably represent closely the percentage of G7 to K9 stars as found from the BSD), may be contrasted with the average of 17% for G8 to K9 stars found above. This shows clearly that the effect of a possible uncertainty of one-tenth of a spectral class in conversion from the HD system to the BSD can be considerable — possibly of the order of 30%, i.e., ± 0.12 in the logarithm of the number of stars.

The example just considered shows clearly the

¹⁾ These results have been deduced from VAN RHIJN and SCHWASSMAN (1935).

extent to which several types of uncertainties may influence any attempt to assess the number of stars (and not only faint stars) of a definite spectral type, or the selection of a spectroscopically homogeneous group of stars, with present-day material.

7. Mean apparent magnitudes

Whilst magnitude groups listed in Table 2, column 5, have been defined in terms of magnitudes, found in each catalogue, it is necessary, for reasons given earlier, that the average apparent magnitude of each group be in a normal system. In this respect neither the Harvard visual magnitudes of the GC stars, nor the photovisual and photographic magnitudes should require significant modification. Some adjustment has however been needed for magnitudes of Yale stars. Here the apparent magnitudes, which have been copied from the AGC, form rather an inhomogeneous system, being, in general, measurements made by the original AGC observers. Quite often these magnitudes have never been related to any extensive magnitude system in common use. In view of these circumstances the following method has been used to find the average apparent magnitudes on the Harvard visual scale for each AGC magnitude group.

Where possible, for the three brightest magnitude groups of a Yale declination zone, the catalogue magnitude limits were converted, via the *Bonner Durchmusterung* and its *Southerly Extension* into the Harvard visual scale using PANNEKOEK's corrections (PANNEKOEK 1924, see Tables 26 and 40). Where no conversion to the BD was given with the AGC, conversion to the Harvard scale was carried out under the assumption that the BD and AGC scales were identical.

Having the apparent-magnitude limits on the Harvard scale, mean apparent magnitudes could be computed readily from the K-star apparent-magnitude distribution given in Figure 3, since the visual-magnitude scale is that of Harvard. This could also be done with the GC material. For the faintest group of Yale stars, which is far from complete, the mean AGC magnitude has been computed from individual magnitudes for each declination zone and converted by one or other of the above-mentioned methods into the Harvard scale.

Concerning all remaining distributions, mean apparent magnitudes have been calculated from individual magnitudes—no corrections have been made for conversion to the international scales since the catalogue magnitudes were deemed sufficiently accurate for the present purposes. Mean apparent magnitudes finally adopted for each magnitude interval have been listed in column 6 of Table 2.

It should be noted that in several cases the extent of the magnitude groups in column 5 of Table 2 is more than the usual one magnitude which was chosen as a compromise between homogeneity and the need for a sufficient number of members. In all these instances, proper motions in the extreme parts of the magnitude interval measured from the adopted mean have been multiplied by factors to convert them to motions expected of them had they the mean apparent magnitude. Thus proper motions of stars in the interval of m_{vis} 3.5 to 4.4 have been multiplied by 0.63 before combining them with the motions in the 4^m.5 to 5^m.5 group. Groningen and Radcliffe-Pulkovo motions for the range m_{pg} 10.5 to 12.4 and McCormick m_{pv} 9.0 to 10.4 and 10.5 to 11.9, are motions combined in a similar fashion.

In concluding this section some comments might be added about the treatment of magnitudes. Firstly, it is undoubtedly true that the whole treatment could be improved upon; particularly that part concerning AGC magnitudes. In this case grouping might have initially been made in terms of BD magnitudes available for all Yale stars, the PANNEKOEK tables then being used for the conversion to the Harvard scale. Such a procedure would have been very lengthy and would not improve the results materially. Similarly the other magnitude systems, including the Harvard magnitudes, might have been converted to the international photovisual or photographic scales. Here again the results would not seem to have warranted the additional labour involved.

8. Systematic corrections applied to proper motions

The system of proper motions adopted is that of the *Dritter Fundamental-Katalog* (FK3) with the modifications to the precession constants¹⁾ used by BLAAUW and DELHAYE (1949) and with the small galactic-rotation effects removed. Since galactic rotation does not produce a distance-dependent effect on proper motions, it seemed simplest to eliminate it by means of small corrections to proper motions. Retention of galactic rotation in proper motions would require considerable modifications to transverse-velocity distributions in order to insert galactic-rotation effects.

Table 1 of BLAAUW and DELHAYE's paper allows GC motions used for the brightest stars to be immediately converted to the adopted standard system. With the exception of Volume 24, motions from the YC may similarly be converted once corrections

¹⁾ These authors have adopted the corrections $\Delta k = -0''/a.0044$ and $\Delta n = +0''/a.0052$ to the precession constants used in constructing FK3 motions. $A = +0''/a.0040$, and $B = -0''/a.0017$ are the constants by which galactic-rotation effects have been removed from proper motions.

given with each volume for conversion to the GC had been applied. Volume 24 of the YC gives motions in the FK₃ system so that in this case the only corrections required are those for precessional constants and removal of galactic rotation. Precession corrections were obtained using

$$\Delta(\mu_{\alpha} \cos \delta) = -\Delta k \cos \delta - \Delta n \sin \alpha \sin \delta,$$

and

$$\Delta \mu_{\delta} = -\Delta n \cos \alpha,$$

where α and δ are the co-ordinates of the centre of the sky zones over which the correction is reasonably constant. Galactic-rotation effects were removed with the aid of Table 2 in BLAAUW and DELHAYE's paper.

For each declination zone, corrections to Yale motions were averaged over intervals of right ascension and the average corrections applied to all stars in each sub-zone so formed. Table 6 gives the average corrections deduced in this manner for each Yale volume used.

TABLE 6
Corrections for converting Yale proper-motion components into the modified FK₃ system.
(unit $\text{o}''/\text{a} \cdot 001$)

R.A. Interval Yale volume	$1^h \rightarrow$ $\Delta(\mu_{\alpha} \cos \delta) \Delta \mu_{\delta}$		$\rightarrow 12^h 30^m$ $\Delta(\mu_{\alpha} \cos \delta) \Delta \mu_{\delta}$		$12^h 30^m \rightarrow$ $\Delta(\mu_{\alpha} \cos \delta) \Delta \mu_{\delta}$		$\rightarrow 1^h$ $\Delta(\mu_{\alpha} \cos \delta) \Delta \mu_{\delta}$	
24			+ 4	+ 5	+ 4	+ 5		
10			+ 3	- 6	+ 3	- 4		
18			+ 4	+ 1	+ 5	+ 4		
19			+ 6	0	+ 5	0		
22 II	+ 2	+ 11	0	- 1	+ 2	+ 2	+ 1	+ 12
22 I	- 5	+ 3	0	- 1	+ 1	- 6	0	+ 5
20	- 3	+ 4	+ 3	- 1	0	+ 2	- 2	+ 7
21	+ 4	- 2	+ 6	+ 7	+ 7	+ 7	+ 10	- 2
17	- 1	0	+ 6	- 3	+ 2	- 3	- 2	+ 2
16	+ 2	+ 4	+ 9	- 5	+ 6	- 6	- 1	+ 2

Note: The arrow ($\rightarrow$) before the α means that the corrections in the column beneath are an average to that right ascension from the nearest zone limit at $b = \pm 50^\circ$ in the direction of smaller right ascension. Likewise an arrow after the α refers to the nearest zone extending to the zone limit at $b = \pm 50^\circ$ in the direction of increasing right ascension.

McCormick relative proper motions may be treated in the same way as Yale Volume 24 since VYSSOTSKY has given corrections for conversion to the FK₃ system.

As regards the Groningen relative proper motions, and the combined Radcliffe-Pulkovo relative motions, systematic corrections for conversion to the Radcliffe relative system were first applied. For each Selected Area in the Groningen material, the required corrections were kindly supplied by Professor VAN RHIJN and Dr PLAUT—they have not been published. Corrections to the combined Radcliffe-Pulkovo motions are readily deduced from data published by BINNENDIJK (1943). The final step was to convert these relative motions, which are now in the Radcliffe relative system, to the standard absolute system by adding the systematic motion of Radcliffe comparison stars which have a mean apparent magnitude of 13.7 pg.

But before carrying out this last step a modification was made to the Radcliffe system of relative motions. This involved inclusion of a correction to the catalogue motions in order to correct for the omission of

comparison stars with total proper motions exceeding $\text{o}''/\text{a} \cdot 05$ in the Radcliffe reductions. A rough but sufficiently accurate estimate of the necessary correction was made in the following manner.

For each Selected Area from which proper motions were taken, the proportion and average relative proper motion of all stars (regardless of spectral type) having apparent magnitudes in the range of $m_{pg} 12.7$ to 14.7 , and total proper motions exceeding $\text{o}''/\text{a} \cdot 05$, were computed and the results combined. In this way a significant average measure of the modification required by the introduction of high-proper-motion stars, which were rejected in the Radcliffe reductions, could be ascertained. It was found that 6% of the reduction stars had been omitted because of their large proper motion, and that their average relative proper-motion components were $\overline{\mu_{\alpha} \cos \delta} = -\text{o}''/\text{a} \cdot 010$ and $\overline{\mu_{\delta}} = -\text{o}''/\text{a} \cdot 024$. This meant that, to the relative motions on the Radcliffe system, a correction of $\overline{\mu_{\alpha} \cos \delta} = +\text{o}''/\text{a} \cdot 001$ and $\overline{\mu_{\delta}} = +\text{o}''/\text{a} \cdot 001$ had to be applied to make the desired modification.

The mean proper motions of Radcliffe comparison stars (high proper-motion comparison stars included)

may be estimated in two ways. In the first place use may be made of the mean secular parallaxes computed by VYSSOTSKY and WILLIAMS (1948) from the "conditioned motions"¹⁾ in the McCormick catalogues. One slight difficulty associated with the use of these results is that conversion has to be made from the photovisual to the photographic magnitude scale—this may be overcome sufficiently well by using the mean colour-index $+0^m.6$ indicated by SEARES (1925). Weighted means of the mean secular parallaxes obtained from the two McCormick catalogues for various apparent magnitudes are shown in Figure 4 by the dots. It is seen from this figure that if these data are to be used for $m_{pg} = 13.7$, some extrapolation is necessary, particularly as VYSSOTSKY and WILLIAMS have noted that their point for $m_{vis} = 12.1$ is not so reliable due to possible selection effects.

The alternative procedure is to compute the required average proper motions from mean parallaxes of the comparison stars and their motion with respect to the sun. For this purpose BINNENDIJK's mean parallaxes are suitable since they refer to sufficiently faint high-latitude stars. However it seemed desirable to modify his data somewhat, particularly in regard to the variation in dispersion of peculiar velocities of stars as a function of distance from the galactic plane. For each group of stars, BINNENDIJK has calculated mean parallaxes with

$$\bar{p} = f \frac{|\mu_i|}{|v'_i|}.$$

Here $|\mu_i|$ is the mean residual proper motion along a direction chosen so that the parallactic motion was as small as possible but also such that it was as near as possible to the direction of the principal axis of peculiar motions. $|v'_i|$ is the average residual velocity of the stars along this direction and is made up of a term due to peculiar motions, $|v_i|$, and another due to the effects of solar motion acting together with the distance dispersion, $2.32 c_A$; c_A being the ratio of the component of solar motion to the full solar motion, thus

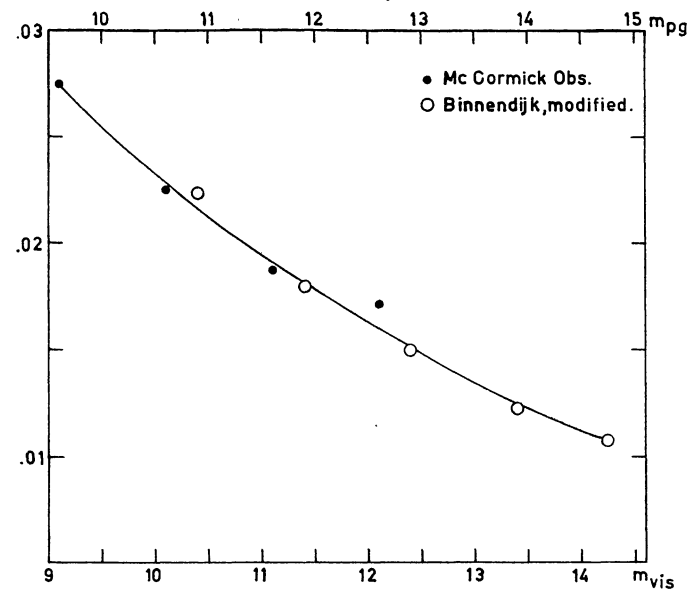
$$|v'_i|^2 = |v_i|^2 + (2.32 c_A)^2.$$

The factor f was a factor introduced to compensate for the fact that $\bar{p}$ and v'_i are not completely independent of one another.

The modifications made to BINNENDIJK's mean parallaxes (for stars with $|b| > 45^\circ$) pertain only to the term $|v'_i|$ in the above expression for the mean parallax which involves the average peculiar motion of the stars along the principal axis, denoted by $a/V\gamma$ in OORT's notation. Thus, instead of using the values

¹⁾ The term "conditioned motions" introduced by VYSSOTSKY and WILLIAMS refers to proper motions in a system almost identical to that adopted in the present work.

FIGURE 4



Mean secular parallaxes of all stars irrespective of spectral class with $40^\circ < |b| < 90^\circ$, as a function of apparent magnitude. The filled circles are those from the McCormick catalogues, and the open circles have been computed from BINNENDIJK's mean parallaxes after modification.

$a/V\gamma$ as plotted by OORT (1936), improved data obtained from work described later have been adopted. Here, by means of curves in Figure 6 together with the average velocities along the principal axis of strong-line, weak-line, and high-velocity stars, 18.4, 23.6, and 40.0 km sec⁻¹, it is possible to compute new values of $a/V\gamma$ for each group using BINNENDIJK's mean parallaxes. No modifications to the relatively unimportant terms $2.32 c_A$ and f have been made. Mean parallaxes so derived have then been converted to secular parallaxes by multiplying by the solar motion as anticipated for each apparent magnitude by finding the proportion of strong-line, weak-line, and high-velocity stars at the distance p^{-1} and using for the solar motion of the three groups, 20, 20, and 32 km sec⁻¹ ²⁾. The mean secular paral-

²⁾ To show the extent to which BINNENDIJK's mean parallaxes have been changed by the computations described above, the table below gives the original values, which are denoted by $\bar{p}_B$, taken from Figure 4 of BINNENDIJK's paper, and the modified values, $\bar{p}_M$, for the stars with $45^\circ \leq |b| \leq 90^\circ$. The parallaxes are in units of seconds of arc.

m_{pg}	$\bar{p}_B$	$\bar{p}_M$	$\bar{p}_M/\bar{p}_B$	m_{pg}	$\bar{p}_B$	$\bar{p}_M$	$\bar{p}_M/\bar{p}_B$
11	.00525	.00530	1.01	13	.00298	.00343	1.15
12	.00398	.00421	1.06	14	.00225	.00279	1.24
				15	.00173	.00230	1.33

Thus the effect of the adjustments to BINNENDIJK's mean parallaxes is to increase them by an amount increasing with the faintness of the stars. The correction factor reaches the not insignificant value of about 1.2 at $m_{pg} = 13.7$.

laxes deduced in this way are illustrated in Figure 4 by the open circles. In view of the fact that no extrapolation is required, and the satisfactory agreement with the McCormick data, where they overlap, the corrected BINNENDIJK mean secular parallax for m_{ps} 13.7 has been accepted.

Systematic corrections to be applied to Groningen and Radcliffe-Pulkovo motions after reduction to the modified Radcliffe relative-motion system are then

$$\Delta(\mu_{\alpha} \cos \delta) = q \sin \Delta \sin \chi,$$

$$\Delta \mu_{\delta} = q \sin \Delta \cos \chi.$$

Here q is the mean secular parallax, Δ the angular distance between the Selected Area and the antapex of solar motion, and χ the angle measured from the northerly direction of the meridian through the Selected Area eastwards to the direction of the antapex of solar motion. Δ and χ were obtained for each Selected Area by means of SMART's graphs (SMART 1923), which use apex co-ordinates $\alpha = 18^h$, $\delta = +34^\circ$.

Mean proper-motion components computed for Radcliffe comparison stars in the Selected Areas are shown in Table 7. In Table 8 the corrections applied

TABLE 7
Mean proper motions, in the modified FK₃ system, of Radcliffe comparison stars.
(unit $\text{o}''/\text{a}.$)

S.A.	$\mu_{\alpha} \cos \delta$	μ_{δ}	S.A.	$\mu_{\alpha} \cos \delta$	μ_{δ}	S.A.	$\mu_{\alpha} \cos \delta$	μ_{δ}	S.A.	$\mu_{\alpha} \cos \delta$	μ_{δ}
5	-13	-1	35	-7	+2	69	+12	-9	92	+12	-9
12	-9	-12	36	-4	+2	70	+10	-10	93	+11	-9
13	-12	-7	53	-10	-11	77	-9	-9	101	-11	-9
14	-12	-1	54	-11	-10	78	-11	-10	102	-12	-8
15	-8	+4	55	-13	-8	79	-13	-9	103	-13	-9
29	-10	-11	56	-13	-7	80	-13	-8	104	-13	-8
30	-12	-9	57	-12	-6	81	-12	-7	105	-11	-9
31	-13	-7	58	-11	-4	82	-11	-7	106	-10	-9
32	-12	-4	59	-9	-3	83	-9	-6	107	-7	-9
33	-11	-2	60	-7	-2	84	-6	-5	115	+13	-9
34	-9	0	68	+12	-9	91	+12	-7			

TABLE 8
Corrections to reduce Groningen motions to the modified FK₃ system. (unit $\text{o}''/\text{a}.$)

S.A.	$\Delta(\mu_{\alpha} \cos \delta)$	$\Delta \mu_{\delta}$
92	+13	-5
93	+10	-10
102	-15	-10
103	-13	-5
104	-17	-6
105	-13	-12
106	-15	-15
115	+14	-7

to Groningen motions to convert them to the adopted absolute system are shown, they are made up of the conversion from the Groningen to Radcliffe relative-motion system and the absolute motions of Radcliffe comparison stars.

Histograms, for each apparent-magnitude interval, of total proper motions were readily obtained from the corrected catalogue proper motions. This has been done, in general, by plotting the corrected proper-motion components, and counting the number of points within concentric rings drawn on a sheet of tracing paper superimposed on the diagram so that the centre of the circles lies over the origin of the

diagram. The entire process is still simpler for Yale stars where large numbers of motions require the same systematic correction—in this case uncorrected motions read directly from the catalogues are plotted and the tracing paper centred on a new origin of the diagram fixed by the systematic correction.

9. Histograms of total proper motion and the estimation of true distributions

Histograms derived in the manner just described are depicted in Figures 5a and 5c—they are all normalized to 100 stars for convenience of intercomparison. To each diagram is appended the number of stars upon which it depends and the mean apparent magnitude. For those wishing to make use of this material, data upon which each histogram has been constructed are given in their entirety in Tables 9a and b.

Computation of the true distribution of proper motions requires the replacement of the histograms by smooth curves from which the influences of histogram grouping and accidental errors in the individual motions have to be removed. Both these influences tend to make the smoothed observed curves more dispersed than the true distribution and will also make the mode of the former occur at a

TABLE 9a
Histograms of total proper motion¹⁾.

μ ($\sigma''/a.01$ unit)	GC Stars		Yale Zone A				Yale Zone B			
	m m 4.5-5.4	m m 5.5-6.4	m m 6.5-7.4	m m 7.5-8.4	m m 8.5-8.9	m m 9.0-9.4	m m 6.5-7.4	m m 7.5-8.4	m m 8.5-8.9	m m 9.0-9.4
0-1	2	25	8	56	91	68	12	62	165	130
1-2	21	49	24	107	215	137	23	153	372	252
2-3	19	60	27	117	159	127	34	150	364	230
3-4	13	77	22	78	90	93	33	128	197	187
4-5	13	62	23	68	57	67	20	88	148	79
5-6	9	35	14	35	31	33	15	47	71	59
6-7	10	27	12	19	22	25	12	35	40	35
7-8	8	34	3	18	15	16	12	17	21	31
8-9	8	35	5	9	9	13	11	13	24	11
9-10	9	18	4	3	7	8	1	13	18	12
10-11	8	10	2	6	4	12	1	7	5	8
11-12	9	12	1	5	7	6	4	6	9	8
12-13	3	8	3	3	5	7	3	7	10	5
13-14	13	9	1	2	4	3	3	3	9	9
14-15	5	7	—	3	5	6	3	—	8	4
15-16	4	4	—	3	2	3	1	3	6	5
16-17	4	4	—	5	4	2	—	1	5	5
17-18	2	4	1	—	3	3	—	—	1	3
18-19	3	4	—	1	4	1	1	—	1	4
19-20	1	1	1	2	2	—	2	—	2	—
	23, 24	21 ² , 22 ²	21, 23	25, 26	21 ³ , 22 ²	21 ³ , 22	74	21, 22	21 ² , 22 ⁴	21 ² , 23
	30 ² , 32	23, 24 ²	49, 69	27, 32	24 ⁴ , 25	24 ² , 30		26, 29	23 ³ , 24 ²	24, 26
	34 ² , 35	26, 27 ²		33 ³ , 34 ²	26 ² , 27 ²	36, 41 ²		30 ² , 31	25, 26 ³	27 ² , 30 ²
	48, 49	30 ² , 37		35, 37	28, 30			33, 34	27 ² , 28	32, 34
	62, 121	50, 53		43 ² , 46	33 ² , 42 ²			43 ² , 45	29, 31	35, 37
	469	59, 73		228	46, 47			48, 50	32, 33 ²	41, 44 ²
		74, 139			51, 83			51, 57	34, 35 ²	61, 63
		235						59, 66	36, 39	81
								67, 75	51, 54	
Totals	178	505	155	555	760	640	192	753	1507	1096

¹⁾ In the first column μ 3-4, for example, indicates the proper-motion interval $\mu = \sigma''/a.030$ to $\sigma''/a.039$ inclusive. Further the tabulation is abbreviated so that for $\mu > \sigma''/a.2$ only proper-motion intervals containing stars are indicated. This has been done by noting 100 times the upper proper-motion limit and giving the number of stars in the interval as a superfix to this limit when the number of stars is greater than 1. For Table 9b the abbreviation commences at $\mu = \sigma''/a.1$, while 200 times the upper proper-motion limit has been noted.

larger proper motion than in the latter distribution.

With the exception of the GC star motions, i.e., groups m_{vis} 4.5 to 5.4, and 5.5. to 6.4, grouping introduces modifications which are minor compared to those due to accidental errors in the motions themselves²⁾. For the exceptional cases, effects of grouping and accidental errors are comparable, but because the dispersion of the motions themselves is so large, neither grouping or accidental errors will cause any significant differences between true and observed GC distributions. Thus the smoothed representations of the GC-star motions are taken to be the true

distributions of total proper motion for these stars. Correction of observed distributions for the effects of accidental errors in proper motions requires the probable errors of the motions. These have been listed in Table 2, column 8, and are generally taken from the material accompanying the catalogue. It should be stressed that these probable errors are average values; for, while generally the probable errors in $\mu_a \cos \delta$ and μ_δ are identical for individual stars, they differ from star to star. To assess the reality of a spread in probable errors of motions in any particular histogram material would however be difficult because further subdivision of the material on the basis of proper-motion probable errors increases the statistical uncertainty of distributions to such an extent that they mask any systematic differences likely to be present.

Several methods have from time to time been

²⁾ A rough but sufficiently precise measure of the grouping effects in comparison with those of accidental errors may be reached by comparing the average deviation from the mean in each histogram group, with the mean error of observation for the distribution. The former is given accurately enough by one quarter of the histogram interval.

TABLE 9b
Histograms of total proper motion¹).

μ ($0''/a.01$ unit)	McCormick		Groningen		Radcliffe-Pulkovo		
	m m 9.0-10.4	m m 10.5-11.9	m m 10.5-12.4	m m 12.5-13.4	m m 10.5-12.4	m m 12.5-13.4	m m 13.5-14.4
0.0- 0.5	10	13	16	10	9	10	5
0.5- 1.0	36	60	37	24	19	21	24
1.0- 1.5	46	59	40	26	16	16	26
1.5- 2.0	37	64	39	28	7	17	15
2.0- 2.5	41	58	27	27	8	11	15
2.5- 3.0	26	25	20	20	4	9	4
3.0- 3.5	18	35	15	10	3	6	6
3.5- 4.0	14	24	8	7	1	3	2
4.0- 4.5	16	23	5	5	1	—	2
4.5- 5.0	15	25	1	8	3	3	4
5.0- 5.5	5	9	4	4	2	—	1
5.5- 6.0	8	6	6	3	1	5	2
6.0- 6.5	4	11	3	1	3	1	1
6.5- 7.0	3	5	2	—	1	1	2
7.0- 7.5	2	6	1	5	1	—	—
7.5- 8.0	2	5	—	—	1	2	—
8.0- 8.5	1	1	4	1	1	2	1
8.5- 9.0	1	7	2	—	—	—	—
9.0- 9.5	2	1	2	—	—	1	—
9.5-10.0	2	1	5	—	—	—	1
	23, 24	21, 22 ²	21 ² , 22 ²	23, 24	21 ² , 23	21, 30	22, 26
	25, 27	23 ³ , 24	26, 27		26, 30	51, 102	28, 40
	28, 31	25 ³ , 26	32, 33		31, 41		82
	33, 37	27 ² , 29	37 ² , 43		52, 65		
	39, 44	30, 32	49, 52				
	50, 55	35 ³ , 40	60, 101				
	59, 62	51, 58	110				
	80	61					
Totals	304	462	253	181	90	112	116

¹) See note to Table 9a.

proposed for freeing observed distributions of the effects of accidental errors. The problem basically involves deriving the true distribution of total proper motions, $f(\mu)$, which is related to the observed total proper-motion distribution, $\varphi(\mu_o)$, by

$$\varphi(\mu_o) d\mu_o = 2h^2\mu_o e^{-h^2\mu_o^2} d\mu_o \int_0^\infty I_0(2h^2\mu\mu_o) f(\mu) e^{-h^2\mu^2} d\mu, \quad (6)$$

where I_0 is the Bessel function of imaginary argument, and h is 0.477 divided by the probable error of the observations. This integral relationship was first derived by KAPTEYN and VAN RHIJN (1920 b), who also showed that if the observed distribution could be represented as the sum of several distributions, like

$$\varphi(\mu_o) = \sum_i 2k_i^2\mu_o e^{-k_i^2\mu_o^2}, \quad (7a)$$

by a suitable choice of the number of terms i , and the

parameters k_i , then the true distribution is given by

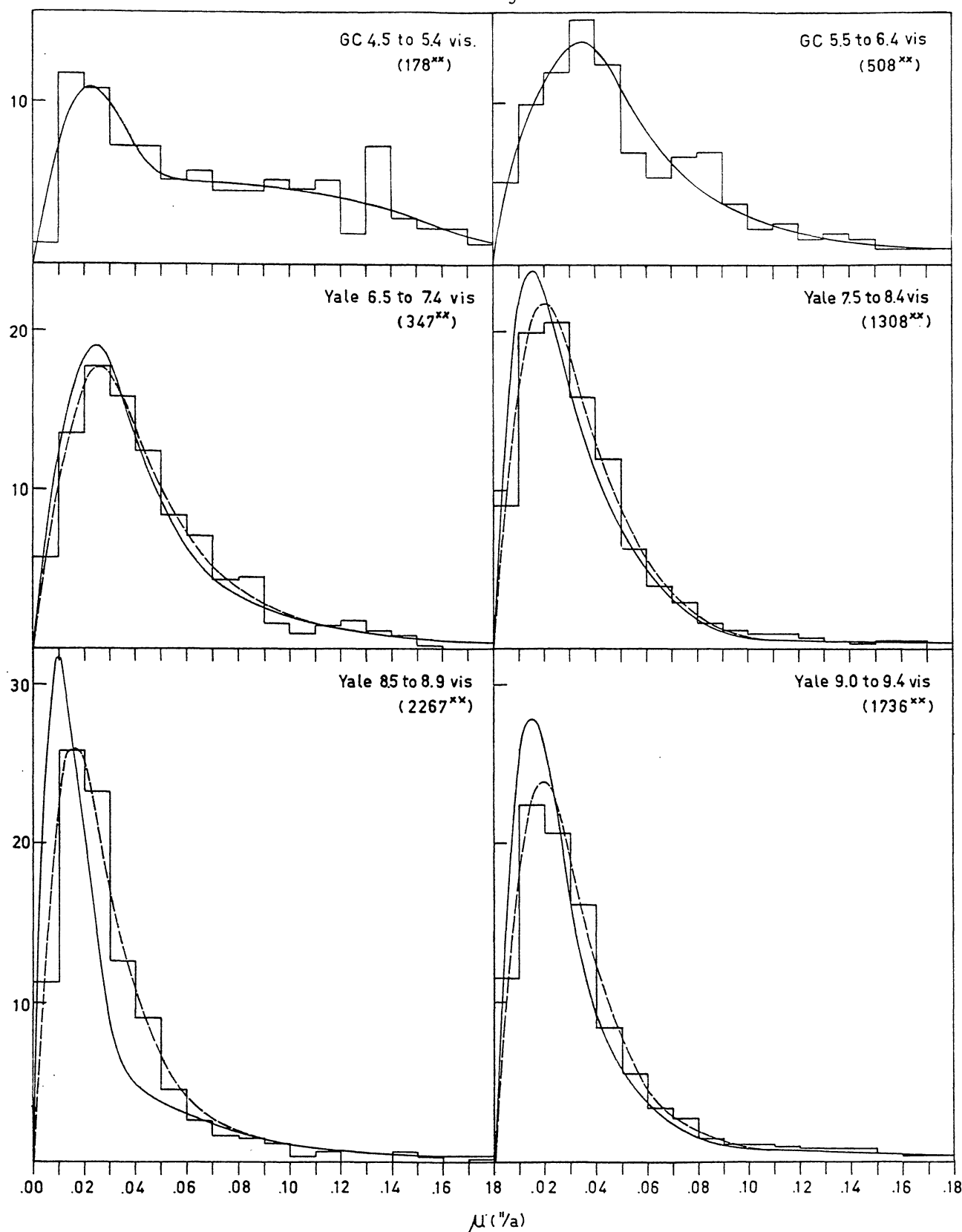
$$f(\mu) = \sum_i 2\beta_i^2\mu e^{-\beta_i^2\mu^2}, \quad (7b)$$

where

$$\beta_i = \frac{hk_i}{\sqrt{h^2 - k_i^2}}. \quad (7c)$$

This procedure has been used in the present work with success — it is far simpler than solving the integral equation by trial and error. The only advantage of the latter procedure is that every significant feature in the observed distribution may be considered. With the KAPTEYN-VAN RHIJN method some detail is lost in each individual proper-motion distribution because of the limited flexibility of the mathematical representation for observed distributions. These shortcomings may be eliminated to some extent by ensuring that the mathematical representations of the observed distributions for

FIGURE 5a



Figures 5a and b: The nine compartments of these figures illustrate the proper-motion data. Broken curves represent distributions of corrected observed total proper motions which are also shown in histogram form. Estimated true distributions are illustrated by continuous curves. To each diagram is appended the source of the data, and the number of stars involved. All distributions have been normalized to 100 stars. Note also the change of scales in passing from Figure 5a to 5b.

FIGURE 5b

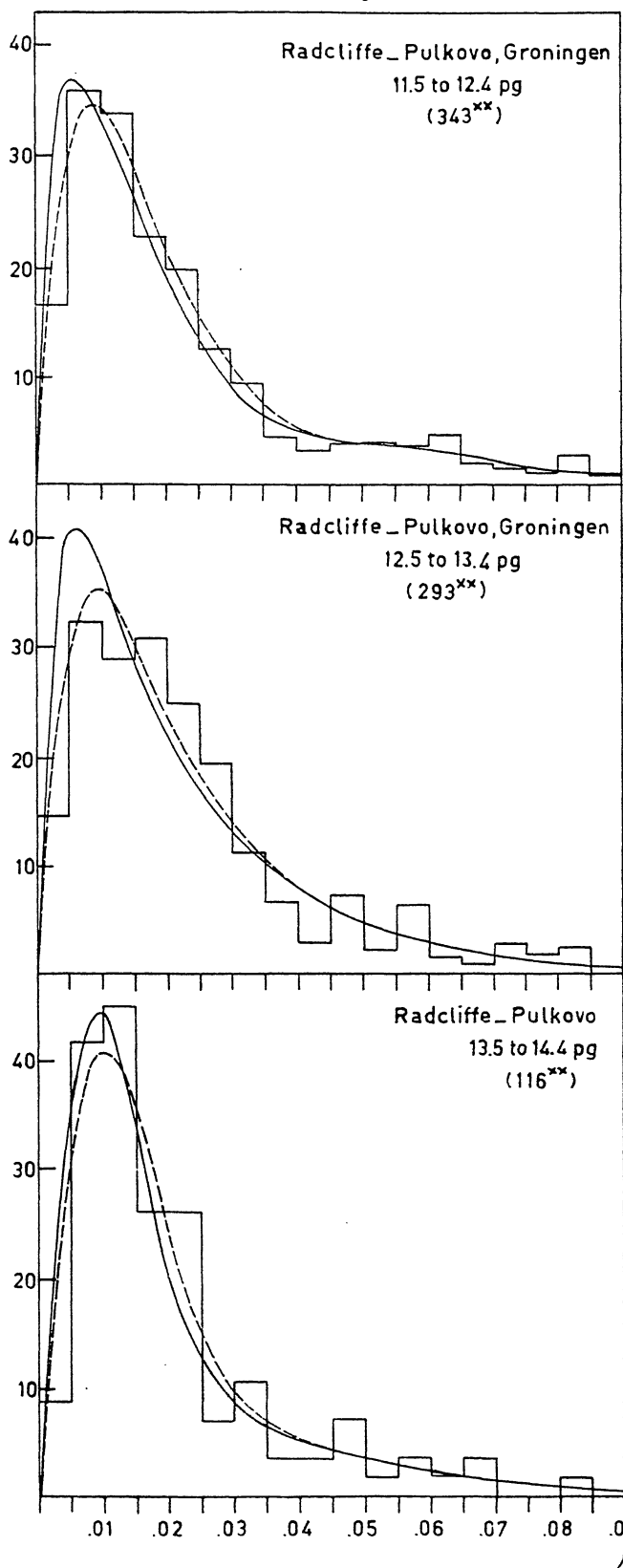


FIGURE 5c

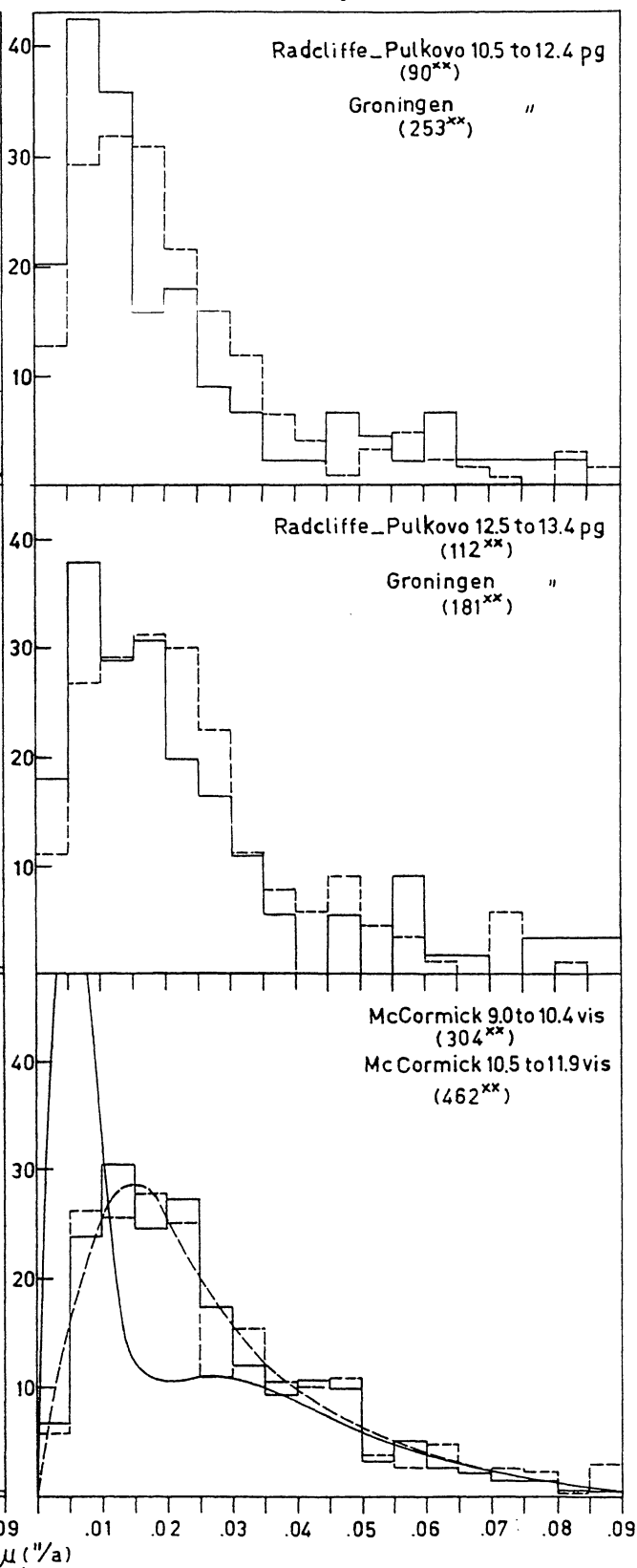


Figure 5c: The separate histograms of Groningen and Radcliffe-Pulkovo motions (full drawn: Radcliffe-Pulkovo, dashed: Groningen) which were combined for Figure 5b, together with the discarded McCormick data. Because of the close resemblance between the histograms compiled from McCormick material, one smooth observed distribution was sufficient to represent both histograms. The kink appearing in the true distribution represented by McCormick data results from the markedly different effects of accidental errors on the component distributions used to represent observed data and is largely fortuitous.

neighbouring apparent magnitudes show few signs of systematic discrepancies¹⁾.

Estimated true distributions are shown in Figure 5a for GC and YC material by heavy continuous curves—the observed distributions are shown by broken curves. Examination of these true distributions reveals several points of interest. In the first place there seems to be an unexpected difference between the proper-motion distributions of the two brightest groups. The maximum frequency for the distribution of the brightest stars seems to appear at too small a proper motion when compared with the distributions for the fainter stars. It seems quite probable that the brightest group has not sufficient stars to give a sufficiently reliable picture of the distribution. Secondly, the close resemblance between Yale motions from Zone A and Zone B indicates that these distributions should be quite reliable. The close agreement is too consistent to be explained in terms of accidental effects covering up systematic differences which might be expected to arise from the ten degrees difference in the mean galactic latitude of the two groups.

In Figure 5c, the histograms derived from the remainder of the material (the McCormick, Radcliffe-Pulkovo, and Groningen catalogues) are illustrated. Regarding the McCormick distributions, results are rather disappointing. The close similarity of the two distributions, which are separated by 1.5 magnitudes, is unlikely. Also the effects of the relatively large accidental errors of observation as illustrated by the estimated true distribution representing the combined McCormick material are too large to be reliably determined—particularly for the height and position of the maximum frequency in the true distribution. For these reasons both distributions were omitted from subsequent computations. The McCormick data show clearly that if reliable true distributions of proper motions for tenth magnitude stars, or fainter, are to be obtained, motions must have probable errors much less than $\pm 0''.008$.

The histograms in Figure 5c from the Groningen and Radcliffe-Pulkovo material for m_{pg} 12.0 and 13.0 show lack of agreement, which is of considerable significance—particularly in view of the high

accuracy (as regards accidental errors) of the motions. Agreement between the two sets of distributions might have been improved if the probable error of the Groningen motions were larger than adopted. Indeed, the assumption of a probable error of $\pm 0''.006$ for Groningen motions would remove most of the discrepancies. However, VAN RHIJN and PLAUT's estimate of the probable error (about $\pm 0''.003$) should be quite reliable since it is based on concordant results from (a) comparison with Radcliffe motions, and (b) comparison of proper motions for the same star on different pairs of plates. Thus as regards individual proper motions, distributions derived from the two catalogues are of comparable quality. A point favouring the Groningen distributions is that the number of stars involved is nearly twice that in the Radcliffe-Pulkovo distributions. But it should be remembered that this has resulted from a much more complete study of only a small number of Selected Areas. Thus if proper-motion distributions fluctuate considerably from one part of the sky to another, then the more extensive sky area covered by Radcliffe-Pulkovo motions would count in their favour. Such circumstances appear to prevail, for BINNENDIJK (1943) has found that mean parallaxes obtained for groups of stars in different parts of the sky deviate from one another by amounts which are frequently much greater than one would attribute to purely statistical uncertainties in observational material. These discrepancies²⁾ increase towards the galactic plane; however they are still quite large for high-latitude stars with m_{pg} from 12.0 to 14.7. From these considerations it seems best to attach the same weight to both the Radcliffe-Pulkovo and Groningen distributions for each of the m_{pg} -groups despite the disparity in the numbers of stars of which they are comprised. The faintest group ($m_{pg} = 13.8$), consisting solely of Radcliffe-Pulkovo motions, must be given relatively low weight because of the smaller numbers of stars.

The combined faint-star proper-motion distributions and that for $m_{pg} = 13.8$ are shown in Figure 5b with the estimated true distributions.

(b) The Velocity Distributions

10. Introduction

In principle, one objective of this section of the work should be to provide distributions of velocities transverse to the lines of sight to the sky zones for which proper-motion distributions have already been furnished. The velocity distributions should be for

²⁾ The exact cause of these effects was not ascertained; the latitude variation might suggest that intervening interstellar clouds might be the cause.

¹⁾ SMART (1928) has proposed a series method for deriving the true distribution. He showed that

$$f(\mu) = \varphi(\mu) - \frac{1}{4h^2} \left\{ \frac{1}{\mu} \frac{d\varphi(\mu)}{d\mu} + \frac{d^2\varphi(\mu)}{d\mu^2} \right\} + \text{terms of higher order in } h^2.$$

However, the derivation of this series relies on the slope of the observed distribution varying only slightly over a range of μ comparable to the probable error. With the exception of the bright-star proper-motion distributions for which the corrections were ignored because of their smallness, this condition was not satisfied by the proper-motion material considered.

K stars of various absolute magnitudes and should include an assessment of their variation with distance from the galactic plane. The second aim is to determine Z^2 for K stars as a function of z . This quantity, along with the density distribution (deduced later), is a primary necessity in calculating K_z .

The most direct way of obtaining transverse-velocity distributions would be to make use of observations of the necessary quantities (absolute magnitude, distance, proper motion and radial velocity) of representative K stars at various distances from the plane. Unfortunately, existing observational data are insufficient in both quality and quantity to attempt an analysis of this nature.

It is therefore necessary to adopt an indirect approach possessing unavoidable tentative aspects. One could envisage studying the transverse-velocity distribution of K stars of various absolute magnitudes, near the galactic plane, as a function of their Z -velocity. Then with an assumed K_z the transverse-velocity distribution at various z -distances may be estimated for stars of each absolute magnitude. Improvements to this computation may be made subsequently when improved estimates of K_z have been computed. These improvements will in turn lead to further, more precise estimates of K_z — the procedure is thus iterative.

However in respect to differentiation between stars of different absolute magnitudes the above procedure cannot be applied as comprehensively as described. Again the reason is lack of observational data — there are not sufficient data to examine variations with absolute magnitude in the important high Z -velocity range which is so sparsely populated near the galactic plane. The most that can be done is to divide the material into the main luminosity classes, and even then it is insufficient to study the distribution of high Z -motions for subgiants and dwarfs. On the other hand, the Z -velocity distributions for K giants and dwarfs for low and moderate Z -velocity may be studied fairly easily. Bok (1935, see Table III) has examined space motions of such stars as found in SCHLESINGER's *Bright Star Catalogue*. This work indicated that there are no great differences in the distribution of low and moderate velocities between K-giant and dwarf stars.

With existing observational material it has been possible to make an investigation of the Z -velocity distribution of K-giant stars with particular emphasis on the high velocities. In the absence of any preferable alternative, it has been assumed that all K stars, whatever their luminosity, possess the same Z -velocity distribution as found for the giants.

Another approximation entering into the computation arises in connection with the study of the space motions of K giants as a function of their Z -velocity.

Instead of arranging the material into groups defined by definite limits of Z , it is much more convenient, and moreover more natural, to analyse the Z -velocity distribution into gaussian components and to derive the transverse-motion distribution representative of each component as a whole. The convenience of such a subdivision of Z -velocity lies in the fact that the proportions with which such components occur at various z -distances are then readily estimated, as are the resulting transverse-velocity distributions. However, in adopting such a procedure, it is not the intention to imply that within each such component there is in reality no correlation of space motion with Z ; it is merely a convenient approximate method for allowing for the gross changes in the transverse-velocity distribution with z .

Thus, in brief, the necessary transverse-velocity distributions have been estimated in the following four stages:

(1) the Z -velocity distribution representative of the class of stars (in this case the K giants) when within about 100 pc of the galactic plane is analysed into gaussian components.

(2) Approximate transverse-velocity distributions characteristic of each component are then obtained from studies of space motions.

(3) From the data provided in the first stage, together with an assumed K_z , approximate proportions with which the components are represented at various z -distances can be computed.

(4) Results of stages (2) and (3) are combined to give transverse-velocity distributions for all relevant z -distances.

Z -velocities of K giants have been obtained from radial velocities of high-latitude stars. Each radial velocity, after correction for solar motion, is then multiplied by a statistical factor which converts it into a Z -velocity after the manner described by OORT (1932). Strictly, this factor will be influenced by changes in the characteristic space motions with z ; thus it can be given correctly only when the analysis of Z -motions is complete. Fortunately this dependence is not very strong and the originally estimated factors need no significant modification.

In order to obtain as much information as possible concerning the high-velocity component, use has been made not only of Z -motions of nearby K giants, but also of stars as far away from the galactic plane as 1 kpc. Here the proportion of high Z -velocity stars has increased to such an extent that an increase in the dispersion of Z -velocities, over and above the dispersion found near the plane, is seen clearly in the observational material. With the help of an assumed K_z , this has provided much valuable information about the high Z -velocity distribution.

From the viewpoint of the density calculations

described later, the most significant features of the calculation of transverse-velocity distributions are the following:

(a) Only the Z -velocity distribution of K giants has been considered in detail; it has been supposed that all K stars irrespective of their luminosity, possess this distribution of Z -motions.

(b) Information about the distribution of high Z -velocities, and the transverse-velocity distribution of these stars, is still very meagre.

(c) The entire analysis is based on an assumed K_z , and is therefore subject to alteration as improved values of K_z are obtained.

11. Analysis of the Z -velocity distribution of K giants in the galactic plane

The most natural form for this analysis seems to be indicated by results of recent work by Miss ROMAN (1952, 1954). In the first paper cited, Miss ROMAN gives a summary of the mean peculiar space motions of four groups of stars defined solely in terms of relatively slight differences in the strengths of metal lines and molecular bands in spectra having a dispersion of 125 Å/mm near $H\gamma$. The stars listed in this paper as belonging to one or other of these groups comprise nearly all stars in the spectral range F5 to K1 brighter than m_{vis} 5.5 and north of declination -20° . The second paper lists members of another group again characterized by distinct spectral and kinematical features. This group was discovered from an examination of spectra of stars in a catalogue of high-velocity stars brighter than m_{vis} 9.5 and north of

declination -20° , and contains F stars only (ROMAN 1955). Values of the mean space motion which Miss ROMAN finds for these five groups of stars are shown in Table 10. It is seen from this table that strong-line, weak-line and weak-CN stars are characterized by distinct differences in average space motions. These results supply indirect but strong confirmation of the reality of the small differences which are found in the spectra of the three groups of stars. Of the other groups, the "4150-stars" seem to differ only in spectrum from weak-line stars; the same may be true for the weak-CN and unnamed groups.

Stars of the first four groups listed in Table 10 have been found amongst giants, subgiants, and dwarfs, and in each of these luminosity classes there appears to be no systematic difference in the mean absolute magnitudes of these star groups. From this it may be concluded that the numbers in Table 10 indicate the relative numbers per unit volume near the sun of the first four of Miss ROMAN's groups of stars. The approximate equality of the numbers of strong- and weak-line stars, which together comprise about 90% of all bright F5 to K1 stars, has no intrinsic significance — it is merely the result of an arbitrary subdivision of line strengths.

It would appear that had Miss ROMAN's survey extended to stars of later spectral type, similar results would have been obtained. There is no reason why star groups with the characteristics of strong- and weak-line stars should exist only amongst stars with spectral type earlier than K2.

All these results indicate that strong- and weak-line stars may profitably be used for a dissection of the velocity distribution into components with different average Z -velocities — they indeed constitute two natural velocity groups.

In Table 11 the observed Z -velocity distributions of strong- and weak-line stars are shown. They have been derived from space motions and are compared in each case with gaussian distributions having the same $\langle Z \rangle$ as the observed distribution²⁾. It is seen from Table 11 that the Z -velocity distribution of strong-line stars is well represented by the gaussian distribution. In the case of weak-line stars the agreement is not so good — particularly at the large Z -velocities where there is a significant excess in the observed number of stars over that computed. The two computed distributions have been taken over as the

²⁾ In computing for Table 11, space-motion material described later in section 12 was used. Peculiar motions were derived by removing the effects of the standard solar motion (20 km sec⁻¹ towards an apex at $\alpha = 18^h40^m$, $\delta = +28^\circ$). The numbers of stars are smaller than given in Table 10 because stars which Miss ROMAN (1949) regards as members of the Ursa Major cluster, as well as several considered by VAN BUEREN (1952) to be members of the Hyades cluster have been omitted.

TABLE 10

Mean peculiar space motions of Miss ROMAN's star groups.

Group name	Number identified	Mean space motion (km sec ⁻¹)
Strong-line	161	26.3
Weak-line	174	41.4
4150	25	42
Weak-CN	12	96
Unnamed ¹⁾	17	probably > 100

¹⁾ These are the stars from the second of Miss ROMAN's papers; as yet no name has been given to the group. On account of the manner in which it was formed, i.e., from a list of high-velocity stars, many of which are quite faint, it is difficult to make an accurate comparison of the group with the preceding four groups. In the first place it would be inappropriate to quote the mean speed of 240 km sec⁻¹ obtained from the 17 stars as being representative of the group as a whole — this mean speed has undoubtedly been influenced by proper-motion selection. The true mean speed is probably less than this value, but almost certainly it is exceptionally high. And secondly, these stars must make only a very small contribution to the numbers of stars near the galactic plane — they must be much rarer in comparison with the other four star groups than is suggested by the numbers given in the above table.

TABLE II

A comparison of Z -velocity distributions of strong- and weak-line stars with gaussian velocity distributions.

$ Z $ (km sec ⁻¹)	strong-line		weak-line	
	obs.	comp.	obs.	comp.
0—5	54	47.7	35	33.2
5—10	33	39.2	34	30.8
10—15	28	26.4	30	26.8
15—20	14	14.6	16	21.6
20—25	5	6.6	18	16.2
25—30	1	2.5	8	11.4
30—35	2	0.8	5	7.4
35—40	—	0.2	2	4.5
40—45	—	—	4	2.5
45—50	1	—	—	1.4
50—55	—	—	2	0.7
55—60	—	—	1	0.3
60—65	—	—	1	0.1
65—70	—	—	1	0.1
Totals $ Z $	138 8.9 km sec ⁻¹		157 14.9 km sec ⁻¹	

nucleus of the analysis of Z -motions of K giants. They will be termed strong- and weak-line distributions with the understanding that complete identity does not exist with the observed distributions. This reservation applies particularly to the weak-line stars, but the resemblances are quite sufficient for the present purposes. When the contributions of the two gaussian components just considered are combined with equal weight and compared with a complete Z -velocity distribution of local K giants (obtained by using all bright K giants for which the requisite velocity data are available) the combined distribution is found to give an inadequate representation of the complete distribution. Of these deficiencies, those which may be remedied by the introduction of groups of stars with higher average Z -velocity than either strong-line or weak-line stars, are by far the most important for this investigation. This is because of their tendency to predominate over the other stars as they get further and further from the galactic plane. Imperfections in the representation of the low-velocity part of the distribution are not likely to be of much consequence.

It is undesirable to adopt parameters for several gaussian distributions to describe the high Z -velocity component based on the data given for weak-CN and the unnamed stars in Table 10. For in the first place it is clear that there are insufficient known members of the groups to allow a satisfactory estimate of their $|Z|$, or their proportionate contribution to the number of stars near the galactic plane. However the main reason against the adoption of such a procedure is that the material is incomplete. Miss ROMAN's search amongst faint high-velocity stars was aimed at finding distinctive spectral features by which they could be identified. Of the many stars considered,

only 17 yielded to this type of analysis — thus a large proportion of the locally rare high-velocity stars are not represented in the spectral groups of Table 10. Furthermore it must be remembered that the lack of high velocities in the gaussian representation of the weak-line star Z -velocity distribution has to be made up.

The complete Z -velocity distribution of K giants has therefore been represented by three gaussian components, the two already described, and a third with an even higher average Z -velocity, which has to account for the important high Z -velocity stars not included amongst the strong- and weak-line gaussian components. Undoubtedly the high Z -velocities will only be superficially represented in this manner, but there is no better alternative. The relative number of stars in these three groups near the galactic plane, and $|Z|$ for the high Z -velocity group, could then be estimated from the available radial-velocity data concerning high-latitude K-giant stars.

The general principles of the method of completing the analysis were as follows. Radial-velocity stars were at the outset grouped according to their distance from the galactic plane. Considering for the moment one such group (whose average distance from the galactic plane is z), let $|Z|_{obs}$ be their average Z -velocity, and $|Z|_{str}$, $|Z|_{wk}$, and $|Z|_{hv}$ those of the strong-line, weak-line, and high-velocity stars. Further, suppose that θ_{str}° , θ_{wk}° , and θ_{hv}° are the fractions belonging to the three velocity groups in the galactic plane, and p , q , and r their relative numbers amongst the stars considered. The quantities p , q , and r are the ratios of the density at height z from the plane, to that in the plane, for each of the three velocity components. Then it is clear that

$$|Z|_{obs} = \frac{p \theta_{str}^\circ |Z|_{str} + q \theta_{wk}^\circ |Z|_{wk} + r \theta_{hv}^\circ |Z|_{hv}}{p \theta_{str}^\circ + q \theta_{wk}^\circ + r \theta_{hv}^\circ}.$$

In this expression, $|Z|_{obs}$, $|Z|_{str}$, $|Z|_{wk}$, p and q may be considered known; p and q being estimated from some plausible K_z . Since it was assumed that strong- and weak-line stars are equally prevalent near the galactic plane, the above expression may be written

$$|Z|_{obs} = \frac{\frac{1}{2} (p |Z|_{str} + q |Z|_{wk}) (1 - \theta_{hv}^\circ) + r \theta_{hv}^\circ |Z|_{hv}}{\frac{1}{2} (p + q) (1 - \theta_{hv}^\circ) + r \theta_{hv}^\circ}. \quad (8)$$

For each of four distance groups lying within 300 pc of the galactic plane, this expression provides a relation between parameters θ_{hv}° and $|Z|_{hv}$. Admittedly there is still the unknown factor r involved, but initially an estimate may be made from a rough guess at $|Z|_{hv}$ and using the approximate K_z as with the calculation of p and q . This estimate of r could be modified later if the provisional estimate was too much in error.

The independently derived relations so found between θ_{hv}° and $|Z|_{hv}$ were then combined with

suitable weighting factors to give the most likely relation. The decision as to which combination of the two unknowns was the most likely was then made by comparing the average Z -speed at a distance of 1000 pc from the galactic plane with that observed.

To implement this scheme a list was prepared from WILSON's catalogue (1953) of Henry Draper K stars lying within 50 degrees of the galactic poles and with radial velocities having probable errors less than ± 5 km sec⁻¹. Whether binary systems were entered into the list depended on the spectrum and luminosity of the brightest component. Subsequently all stars in the list likely to be members of the Ursa Major and Hyades clusters were deleted along with several stars showing every indication of belonging to the Coma Berenice cluster. The final list, in so far as it concerns stars brighter than m_{vis} 6.5, could be considered to be fairly free of cluster stars. The effects of proper-motion selection on the material so formed was assessed by comparing the total proper-motion distribution of the stars with that given by VAN RHIJN (1925a, Table 4) for all K stars at these high latitudes. Marked selection effects were found to occur for $m_{vis} > 6.0$ but only for stars with $\mu > 0''.20$, which is far in excess of the great majority of K-giant proper motions. Thus no selection effects need be allowed for in the K-giant material contained in this list.

For each K-giant star¹⁾ the influence of the standard solar motion on the radial velocity was removed, leaving the peculiar radial velocity. These stars were then grouped according to distance from the galactic plane. For stars within 35 degrees of the galactic poles this was carried out by grouping stars into apparent-magnitude intervals $m_{vis} < 5.5$, 5.5 to 6.5, 6.5 to 7.5, and 7.5 to 8.5. While for stars between 35 and 50 degrees from the poles the magnitude

groups were $m_{vis} < 6.0$, 6.0 to 7.0, 7.0 to 8.0 and 8.0 to 9.0. In this manner four distance groups were formed, since with this arrangement of magnitude limits, stars of the central-zones with $m_{vis} < 5.5$ had about the same z -distance as stars with $m_{vis} < 6.0$ of the outer zone, central-zone m_{vis} 5.5 to 6.5 stars about the same average z -distance as m_{vis} 6.0 to 7.0 stars of the outer zone, and so on. Sufficiently precise average z -distances could be computed for each group assuming all stars to have an apparent magnitude equal to the average of the group and mean visual absolute magnitude of +0.6, which corresponds to the apparent luminosity of bright K giants. Estimated mean z -distances for the four groups are given in Table 12, column 4.

Assuming K giants to have peculiar space motions which may be well represented by one Schwarzschild ellipsoidal velocity distribution, the factors f by which the peculiar radial speeds must be multiplied in order to obtain statistically accurate peculiar Z -speeds may be calculated for each star from²⁾

$$f = \left\{ l^2 \left(\frac{A}{C} \right)^2 + m^2 \left(\frac{B}{C} \right)^2 + n^2 \right\}^{-\frac{1}{2}}. \quad (9)$$

Here l , m , and n are the direction cosines of the lines of sight to the star with respect to the principal axis of the velocity ellipsoid, whilst A , B , and C are the velocity dispersions. Factors computed from (9) using the average axial ratios given in the Appendix for strong- and weak-line stars have been used to convert peculiar radial velocities into peculiar Z -velocities. But it should be remembered that with increasing distance from the galactic plane, the high Z -velocity stars will have a greater say in determining these f -factors. Consequently, for the faint-star radial velocities, conversion factors f should contain more and more of the influence of the high-velocity stars. (The accompanying change in the proportion of strong- and weak-line stars is not so important

¹⁾ The majority of giant stars were identified by the luminosity classification given in the WILSON catalogue; however some K stars without such classifications were added to the list on account of their small proper motion.

²⁾ This expression may be deduced from CHANDRASEKHAR (1942, relation 1.305).

TABLE 12
Data relevant to the estimation of θ_{ho}^2 and $|Z|_{ho}$.

	Distance group		$\bar{z}$ pc	n	estimated proportions			$ Z _{obs}$ p.e.
	$40^\circ \leq b \leq 55^\circ$	$ b > 55^\circ$			str-l	wk-l	hv	
	m	m						
1	< 6.0	< 5.5	0	259	.4	.4	.2	14.0 $\pm$.4
2	6.0-7.0	5.5-6.5	120	324	.34	.42	.24	13.6 $\pm$.4
3	7.0-8.0	6.5-7.5	200	141	.25	.45	.30	15.6 $\pm$.6
4	8.0-9.0	7.5-8.5	300	42	.11	.45	.44	21.2 $\pm$ 1.7
5			1000	78	.00	.19	.81	24.1 $\pm$ 1.3

Note: Three of 42 stars of the fourth group had $|Z|$, 85, 100, and 203 km sec⁻¹. Since this seemed an unusually large proportion of high velocities when considered in relation to the other groups, they have been given half weight in computing $|Z|_{obs}$.

because their axial ratios are very similar). However this weakness is unlikely to be very detrimental for the four distance groups considered here, despite the fact that nothing is known of the axial ratios for the high-velocity group being sought. Assuming tentatively that strong-line, weak-line and high-velocity stars are in the proportions 0.4:0.4:0.2 in the galactic plane¹⁾ and that these groups have average Z -velocities of 8.9, 14.9, and ∞ , then the proportions of stars belonging to the three components at the relevant distances are to be found in columns 6, 7 and 8 of Table 12. This computation, in illustrating a rather extreme case, shows that in the nearest three groups the high-velocity stars can have no decisive influence on the average conversion factors unless they have peculiar space motions distributed in a fashion markedly different from those of the other components. It would be amongst the fourth and particularly the fifth distance group that their influence could be felt most strongly. As indicated by column 4 of Table 12, not so many stars fall in the fourth group so that the overall effect of the high-velocity stars on conversion factors for radial velocities required in this initial stage of the analysis should not be very great. On the other hand the conversion factors computed in the above manner and used for the fifth distance group may be considerably in error since Table 12 shows that this group may well contain a very considerable proportion of the high Z -velocity stars. The error, if any, incurred here cannot unfortunately be assessed at present.

The observed average Z -velocity obtained for each distance group is to be found in the last column of Table 12. There seems to be evidence here of the existence of a small increase in $|\bar{Z}|_{obs}$ within 300 pc of the plane — but the most interesting feature of these data is the distinct increase in $|\bar{Z}|_{obs}$ over the first 1000 pc from the plane. This increase is undoubtedly a manifestation of the greater influence of high Z -velocity stars in regions remote from the plane.

For each of the first four distance groups the relation between θ_{hv}° and $|\bar{Z}|_{hv}$ may now be computed from equation (8). In this calculation the OORT-VAN WOERKOM K_z given in Figure 14, was used. Parameters p and q have been computed with $|\bar{Z}|_{str} = 8.9$

¹⁾ It has already been noted in connection with Table 10 that strong- and weak-line stars probably constitute about 90% of the F5 to K1 stars. Thus in this tentative computation, in which the OORT-VAN WOERKOM K_z -curve of Figure 14 was used, it seemed reasonable to suppose that 20% of local stars belong to the high-velocity group. This is probably an overestimate, but it should indicate the maximum proportion of high Z -velocity stars that are likely to influence calculations of f -factors. The extremeness of the conditions for this computation has been further accentuated by the assumption that $|\bar{Z}|_{hv}$ is infinite.

and $|\bar{Z}|_{wk} = 14.9$ km sec⁻¹; r was estimated assuming $|\bar{Z}|_{hv} = 25$ km sec⁻¹. For the tentative values of θ_{hv}° of 0.05, 0.10, 0.15 and 0.20, the values of $|\bar{Z}|_{hv}$ computed from each of these equations are shown in columns 2 to 5 in Table 13. With the exception of data from the last of the four distance groups, the agreement of results is surprisingly good. For each value of θ_{hv}° , a weighted average of the four estimates of $|\bar{Z}|_{hv}$ has been adopted — these are shown in Table 13, column 6. In this computation half weight has been given to the most distant of the first four groups, for despite the

TABLE 13

Estimates of $|\bar{Z}|_{hv}$ for several values of θ_{hv}° together with corresponding $|\bar{Z}|$ expected at $z = 1$ kpc

θ_{hv}°	$ \bar{Z} _{hv}$ distance group				$ \bar{Z} _{hv}$ (average)	$ \bar{Z} $ at $z = 1$ kpc
	1	2	3	4		
.05	54	36	52	84	53 ± 3.8 p.e.	42.1 ± 3.5 p.e.
.10	33	24	33	51	33 ± 1.5	29.7 ± 1.8
.15	23	20	26	40	26 ± 1.1	24.0 ± 1.3
.20	22	18	23	34	23 ± 1.0	21.6 ± 1.0

fact that small numbers of stars in the group would suggest an even smaller weight, this is offset by the fact that they should contain a greater proportion of the high-velocity component, and hence should provide more reliable information about its presence. The precision of these estimates of $|\bar{Z}|_{hv}$ might be improved further by repeating the calculation for each θ_{hv}° using values of r calculated with the new $|\bar{Z}|_{hv}$; such computations however produce no significant alterations.

Finally the probable errors given with the values are based solely on the residuals of the four average values of $|\bar{Z}|_{hv}$ obtained from each distance group from the weighted mean — they take no account of the systematic errors introduced into the estimated p , q , and r by uncertainties in the K_z adopted at this stage.

A decision as to which combination of θ_{hv}° and $|\bar{Z}|_{hv}$ is most satisfactory has been reached on the basis of the average Z -velocity each combination would indicate at a distance of 1 kpc from the galactic plane. This mean velocity may be estimated with the aid of the OORT-VAN WOERKOM K_z and is shown in column 7 of Table 13 for each value of θ_{hv}° considered.

For the distance of 1 kpc from the galactic plane the average Z -velocity has been computed using radial-velocity material kindly provided by Dr EDMONDSON of the Goethe Link Observatory. This material provides, in addition to the radial velocities, accurate photovisual magnitudes, spectral types, and luminosity classifications of many high-latitude stars. 78 K stars of luminosity class III were used, their

distances ¹⁾ from the galactic plane ranging from 700 to 1300 pc. After correction for solar motion, the Z-velocities were estimated using conversion factors. The average Z-velocity so obtained was 24 ± 1.3 p.e. km sec⁻¹. Comparison of this observed result with the computed estimates in column 7 of Table 13 would indicate the values $|\bar{Z}|_{hv} = 24 \pm 1.8$ (p.e.) km sec⁻¹. This result is quite independent of any systematic error which may be present in EDMONDSON's velocities, since we are concerned only with dispersions.

The complete set of parameters used to describe the distribution of Z-velocities of giant K stars in the galactic plane is given in Table 14.

TABLE 14

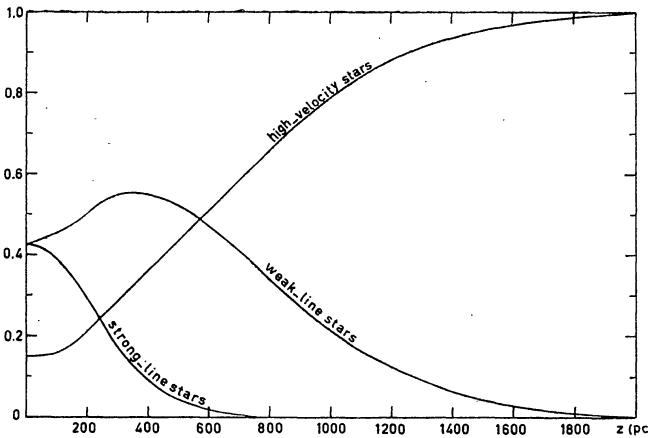
Finally adopted parameters for the gaussian components of the K-giant Z-velocity distribution at the galactic plane.

Star group	θ°	$ \bar{Z} $
Strong-line	0.425	8.9
weak-line	0.425	14.9
high-velocity	0.150	24.0

By means of expression (3b), the data contained in Table 14, and the initially adopted K_z , the relative proportions in which the three Z-velocity components occur at various z-distances from the galactic plane

¹⁾ These distances could be estimated using absolute magnitudes given by W. W. MORGAN, KEENAN and KELLMAN in "An Outline of Spectral Classification" accompanying their "Atlas of Stellar Spectra". Allowance for interstellar absorption was made using the smoothed-out model of the absorbing material given by VAN RHIJN (1949b).

FIGURE 6



Computed proportions of strong-line, weak-line, and high-velocity stars to be expected at various z-distances to 2 kpc. The computations were carried out with the data of Table 14 and the OORT-VAN WOERKOM K_z shown in Figure 14.

may be computed. The results of this computation are shown in Figure 6. Although the curves shown in this figure are based on a very simplified picture of the K-giant star Z-motions and so are only approximate, they should give quite reliable impressions; providing of course that the K_z used is reasonably accurate. It is seen for example that the proportion of strong-line stars diminishes very rapidly with increasing z, and that beyond $z = 400$ pc they play no vital role. On the other hand, weak-line stars do not relinquish their claim to importance until z reaches about 1 kpc. The feature of this diagram which is of most interest however is the rapid rise to ascendancy by high Z-velocity stars. Within 500 pc of the galactic plane the proportion of high-velocity stars rises from

TABLE 15

A comparison between the five observed Z-velocity distributions and corresponding distributions computed from the analysis of Z-motions near the galactic plane.

Z-interval km sec ⁻¹	1		2		3		4		5	
	obs.	comp.	obs.	comp.	obs.	comp.	obs.	comp.	obs.	comp.
0-5	70	66.4	86	80.1	34	32.8	7	9.0	8	11.0
5-10	52	56.8	61	69.2	32	28.8	5	7.9	9	10.7
10-15	44	44.5	61	55.1	25	23.3	5	6.6	13	9.9
15-20	31	31.0	40	39.1	8	17.1	7	5.2	14	8.8
20-25	20	20.2	26	26.3	8	11.9	6	3.8	3	7.4
25-30	18	13.4	20	17.8	11	8.4	6	2.8	8	6.5
30-35	8	8.7	12	11.8	9	5.8	1	2.0	10	5.4
35-40	7	5.8	7	7.9	5	4.0	1	1.4	5	4.3
40-45	2	3.9	3	5.3	3	2.7	1	1.0	2	3.6
45-50	1	2.6	5	3.6	3	1.9	0	0.7	1	2.8
>50	6	5.7	3	7.8	3	4.3	3	1.6	5	7.6
Stars	259		324		141		42		78	
$ \bar{Z} $ km sec ⁻¹	14.0		13.6		15.6		21.2		24.1	
p.e.	± 0.4		± 0.4		± 0.6		± 1.7		± 1.3	

15% to about 50% and at $z \simeq 1200$ pc such stars comprise about 90% of all K-giant stars. This means that in calculations of the z -density distribution of K stars it is essential to know as much as possible about the characteristics of high-velocity stars if z -distances greater than about 200-300 pc are contemplated. The paucity of high-speed stars near the sun is a serious obstacle to progress in studying their role in the structure of the Galaxy and until efficient means are available for selecting, for closer examination, representative samples of them from the faint stars, calculations of K_z must be somewhat tentative.

To exhibit the extent to which the analysis of Z -velocities near the galactic plane has been successful in representing the observational material, expected distributions of Z -velocity at z -distances corresponding to the five distance groups have been computed on the basis of the velocity distribution given in Table 14. Comparison has been made in Table 15 with the observed distributions. On the whole the computed distributions seem to be very satisfactory. The only significant discrepancies occur in the numbers in the second and fourth distance groups of the stars with $|Z| > 50$ km sec⁻¹. But since these defects are in opposite directions, they do not detract greatly from the overall agreement.

12. Transverse-velocity distributions of strong- and weak-line stars

At the outset it should be realized that strictly what is needed are the transverse-velocity distributions of the groups of stars defined by the gaussian Z -velocity distributions, and not those of Miss ROMAN's strong- and weak-line stars. This objective might have been

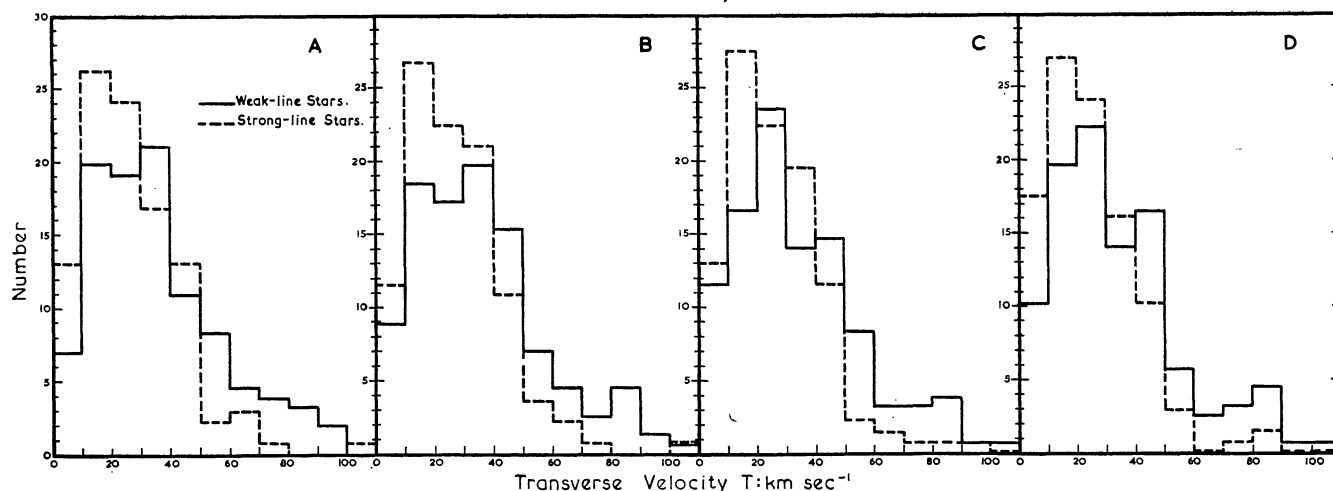
achieved by calculating the transverse-velocity distributions of strong- and weak-line stars for Z -intervals of Table 11, and combining them with weights proportional to the numbers in the two "computed" columns of that table. However the resemblances between the two Z -velocity distributions for each star group are sufficiently great to render such a detailed treatment quite unnecessary.

Therefore, since space motions are available for all stars classified as either strong- or weak-line stars, it is possible to obtain the relevant transverse-velocity distributions characteristic of each of these groups by combining data from individual members with equal weight. This method involves the assumption, already made when Z -velocities of K giants were being studied, that there is no variation with spectral or luminosity class in kinematical properties within either the strong- or weak-line group. This assumption is necessitated by the limited number of recognized members of these groups.

Space motions, relative to the sun¹⁾ have been taken from the unpublished list compiled by Dr J. E. baron DE VOS VAN STEENWIJK and kept at the Leiden Observatory. This list gives space motions in the form of components in the directions $l = 0^\circ$, $b = 0^\circ$; $l = 90^\circ$, $b = 0^\circ$; $b = 90^\circ$, and uses parallaxes obtained from Mount Wilson spectroscopic absolute magnitudes (ADAMS, JOY, HUMASON and BRAYTON 1935) with corrections suggested by VAN RHIJN (1949a).

¹⁾ It is clear that solar motion should not be removed from these motions—its influence is of course present in the proper motions and its removal from the latter, if that were possible, would only deny the use of one of the two means of gauging distance.

FIGURE 7



Histograms of the projections of space motions of known strong- and weak-line stars onto planes perpendicular to the four lines of sight noted. Known members of clusters have been omitted. These histograms were derived to ascertain the extent of the variation in the transverse-velocity distributions over the sky regions from which proper-motion material was drawn.

Velocity distributions of strong- and weak-line stars transverse to the directions

	1	2	3	4
l	197°	237°	297°	341°
b	$+69^\circ$	$+60^\circ$	$+62^\circ$	$+70^\circ$

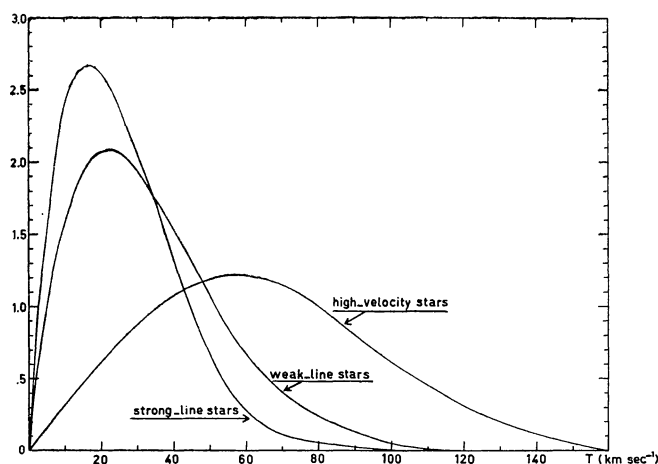
have been computed. These directions are indicated on Figure 2 by small squares and are the centroids of the four sky zones created in the north galactic polar zone $b \geq 50^\circ$ by the lines $\alpha = 12^h 30^m$ and $\delta = +10^\circ$. Distributions of transverse velocity computed for these directions should be representative of the whole sky region from which proper motions are drawn — they are shown in Figure 7.

Since no outstanding differences are present between either of the two sets of distributions they have been averaged into two distributions — one representative of weak-line stars, the other for strong-line stars — over the sky regions from which proper motions are available. These average distributions after smoothing, and correction for small grouping effects, are shown in Figure 8. They have been normalized to 100 stars. No attempt has been made to correct these distributions for accidental errors in proper motions, parallaxes, or radial velocities.

13. Transverse-velocity distributions for the high Z -velocity stars

All that has been revealed concerning the high Z -velocity stars in the study described in section 11, has been a rough estimate of the proportion near the galactic plane, and an equally tentative estimate of their mean Z -velocity. Because of the statistical

FIGURE 8



Adopted transverse-velocity distributions of strong-line and weak-line stars and high-velocity stars for use in the analysis of proper-motion distributions of K stars with $|b| > 40^\circ$.

All distributions are normalized to 100 stars.

character of the analysis, it is not possible to pick out a representative group of high Z -velocity stars and derive the necessary transverse-velocity distributions from their space motions, as has already been done for strong- and weak-line stars.

It seems therefore that a guess must be made concerning the motions of high Z -velocity stars parallel to the galactic plane. In the first place it has been assumed that their peculiar space motions may be represented by a velocity ellipsoid with two of its axes lying in the galactic plane (the principal axis being in the direction of the galactic centre at $l = 327^\circ$), and that the required transverse-velocity distributions are represented sufficiently well by the components parallel to the galactic plane¹). The velocity dispersion for the ellipsoid axes in the galactic plane have been chosen so that they bear about the same ratios to the already computed Z -velocity dispersion as have been found in the Appendix for the weak-line stars. Having thus assessed the velocity dispersions along the principal axes of the velocity ellipsoid, the solar motion may be deduced from the relation between the dispersion of peculiar velocities along the B -axis, i.e., the θ -direction in the notation of section 1, and solar motion which has been obtained by STROMBERG (1925, see formula 9).

The required distribution of transverse velocities T , is then given by

$$F_{ho}(T) = \frac{1}{2\pi AB} e^{-\frac{1}{2}\left(\frac{a^2}{A^2} + \frac{b^2}{B^2}\right)} T \times \quad (10)$$

$$\int_0^{2\pi} e^{-\frac{1}{2}\left\{T^2\left(\frac{\cos^2\theta}{A^2} + \frac{\sin^2\theta}{B^2}\right) - 2T\left(\frac{a\cos\theta}{A^2} + \frac{b\cos\theta}{B^2}\right)\right\}} d\theta,$$

where A and B are the velocity dispersions along the principal axes of the velocity ellipsoid which lie in the galactic plane; a and b are the components of the solar motion along the A - and B -axes. This expression is normalized to unity.

Using the parameter values

$$\begin{aligned} a &= 0 \text{ km sec}^{-1} & b &= 38 \text{ km sec}^{-1} \\ A &= 50 \text{ km sec}^{-1} & B &= 38 \text{ km sec}^{-1} \end{aligned}$$

in expression (10), the required transverse-velocity distribution of high-speed stars is obtained. It is shown in Figure 8, normalized to 100 stars.

¹) There seems to be little point in carefully projecting the velocity ellipsoid into planes transverse to the various proper-motion zones and deriving the corresponding distributions of transverse velocity. The uncertainties associated with the representation of the high velocities and parameters used in the representation smother any advantages that such a course of action may possess.

14. Estimation of the transverse-velocity distributions of giant K stars as a function of z -distance

With the data given in Figures 6 and 8, the transverse-velocity distribution of K giants, at any z -distance up to 1400 pc, can be computed from

$$F(T, z) = \theta_{str}^z F_{str}(T) + \theta_{wk}^z F_{wk}(T) + \theta_{hv}^z F_{hv}(T), \quad (11)$$

where

$$\theta_{str}^z = \frac{p \theta_{str}^0}{p \theta_{str}^0 + q \theta_{wk}^0 + r \theta_{hv}^0}, \text{ etc.} \quad (12)$$

The factors, p , q and r may be estimated at any distance from the galactic plane for any K_z by means of

$$p = e^{\frac{1}{Z_{str}^2} \int_0^z K_z dz}, \text{ etc.} \quad (13)$$

Since $\theta_{str}^z + \theta_{wk}^z + \theta_{hv}^z = 1$, and $F_{str}(T)$, $F_{wk}(T)$, and $F_{hv}(T)$ are each normalized to 100 stars, $F(T, z)$ will also be so normalized. Four sample distributions of $F(T, z)$ are shown in Figure 9 to illustrate the considerable change which the transverse-velocity distribution of K giants undergoes in the first 1400 pc from the galactic plane as a result of the changing population of Z -speed groups. They have been calculated with the OORT-VAN WOERKOM field. The velocity distribution $F(T, z)$ will not change significantly beyond $z = 1400$ pc because already at this distance nearly all the stars belong to the high-speed group. (This last remark applies strictly only to the simplified representation of the Z -speed distribution in this work — in

reality considerable variations in $F(T, z)$ may occur at great z -distances due to the details in the high Z -speed distribution which cannot be examined very precisely).

Velocity distributions, $F(T, z)$, computed in this manner constitute the nucleus of all subsequent calculations of total proper-motion distributions, whether they be for giant, or dwarf K stars.

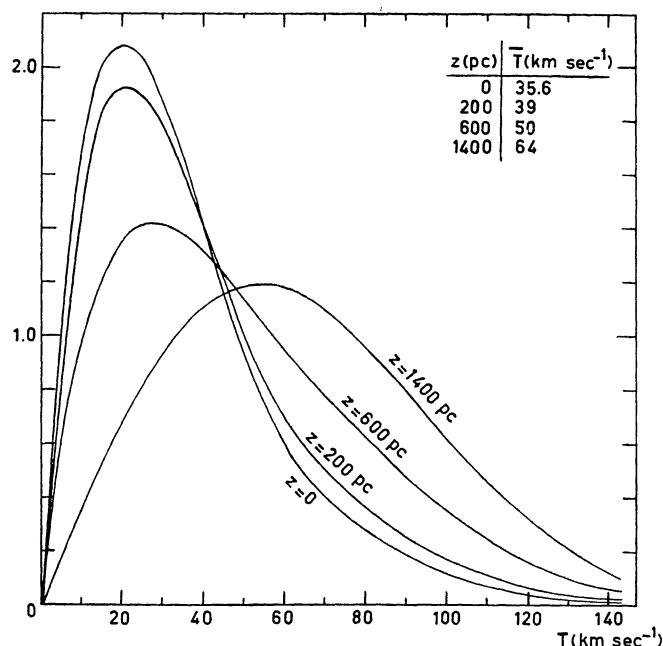
III. THE COMPUTATION OF K_z

15. Introduction

In essence the method described in the next three sections for computing a new K_z is the following. The observed total proper-motion distributions, corrected for accidental errors of observation, have been analysed in such a way as to yield the percentage of stars in several luminosity groups at various apparent magnitudes¹⁾. Two luminosity groups have been considered; one comprising dwarfs and "subgiants", later called the dwarf-"subgiant" group, the other composed of giants, bright giants, and supergiants, called the giant group.

Percentages of stars in the luminosity groups at the mean apparent magnitudes of the observed proper-motion distributions are deduced by finding the combination of proper-motion distributions characteristic of each group which agrees most satisfactorily with the true distributions. Estimates of proper-motion distributions expected of stars of each luminosity group at each apparent magnitude are made with the aid of an adopted luminosity function together with transverse-velocity distributions as a function of z given earlier, and with a tentative z -density distribution obtained from the OORT-VAN WOERKOM K_z and the Z -velocity distribution arrived at in section 11. The computed proper-motion distributions must at this stage be considered with caution, particularly in respect to the extent to which they depend on the high-velocity group of stars whose proportion both in and away from the plane, and whose transverse-velocity distribution are necessarily more uncertain than those of the other two velocity groups. Furthermore, the fact that much of the preliminary computations depends on a preliminary K_z , e.g. the analysis of the Z -velocity distribution, contributes to

FIGURE 9



Distributions of transverse velocity of K stars computed from data of Figures 6 and 8, at distances 0, 200, 600, and 1400 pc from the galactic plane. All distributions are normalized to 100 stars.

¹⁾ In general the different luminosity classes do not appear in the same proportions over the entire range of apparent magnitude. For example, the proportion of dwarf K stars may be expected to increase towards the fainter apparent magnitudes. This comes about because of the rapidity with which the star density falls off with increasing distance from the galactic plane at the great distances from which the giants are observed. Differences possibly existing in the density distributions of giants and dwarfs would of course play a part in determining the relative percentages of the two groups at various apparent magnitudes; this question cannot be investigated for reasons already given in section 10.

the provisional nature of computed proper-motion distributions.

With estimates of the percentages of the luminosity groups present at various apparent magnitudes, numbers of stars for each group separately may then be computed from the counts of all K stars derived in section 6. For the purposes of estimating K_z the counts for K giants alone have been used.

The main reason for using K giants is that the K-star counts on which the present work is based may include a considerable number of G stars (cf. section 6). This admixture of G stars will much less influence the counts and luminosity distribution in the giant group than it will in the dwarf group. Thus the separation of the luminosity groups through the transverse-velocity and proper-motion distributions should yield a giant group which closely resembles the HD K giants. The resemblance should be considerably better than that between the entire group classified as K from the BSD and the HD K stars.

A new K_z may now be arrived at by trial and error, namely by computing a K_z which will make the density distribution as derived from the Z-velocity distribution of section 11 and the relation (4b), yield the above counts of K giants when combined with the luminosity function of the giants. This K_z should be an improvement over the provisional one in so far as it satisfies additional observational data, i.e. the K-star counts and the observed proper-motion distributions.

Finally there is some discussion in sections 18 and 19 of the comparisons of present results with the work of others, and of possible work which might be pursued in the future.

16. The percentage of K stars of several luminosity classes present at various apparent magnitudes at high galactic latitudes

It was mentioned in the introduction above that K stars have been subdivided into several groups on the basis of luminosity. In the actual calculations the groups defined in terms of absolute visual and photographic magnitudes in Table 16 were considered.

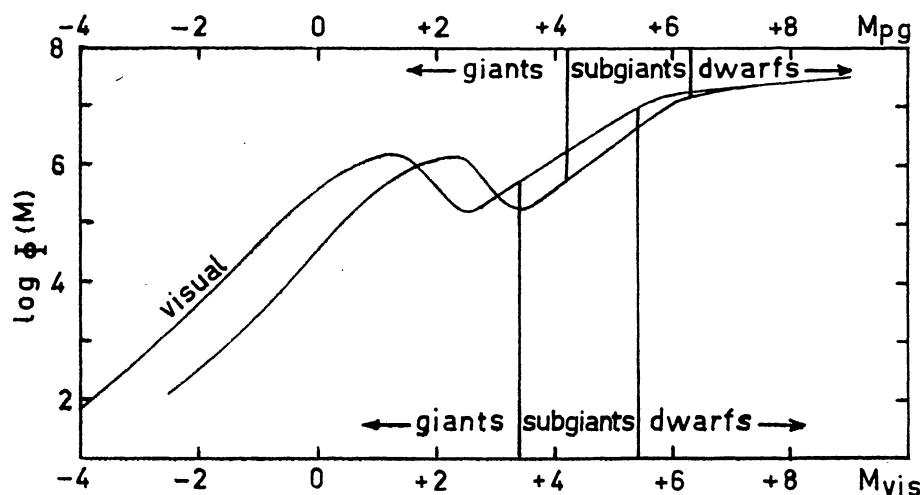
TABLE 16

Absolute magnitude limits for luminosity groups used in density computations.

Groups	M_{vis}	M_{pg}
dwarfs	$\geq +5.4$	$\geq +6.3$
"subgiants"	$+3.4$ to $+5.4$	$+4.2$ to $+6.3$
giants	$\leq +3.4$	$\leq +4.2$

It should be pointed out that the "subgiants" introduced here are not identical to spectroscopic subgiants. Spectroscopic "subgiants" of type K are non-existent or extremely rare (cf. JOHNSON and MORGAN 1953). The "subgiants" of Table 16 must be mainly G dwarfs erroneously classified as K dwarfs (cf. previous section and Section 6). For computing the percentage of giants, the dwarfs and "subgiants" were combined, thus forming the dwarf-"subgiant" group. In some subsequent tables, items relating to the separate components of the dwarf-"subgiant" group are given for the sake of completeness. Their relation to the luminosity function, $\Phi(M)$, of K stars near the sun, as accepted by VAN RHIJN and SCHWASSMANN (1935), and adopted later in the present study, is illustrated in Figure 10.

FIGURE 10



The VAN RHIJN-SCHWASSMANN luminosity function, $\Phi(M)$, (K stars per cubic kpc between $M - \frac{1}{2}$ and $M + \frac{1}{2}$) as a function of both visual and photographic magnitude. Indications of the luminosity groups into which these stars were divided are also given.

From the luminosity function shown above, only the giants appear quite clearly about $M_{vis} = +1.2$. Of the dwarfs, the luminosity function would seem to convey little information except of their existence; towards the fainter end there are clear signs of a flattening of the function after a steady rise, but the data are not extensive enough at the fainter end to get as precise information as yielded by the giants. However this is not a serious defect because it is only the brighter and better defined part of the dwarf section of the luminosity function that is of importance for the present work. The number of the brighter dwarfs seen at any apparent magnitude is so accentuated by the larger volume they occupy, due to the considerable dispersion in absolute magnitudes, that the lack of accurate knowledge about the faint ones is not troublesome. A more reliable impression of the absolute magnitudes populated by K dwarfs may be derived from the study by KUIPER (1942) of the observational data available for stars within 10.5 pc of the sun. His results show that the mean visual absolute magnitude of K dwarfs runs from +6.0 at K₀ to +8.8 at K₉. With a dispersion of absolute magnitudes of ± 0.5 (this is little more than a guess) about the mean it would appear that K dwarfs extend from $m_{vis} + 5.5$ to about +9.5. For the present purposes, all K stars with $M_{vis} \geq 5.5$ have been grouped together as dwarfs.

Of the remaining stars contributing to the luminosity function, the giants, which may be considered to occupy the region between M_{vis} 0.0 to +2.3, are combined on the one side with the comparatively rare stars brighter than 0.0 and with stars having $M_{vis} + 2.3$ to +3.4 to form another group. The remainder of the stars from $M_{vis} + 3.4$ to +5.5 have been called "subgiants".

Limiting magnitudes in the photographic system are obtained with adequate precision by using an average of the colour-indices of K giants and dwarfs as found in ALLEN's tables (1955).

Computations of the relative distance distributions of stars in each of the three luminosity groups for the apparent magnitudes of the observed total proper-motion distributions may be made quite simply. The luminosity functions derivable from Figure 10 for each group are used and allowances made for the star-density gradient and interstellar absorption. Imagining the volume of a cone of small solid angle directed at $b = 59^\circ$ (the average galactic latitude of the observed total proper-motion distributions) to be split up into elements of thickness d ($\log r$), the relative number of K stars in the luminosity group seen at apparent magnitude m and contained in one such volume element is

$$n(r) d \log r = \frac{\Phi(M) \frac{\Delta(z)}{\Delta(0)} r^3 d \log r}{\int_{\log r_1}^{\log r_2} \Phi(M) \frac{\Delta(z)}{\Delta(0)} r^3 d \log r}, \quad (14)$$

where $M = m + 5 - 5 \log r - \alpha(r)$, $z = r \sin 59^\circ$, and $\alpha(r)$ is the integrated interstellar absorption along the axis of the cone. The limits r_1 and r_2 are the distances corresponding to the limiting absolute magnitudes of the group in question. The distribution of the integrated absorption, $\alpha(r)$, may be found accurately enough from the smoothed-out picture of the absorbing layer about the galactic plane which has been prepared by VAN RHIJN (1949b). $\alpha(r)$ derived in this way is tabulated in Table 17. The relative

TABLE 17

Integrated interstellar absorption from the galactic plane to various z -distances at $b = 59^\circ$.

z (pc)	α	z (pc)	α
	m		m
20	.03	160	0.18
40	.07	320	.23
80	.13	640	.25

Note (i) Since the absorptions are small, the difference between visual and photographic absorptions are minor so that the averages of the two, which are given above, are adequate.

(ii) Beyond $z = 640$ pc there is no absorption.

density factor $\frac{\Delta(z)}{\Delta(0)}$ must be inserted in the above expression since the luminosity function $\Phi(M)$ gives the numbers of K stars per cubic parsec near the plane. The density factor may be estimated with ample precision using the Z -velocity distribution proposed in section 11, with the percentages given in Table 14, together with the OORT-VAN WOERKOM K_z . These data inserted in relation (4b) allow $\frac{\Delta(z)}{\Delta(0)}$ to be estimated; it is illustrated in Figure 13.

Before proceeding it might be mentioned that both the absorption, amounting to about 0.25 mag., and density-gradient corrections to the distance distributions need only be known approximately; the outcome of the present computations is not significantly affected by possible errors in these quantities.

Once in possession of the distance distributions, distributions of total proper motion are readily obtained using the estimated proportions of the three velocity groups present at each z -distance (see Figure 6), in conjunction with the transverse-velocity distributions appropriate to each velocity

group (see Figure 8). Clearly the total proper-motion distribution of each luminosity group at apparent magnitude m is given by

$$n(\mu) d\mu = 4.74 \int_{\log r_1}^{\log r_2} r n(r) F(T, z) d \log r d\mu, \quad (15)$$

where $T = 4.74 \mu r$; $n(r)$ and $F(T, z)$ are defined by the relations (14) and (11) respectively.

In this manner the complete array of total proper-

motion distributions was computed for each of the three luminosity groups at the nine apparent visual and photographic magnitudes of the observational material. The computed distributions are summarized in Table 18.

Since it is essential to have some impression of the extent to which the high-velocity stars, which form the least known group, contribute to the frequencies of proper motions, rough average z -distances of stars contributing to each frequency have been inserted in italics beneath the frequencies in Table 18.

TABLE 18

Distributions of total proper motions as computed for K dwarfs, "subgiants", and giants, for nine apparent magnitudes at galactic latitude 59°

	5.1 _{vis}			6.1 _{vis}			6.9 _{vis}			8.1 _{vis}			8.9 _{vis}			9.4 _{vis}			12.0 _{pg}			12.9 _{pg}			13.8 _{pg}		
	d	sg	g	d	sg	g	d	sg	g	d	sg	g	d	sg	g	d	sg	g	d	sg	g	d	sg	g	d	sg	g
$n/a_{.000-.005}$	—	—	0.2	—	—	0.7	—	—	1.7	—	—	4.8	—	—	7.3	—	—	8.7	—	1.4	15.8	—	4.6	20.9	2.3	7.1	27.3
									<i>270</i>			<i>320</i>			<i>440</i>			<i>500</i>		<i>210</i>	<i>900</i>		<i>270</i>	<i>1260</i>	<i>220</i>	<i>390</i>	<i>1600</i>
$.005-.010$	—	—	2.3	—	—	5.1	—	—	7.3	—	—	11.0	—	1.2	15.1	—	2.6	17.4	2.3	7.1	27.9	4.8	10.4	36.1	7.2	14.8	41.7
									<i>220</i>			<i>310</i>			<i>85</i>			<i>115</i>		<i>100</i>	<i>180</i>		<i>130</i>	<i>270</i>	<i>190</i>	<i>390</i>	<i>1500</i>
$.010-.020$	—	—	7.0	—	—	11.2	—	—	16.3	—	2.6	24.2	—	5.0	30.0	—	6.4	32.5	5.9	16.2	38.8	10.7	24.1	35.6	17.2	30.0	28.0
									<i>210</i>			<i>290</i>			<i>70</i>			<i>90</i>		<i>90</i>	<i>170</i>		<i>130</i>	<i>260</i>	<i>190</i>	<i>370</i>	<i>1260</i>
$.020-.030$	—	—	8.2	—	—	12.5	—	1.9	16.4	0.6	4.8	20.4	2.9	6.2	20.7	3.6	8.3	20.3	7.8	17.1	13.5	12.6	21.1	6.5	17.7	21.3	2.4
									<i>190</i>			<i>270</i>			<i>30</i>			<i>50</i>		<i>90</i>	<i>170</i>		<i>130</i>	<i>250</i>	<i>190</i>	<i>350</i>	<i>730</i>
$.030-.060$	—	2.2	24.0	—	5.1	31.2	2.4	7.3	33.6	5.4	13.7	29.0	7.6	21.4	22.5	11.3	26.1	18.8	25.2	35.4	3.9	33.2	30.0	0.9	32.9	23.0	0.6
									<i>170</i>			<i>240</i>			<i>30</i>			<i>50</i>		<i>80</i>	<i>160</i>		<i>120</i>	<i>240</i>	<i>160</i>	<i>330</i>	<i>540</i>
$.060-.100$	—	4.3	23.8	2.5	5.2	22.2	3.1	11.2	16.9	7.8	19.8	8.8	12.4	25.5	3.9	17.2	25.7	2.1	25.9	16.8	0.1	22.2	8.6	—	15.4	3.5	—
									<i>150</i>			<i>210</i>			<i>30</i>			<i>50</i>		<i>80</i>	<i>150</i>		<i>110</i>	<i>220</i>	<i>160</i>	<i>270</i>	
$.100-\infty$	100.0	93.5	34.5	97.5	89.7	17.1	94.5	79.7	7.8	86.2	59.1	1.8	77.1	40.7	0.5	67.9	30.9	0.2	32.9	6.0	—	16.5	1.2	—	7.3	0.3	—
									<i>120</i>			<i>160</i>			<i>30</i>			<i>40</i>		<i>70</i>	<i>140</i>		<i>90</i>	<i>210</i>	<i>110</i>	<i>220</i>	
Percentages	0.6	1.4	98.0	1.2	1.5	97.3	1.3	2.1	96.6	1.7	3.1	95.2	3.1	4.8	92.1	3.6	5.8	90.6	16.7	22.7	60.6	29.8	30.8	39.4	46.3	34.3	19.4

Notes:

1. The proper-motion distribution for each luminosity group is normalized to 100 stars.
2. Entries in italics are rough estimates of mean z -distances of the stars.
3. Numbers at the foot of each column give the relative numbers of the three luminosity groups as computed with the OORT-VAN WOERKOM K_z , VAN RHIJN-SCHWASSMANN luminosity function, and the estimated density distribution $\Delta(z)/\Delta(0)$.
4. The horizontal lines indicate the μ' used in the calculation of percentages of giants.

In the last line of Table 18 the relative numbers of giants, "subgiants", and dwarfs present at each apparent magnitude are given for the adopted K_z , the VAN RHIJN-SCHWASSMANN luminosity function, and the estimated density distribution $\Delta(z)/\Delta(0)$.

Remembering from Figure 6 that already at $z = 600$ pc, 50% of the stars belong to the high-velocity group, a glance at the entries in Table 18 will show which frequencies are liable to be most uncertain. In particular, proper-motion frequencies of giants rely heavily on the high-velocity star data for $m_{pg} \geq 12$; those of dwarfs and "subgiants" never depend greatly on high-velocity stars, since their maximum distance is about 390 pc. These uncertainties will manifest themselves initially among frequencies of high proper motions of nearby bright stars and spread steadily towards the lower proper motions as one considers the more distant fainter stars of which high-velocity stars form an increasing proportion.

We turn now to assessing the fractions of K giants by means of the proper motions. For a given apparent magnitude denote the fraction of the dwarfs and "subgiants" which has proper motions in excess of μ' by d' , and, similarly, the fraction of the giants which has proper motions in excess of μ' by g' . These fractions can be found from Table 18. Denote, further, the observed fraction of stars with proper motions in excess of μ' by α' . Then it is clear that, if d is the total fraction of dwarfs + "subgiants" among stars of the apparent magnitude considered,

$$\alpha' = d'd + g'(1 - d). \quad (16)$$

The two terms on the right-hand side of this expression give merely the fraction of stars, with $\mu > \mu'$, which are dwarfs and "subgiants", and giants respectively. The desired fraction of stars is then

$$g = 1 - d = \frac{d' - \alpha'}{d' - g'}. \quad (17)$$

The proper-motion limit μ' is to some extent arbitrary — all that its choice depends on is the obtaining of sufficient numbers of stars to provide adequate estimates of the three factors d' , g' and α' . The following values of μ' were used.

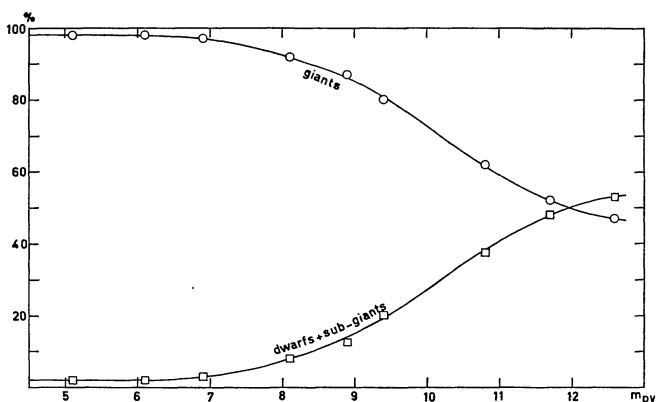
M_{vis}		M_{pg}	
5.1, 6.1 and 6.9	$0''/a.1$	12.0, 12.9 and 13.8	$0''/a.03$
8.1, 8.9 and 9.4	$0''/a.06$		

There is no reason why slightly different values should not be used; the results are insensitive to a reasonable choice of μ' .

In the above manner percentages of K giants amongst stars of various proper-motion distributions have been computed. Results are illustrated in Figure 11, in which they are shown as a function of m_{vis} ; a colour-index of $+1^m.2$ was used to convert photographic data to the visual scale. In Table 19 they are shown together with the ensuing estimates of K-giant counts; the latter are depicted in Figure 12.

A comparison of the distributions of total proper

FIGURE 11



Percentages of giants (bright, normal, and faint giants) and dwarfs and "subgiants" as a function of apparent magnitude, as computed from total proper-motion distributions. In uniting all the results in the m_{pv} scale, a colour-index of $+1^m.2$ was assumed for all luminosity groups.

TABLE 19

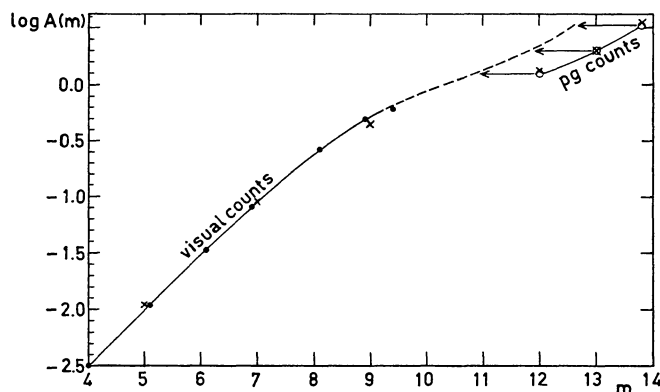
Estimates of the percentages of K giants at various apparent magnitudes and logarithms of the frequency per square degree per unit apparent-magnitude interval for $|b| = 59^\circ$

m_{vis}	% giants	$\log A(m)$	m_{pg}	% giants	$\log A(m)$
5.1	98	-1.97	12.0	62	0.09
6.1	98	-1.48	12.9	52	0.29
6.9	98	-1.09	13.8	47	0.51
8.1	93	-0.57			
8.9	88	-0.30			
9.4	80	-0.22			

Note: The proportions of giants were derived from the percentages of stars observed with $\mu' \leq 0''/a.1$ for $m_{vis} \leq 6.9$, $\mu' \leq 0''/a.06$ for m_{vis} 8.1, 8.9 and 9.4, and $\mu' \leq 0''/a.03$ for m_{pg} 12.0, 12.9 and 13.8 as computed directly from the histograms of Tables 9a and b. Yale-zone material was combined in proportion to the star numbers, whilst photographic data from the two sources were combined with equal weights.

motion computed with the percentages just derived and the proper-motion distributions of stars in the luminosity groups (see Table 18), with observed

FIGURE 12



The apparent-magnitude distribution of K giants at $|b| = 59^\circ$ as computed from data in Figures 3 and 11. Continuous curves represent visual and photographic counts, while the broken line continuing on from the visual-count curve is deduced from photographic counts with colour-index $+1^m.2$. Crosses show the counts as computed with the "K giant" K_z of Figure 14 and the Z-velocity distribution of Table 15.

TABLE 20

A comparison between computed and true total proper-motion distributions

μ -interval	5.1 _{vis}		6.1 _{vis}		6.9 _{vis}		8.1 _{vis}		8.9 _{vis}		9.4 _{vis}		12.0 _{pg}		12.9 _{pg}		13.8 _{pg}	
	obs.	comp.	obs.	comp.	obs.	comp.	obs.	comp.	obs.	comp.	obs.	comp.	obs.	comp.	obs.	comp.	obs.	comp.
$\mu/a \mu/a$																		
.000-.005	1.1 ± 0.8	2.5	4.9 ± 1.0	5.6	1.7 ± 0.7	1.6	3.7 ± 0.5	4.5	6.7 ± 0.5	6.4	3.6 ± 0.5	6.9	12.5 ± 1.8	10.1	10.7 ± 1.9	12.0	9.0 ± 2.8	14
.005-.010	1.8 ± 2.6	6.8	9.7 ± 1.4	11.0	4.7 ± 1.2	7.1	9.3 ± 0.8	10.1	14.8 ± 0.8	13.3	9.8 ± 0.8	14.2	16.7 ± 2.1	19.3	19.0 ± 2.6	22.5	20.6 ± 4.2	25
.010-.020	10.7 ± 2.4	8.1	11.9 ± 1.5	12.3	16.0 ± 2.1	15.8	23.2 ± 1.3	22.4	26.8 ± 1.1	26.6	26.8 ± 1.2	26.6	26.9 ± 2.7	28.4	27.1 ± 3.1	26.9	33.9 ± 5.4	26
.020-.030	19.7 ± 3.3	23.5	34.3 ± 2.6	30.6	18.9 ± 2.3	16.0	19.4 ± 1.2	19.0	17.3 ± 0.9	18.7	21.2 ± 1.1	17.5	14.6 ± 2.0	13.4	17.4 ± 2.5	11.5	13.0 ± 3.3	10
.030-.060	19.7 ± 3.3	23.5	22.8 ± 2.1	21.9	34.9 ± 3.2	32.8	29.0 ± 1.5	27.6	21.6 ± 1.0	21.7	23.9 ± 1.2	19.1	14.0 ± 1.9	14.3	18.9 ± 2.6	15.6	14.6 ± 3.6	14
.060-.100	19.7 ± 3.3	23.5	16.4 ± 1.8	18.6	14.6 ± 2.0	16.6	8.3 ± 0.8	9.3	6.3 ± 0.5	6.0	7.2 ± 0.6	6.3	7.0 ± 1.4	7.9	4.7 ± 1.3	7.3	4.4 ± 1.9	13
>.100	37.0 ± 4.5	35.6			9.2 ± 1.6	10.1	7.1 ± 0.7	7.1	6.5 ± 0.5	7.3	7.5 ± 0.7	9.4	8.3 ± 1.5	6.6	2.2 ± 0.9	4.2	4.5 ± 2.0	5

distributions, after correction for accidental errors, is made in Table 20. Taking into account the indicated mean errors of observed frequencies, and the fact that the transverse-velocity distribution used for high-velocity stars was little more than a guess, the agreement is quite good. If the differences between the two sets of data were purely accidental, they should lie within the mean errors in 68% of cases. Actually 62% lie *outside* the mean errors — many of these lying in the $\mu = 0''/a.000$ to $0''/a.010$ interval where the corrections given to the observed distributions for accidental errors are large and not as reliable as indicated by the mean errors. However since in most of these cases the computed frequencies are too high, it is possible that the frequency of small transverse velocities in the transverse-velocity distribution of high-velocity stars may require a small modification. This could be readily achieved using slightly larger peculiar-velocity dispersions A and B than those adopted.

The giant-percentage results deserve some comment. In the first place, already at m_{vis} 9.4, the percentages of giants given in Figure 11 are seen to begin to depart considerably from those computed from $\Phi(M)$ and $\Delta(z)$ and shown at the bottom of Table 18. This is the influence of the introduction of proper-motion distributions in assessing the extent to which luminosity groups are present at various apparent magnitudes. Secondly, it must be remembered that the percentage for m_{vis} 13.8 in particular and possibly to a lesser extent that for $m_{pg} = 12.9$ are somewhat more uncertain than the others, due to the proper-motion distribution for these stars not being as representative as the distributions for brighter stars. Thus the indication of a pause between m_{vis} 11.7 and 12.6 in the steady decline of the percentage of giants, which begins around m_{vis} 8, need not be taken too seriously.

Finally, several remarks must be made concerning the reliability of the percentages as computed above. It should be stressed at the outset that the percentages of giants have been computed via estimates of the percentages of dwarfs and "subgiants". This is an important feature to note, since in so doing, the percentages are far less dependent upon knowledge of transverse-velocity distributions of high-velocity stars than would have been the case had they been deduced directly from the investigations of giant stars. However, it is difficult to assess precisely the extent to which the inadequacies in present knowledge of high-velocity stars influence the percentages computed above. For although some idea of the uncertainties of the proportion of high-velocity stars in the plane, and their mean Z -velocities, can be obtained, there is nothing to indicate the reliability of the transverse-velocity distribution used for the

high-velocity stars. Therefore, at most it is only possible to give impressions of what percentages are yielded when the assumed Z -velocity distribution is modified within what would appear reasonable limits.

The percentages of giants at m_{pg} 12, 12.9 and 13.8 have been recomputed for the case in which strong-line, weak-line and high-velocity stars are present in the proportions 45:45:10 in the galactic plane instead of 42.5:42.5:15 as adopted in the preceding calculations.

If no modification is made to the transverse-velocity distributions of the three groups of stars, then the percentages of giants in the third column of Table 21 are obtained. Further, if the transverse-

TABLE 21

Percentages of K giants at photographic magnitudes as computed for several Z -velocity and transverse-velocity distributions

m_{pg}	percentage of giants		
	(1)	(2)	(3)
12.0	62	62	62
12.9	52	49	48
13.8	47	45	43

Note: The velocity distributions are as follows:

- (1) The data used for Table 19.
- (2) The data used for Table 19 with only 10% h.v. stars in the galactic plane.
- (3) The data used for Table 19 with 10% h.v. stars at $z = 0$, and a transverse-velocity distribution with $A = 40$ km/sec and $B = 30.4$ km/sec.

velocity distribution of high-velocity stars previously used is replaced by one in which the dispersions along the principal axes in the galactic plane are $A = 40$ km/sec and $B = 30.4$ km/sec, then the percentages given in column 4 of Table 21 result. The table shows that the combined effect of both modifications on the percentages of giants is no more than about 10% at the faintest magnitude, and nil at $m_{pg} = 12$. Similar alterations occur in the percentages in the opposite sense if modifications made to the velocity distributions are reversed.

To place the contents of Table 21 into perspective, data upon which some of the results are based are given in Table 22. The first five columns concern the distance distribution of "subgiants" and dwarfs as computed with the density distribution based on the Z -velocity distribution of Table 14, and the OORT-VAN WOERKOM K_z . The sixth column gives the minimum transverse velocity possessed by stars in the distance intervals with $\mu > 0''/a.03$ ¹⁾. The following three columns give the percentages of weak-line,

¹⁾ This is the proper-motion range of dwarfs and "subgiants" which decide the giant-star percentages among faint stars.

TABLE 22
(see text for explanation)

log <i>r</i>	<i>z</i> (pc)	contributions to percentage of dwarfs and "subgiants" with $p > 0.1203$			T_{lim} km/sec	percentages of vel. groups and for each group the % with $T > T_{lim}$			(hv)'
		$m_{pg} = 12$	$m_{pg} = 12.9$	$m_{pg} = 13.8$		str-l	wk-l	hv	
1.5-1.7	34	1.9	0.8	0.4	5.6	40 (93)	49 (96)	11 (99)	99
1.7-1.9	54	7.6	2.5	1.3	8.3	37 (86)	51 (92)	12 (98)	98
1.9-2.1	86	22.9	8.6	4.0	14.2	34 (73)	53 (82)	13 (97)	96
2.1-2.3	136	25.9	21.7	10.6	22.4	28 (51)	56 (66)	16 (92)	90
2.3-2.5	215	9.0	15.2	19.0	35.7	20 (24)	61 (41)	19 (81)	76
2.5-2.7	340	2.4	3.4	6.5	56.5	11 (4)	63 (14)	26 (58)	48
		69.7	52.2	41.8					

strong-line, and high-velocity stars as computed with 10% high-velocity stars in the plane, and the OORT-VAN WOERKOM K_z . In brackets are the percentages of stars of the three velocity groups with $T > T_{lim}$. If conversion is made to the high-velocity transverse-velocity distribution with $A=40$ km/sec, $B=30.4$ km/sec, the values of the percentage of high-velocity stars with $T > T_{lim}$ are given in the last column.

According to the above estimates, systematic errors amounting to about 10% might be expected in the percentages and counts of giant stars in Figures 11 and 12 because of doubts regarding velocity distributions. These errors should be quite small for the visual magnitudes, but grow steadily towards the faintest photographic magnitudes considered.

It would appear that the principal cause for error in the giant counts arises not so much with the uncertainties about the high-velocity stars but rather with the classifications of K stars. Towards the end of the discussion of K-star counts in section 6, it was noted that (a) an error of one tenth of a spectral class in the conversion from HD to BSD classifications could cause a 30% alteration in the K-star counts, and (b), that there were considerable differences between the classifications by SCHWASSMANN and WACHMANN (the Bergedorf observers). The latter differences appear to vary with apparent magnitude, and become of the order of 30% for the faintest groups. Both these effects might well be present to some degree in the photographic counts adopted earlier, and consequently in the adopted giant-star counts. Therefore, when considering the K-giant counts, one must bear in mind the possibility of systematic errors in the photographic counts which may be magnitude dependent.

17. The computation of the z -density distribution of K giants, and K_z

With the luminosity function of K giants (as given in Figure 10) and the visual and photographic

counts of Figure 12, the relative z -density distribution, $\Delta(z)/\Delta(0)$, may be computed from

$$A(m) = 7.01 \times 10^{-4} \int_{\log r_1}^{\log r_2} \Phi(M) \frac{\Delta(z)}{\Delta(0)} r^3 d \log r,$$

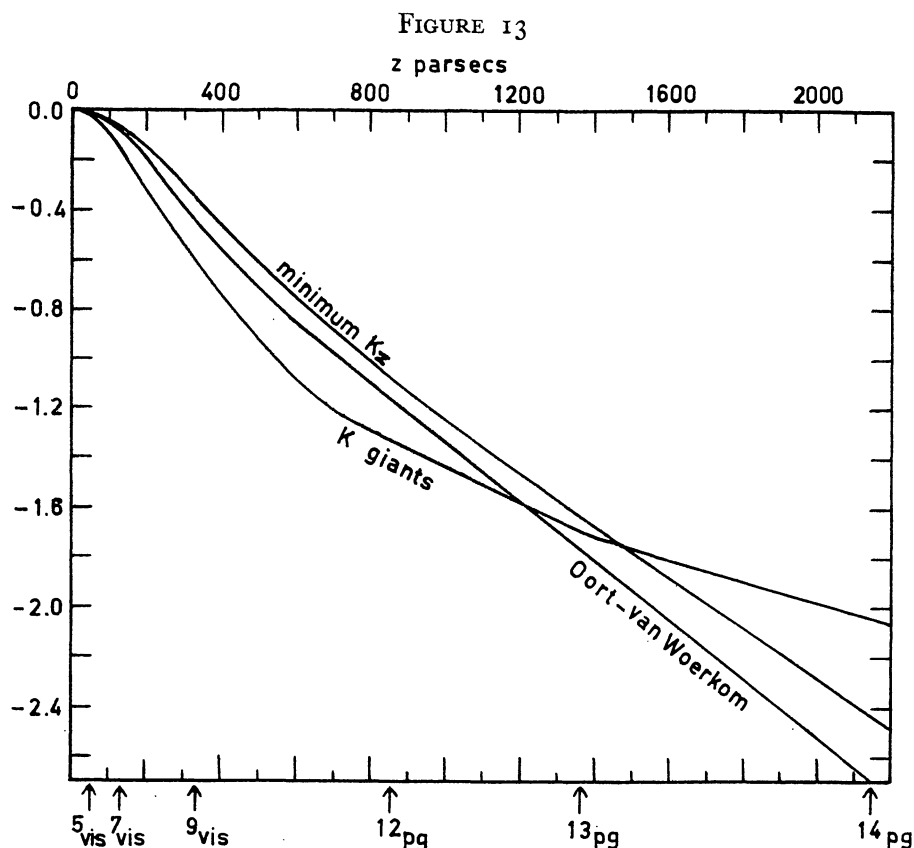
simply by trial and error. Such a density distribution can be considered a modification of that computed from the OORT-VAN WOERKOM K_z ; a modification which is more in accord with the available proper-motion data and numbers of K giants. The density distribution best fitting the star counts of Figure 12 is shown in Figure 13; giant-star counts derived from this density distribution are shown by the crosses in Figure 12. On Figure 13 indications are given as to the z -distances at which giant-star counts of several apparent magnitudes convey the most reliable densities¹⁾.

Cursory examination of Figure 13 at once reveals marked deviations between the originally assumed density distribution and the new distribution. The more rapid decline in density within 600 pc of the plane indicates that the new density distribution must yield a larger K_z than shown by the OORT-VAN WOERKOM curve. The much diminished rate of decrease in density beyond 600 pc implies a field weaker than OORT and VAN WOERKOM's in this region.

Detailed computations of K_z from the new density distribution confirm the indications given by the latter. These calculations may be carried out directly, as discussed earlier in a footnote to page 4, using

the relation between $\int_0^z K_z dz$ and $\log \Delta(z)/\Delta(0)$ given by the Z -velocity distribution of Table 14. This relation is given in Table 23. The field which they yield is illustrated in Figure 14, where it is marked "K giants".

¹⁾ These positions approximately locate the average distances of the stars.



Relative densities, $\Delta(z)/\Delta(0)$, computed for giants. One is based on the OORT-VAN WOERKOM field of Figure 14, and the Z-velocity distribution of Table 15. This has been used in all preliminary computations. The curve marked "K giants" has been deduced from the K-giant-star counts of Figure 12. The remaining curve marked "minimum K_z " was deduced from the field so marked in Figure 14. z -distances at which stars of several apparent magnitudes contribute most reliable densities are shown by arrows.

Comparison of the new field with that of OORT and VAN WOERKOM shows that the former increases near the galactic plane more rapidly by a factor of 1.5. This means that the new field will give a total density near the galactic plane correspondingly

TABLE 23

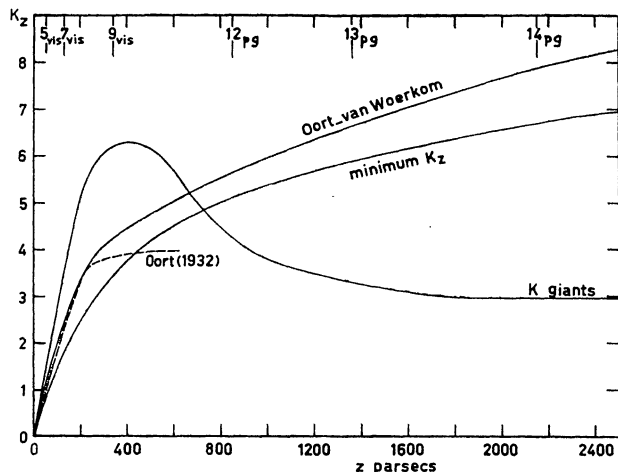
The relation between $\int_0^z K_z dz$ and $\log \frac{\Delta(z)}{\Delta(0)}$ for the Z-velocity distribution of Table 14.

$\int_0^z K_z dz$ ($\text{km}^2\text{sec}^{-2}$)	$\log \frac{\Delta(z)}{\Delta(0)}$	$\int_0^z K_z dz$ ($\text{km}^2\text{sec}^{-2}$)	$\log \frac{\Delta(z)}{\Delta(0)}$
0	0.00	400	-0.61
25	-0.05	500	-0.71
50	-0.10	600	-0.81
100	-0.19	800	-0.98
150	-0.27	1000	-1.13
200	-0.35	1500	-1.46
250	-0.42	2000	-1.75
300	-0.49	2500	-2.02
400	-0.61	3000	-2.27

larger than the $5.8 \times 10^{-24} \text{ gm cm}^{-3}$ quoted earlier from the older field. Beyond $z = 500 \text{ pc}$ serious divergences occur. Here, as a result of density gradients based on giant-star counts for $m_{\text{vis}} 9.4$ and the photographic magnitudes, K_z descends rapidly towards a steady value of $3 \times 10^{-9} \text{ cm sec}^{-3}$.

It appears very likely, however, that beyond $z = 600 \text{ pc}$ the new K_z is impossible. For, provided with $\frac{\partial K_z}{\partial \varpi} + \frac{K_z}{\varpi}$ as a function of z from SCHMIDT's (1956) model of the Galactic System, and minimum densities of ordinary stars and gas at various z -distances, it is possible to estimate a minimum K_z that might be anticipated by integrating Poisson's equation. For the relative z -density distributions of ordinary stars, that given by OORT (1936) for stars with $M > +4.5$ was used, while for interstellar material and extreme population I objects, data given for interstellar gas by VAN DE HULST, MULLER and OORT (1954) in their Table 4 has been adopted. A total density of ordinary stars of $3.3 \times 10^{-24} \text{ g cm}^{-3}$ was adopted. GLIESE (1956) gives $3.9 \times 10^{-24} \text{ g cm}^{-3}$, but here was included a

FIGURE 14



Illustrates several K_z -curves derived for z -distances to about 3 kpc. The broken curve is the original OORT (1932) result. The "OORT-VAN WOERKOM" curve is a compound of the broken curve and that computed by OORT and VAN WOERKOM from a model galaxy. The third curve is that deduced in this paper from K-giant-star counts, whilst the fourth is considered to be the weakest likely K_z . Arrows along the z -scale are used for the same purpose as in Figure 13.

contribution of $0.6 \times 10^{-24} \text{ g cm}^{-3}$ originating from Me stars which may be extreme Population I objects (DELHAYE 1953). Including these Me stars with the contribution of interstellar material as estimated in section 2, the $2.2 \times 10^{-24} \text{ g cm}^{-3}$ used here for material other than ordinary stars is obtained. With these data, the minimum K_z shown in Figure 14 was

TABLE 24

Giant K-star counts at photographic magnitudes compared with computed numbers for several fields K_z .

m_{pg}	log $A(m)$ adopted for K giants	log $A(m)$ for K giants with several fields		
		(a)	(b)	(c)
12.0	+ 0.09	0.11	0.35	0.22
13.0	+ 0.29	0.34	0.43	0.26
13.8	+ 0.51	0.56	0.35	0.11

Field (a) is that marked "K giants" on Figure 14, (b) is the minimum K_z , and (c) is 1.20 times the minimum K_z .

derived. This is seen to be much higher than the field computed from K giants beyond $z = 500 \text{ pc}$.

Now it was mentioned previously that two types of errors are likely to be present in the photographic star counts upon which K_z for $z > 800 \text{ pc}$ depends. These are systematic and magnitude dependent errors. Although it now appears impossible to use these data to derive a new K_z , the minimum K_z just computed may be used to obtain some impression of the magnitude of errors in the photographic star counts.

The density distribution of K giants corresponding to the minimum K_z is shown in Figure 13, and the numbers of giants resulting from this are tabulated in Table 24, column (b). It is seen that the deviations are rather larger than expected, and that the computed frequencies start declining around $m_{pg} = 13$. If in these computations, the minimum K_z is replaced by one which is greater by a factor 1.2, the frequencies given in column (c) result. Here the deviations from the adopted values are not so large but disagreement at $m_{pg} 13.8$ is greatly enhanced.

The results of these computations indicate that, with reasonable fields, the trend in the faint photographic star counts differs somewhat from that observed. Moreover, in all probability there is a magnitude dependence in the observations which removes any likelihood of reaching any specific conclusions regarding K_z for $z > 500 \text{ pc}$ from K-giant stars.

18. Discussion

Although it does not appear that the above analysis yields any definite conclusions regarding K_z for $z > 500 \text{ pc}$, definite results have been obtained for smaller z -distances and these are worthy of more detailed discussion.

In so far as interest remains at present only in the observational data for stars with $m_{vis} \leq 9.4$, the replacement by the new K_z of the OORT-VAN WOERKOM field introduces no significant modifications to the calculations. The dwarf-"subgiant" group of stars at $m_{vis} 9.4$ is so close to the galactic plane (within $\sim 250 \text{ pc}$) that the difference between the two fields does not build up sufficiently to influence the percentages already computed for the three velocity groups. Moreover the transverse-velocity distribution of the dwarf-"subgiant" group, from which the percentages of giants were derived is not influenced greatly in this way. Herein lies the most important attribute of the present method for computing the density distribution of the giants, and K_z . The only respect in which some hesitancy may remain in accepting the above claim concerns the analysis of Z-motions wherein the mean Z-velocity of the high-velocity group was derived from the integral of the OORT-VAN WOERKOM K_z to $z = 1 \text{ kpc}$. But here again deviations of the derived field from that assumed would need to be considerable, of the order of 50% at least, before much alteration would be needed to the analysis.

As already noted, the new $\frac{\partial K_z}{\partial z}$ is somewhat larger than obtained by OORT in 1932. It yields a local density in the neighbourhood of the sun of 0.13 solar masses per pc^3 (or $9.2 \times 10^{-24} \text{ g cm}^{-3}$) using relations (5a) and (5b) with the values of A and B quoted.

This leaves a large discrepancy with the directly observed density of $5.5 \times 10^{-24} \text{ g cm}^{-3}$ deduced earlier.

In the next article OORT has given a different analysis, in which full use is made of the conditions imposed by Poisson's law. In this case it is possible to utilize also the K-star counts for fainter magnitudes. The resulting density, which is in a large measure independent of mine, is 0.15 solar masses per pc^3 .

Any attempt to compare the percentages of HD K giants computed above with results available from direct observation would inevitably involve some adjustment of one or the other in regard to spectrum and luminosity to achieve conformity between the two sets of data. However, for the present, obser-

vational data as published will be used, so that the following comparison is somewhat provisional.

For intermediate and high galactic latitudes there are three sets of observations of luminosity classes with which our computations may be compared, i.e. MOORE and PADDOCK (1950), EDMONDSON, VYSSOTSKY and JANSSEN (1949) and ELVIUS (1951). Results obtained by these investigations are set out below in tabular form together with relevant information regarding the observational data. Percentages in the second last column of the table may be compared with Table 19 and Figure 11. In the circumstances it is seen that agreement is quite good, but many more observations of the very faint higher-latitude K stars

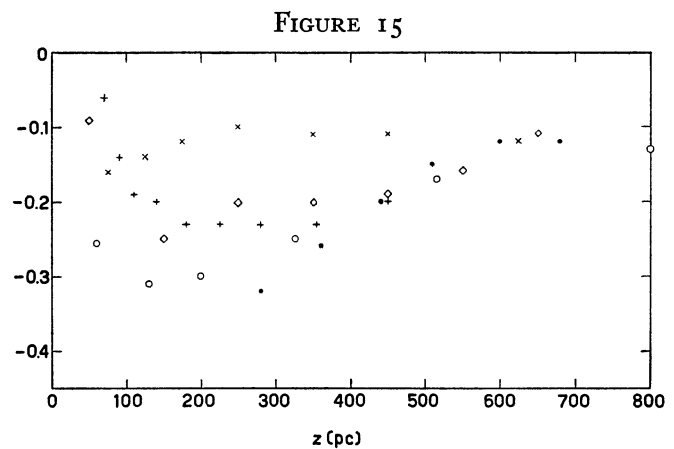
Observers	Classification system	Stars	$\bar{m}$	% K giants	number of stars
MOORE and PADDOCK	HD and MKK	$ b > 40^\circ$ $\delta > -21^\circ$	7.5 vis.	95	234
EDMONDSON, VYSSOTSKY and JANSSEN	MKK	$ b > 31^\circ$ ($ b = 51^\circ$)	10.4 vis.	75	322
ELVIUS	Stockholm	S.A. 6, 15, 16 ($ b = 39^\circ$)	13.0 pg	53	29

are needed to supply reliable confirmation of percentages computed for such stars. In any case, available observations undoubtedly give some confidence in this aspect of the computations.

Further comparisons may be made in respect to the z-density distribution of K giants. In this connection the most suitable quantity to use as a basis for comparison is the density gradient, $\frac{d \log \Delta(z)}{dz}$. For not only is this quantity fundamental to the K_z question, it is often the only meaningful result of density computations because local densities are frequently not too well determined and absolute densities depend on uncertain luminosity limits of spectral groupings.

There are numerous density gradients with which comparison may be justifiable, but only the more recently published determinations will be considered. These are, for K giants, those deduced from results of VAN RHIJN and SCHWASSMANN, and ELVIUS; for stars with $M_{pg} \simeq +2$ (which comprise K giants among other stars), the results of BOK and MACRAE, and OORT. It might be noted that the only modification to published data was an adjustment of ELVIUS's results to a mean M_{pg} of $+2.2$ in place of $+1.7$ which he used. Density gradients so found are shown with the present results for z up to 700 pc in Figure 15. For reasons mentioned earlier, there is no point in extending the comparison further. Of the five sets of results four are seen to agree reasonably well; those due to VAN RHIJN-SCHWASSMANN however deviate

considerably from the others over a considerable range of z -distances. The reason for this disagreement is not clear. It would appear that an average of the remaining data might yield the best estimate of the density gradient with z as yet available.



Comparison between various estimates of $\frac{d \log \Delta(z)}{dz}$ for K giants and stars of similar luminosity.

- K giants: • T. ELVIUS, *Stockholm Ann.* **19**, No. 3, 1956.
K giants: × P. J. VAN RHIJN and A. SCHWASSMANN, *Zeits.f. Ap.* **10**, 161, 1935.
K giants: ♦ Present calculations.
K giants: + J. H. OORT, *B.A.N.* **8**, 75, 1936 (No. 290).
Stars with $M_{pg} \simeq +2$: ○ B. J. BOK and D. A. MACRAE, *N.Y. Acad. Sci. Ann.* **42**, 219, 1941.

19. Scope for future investigations

There appears to be scope still for further investigations of the dynamics of high-latitude stars with existing observational material. Further there is hope of obtaining, in the future, results of greater reliability from improved observational techniques being developed at the present time for various measurements, in particular for the determination of spectral classes and luminosities. Besides these opportunities there is a need for reconsideration of the foundations of the theory upon which computations are based. One aspect of the last-mentioned problem concerns the validity of the well-mixed condition, which has only been tested for relatively near-by stars¹⁾. Another feature of considerable interest and importance which requires investigation concerns the applicability of equation (4c) at large distances from the plane—this question has already been commented upon in a footnote to page 3.

Returning to the uses to which existing observational data might be put, the following applications may be cited. Next to the K giants, F stars constitute the most favourable source of information regarding K_z to moderately large z -distances. These stars are quite plentiful at high latitude; they are predominantly main-sequence stars with luminosities somewhat lower than the K giants. Considering that counts and proper-motion distributions of F stars are available to $m_{pg} \sim 13$, and that the mean absolute magnitude of the apparent luminosity function of F stars is about $M_{pg} = +3$, it should be possible to obtain densities to about 1000 pc from the plane. This would provide a valuable addition to results from K stars. Photographic counts of F stars of better quality than those obtained for K stars above would be required.

With respect to K_z within 200 pc or so from the galactic plane, it is only from stars with $|\bar{Z}|$ somewhat lower than that of F, G, or K stars that accurate results might be obtained—as OORT has pointed out. This point is illustrated in Table 25, where relative densities have been computed at $z = 50, 100, 150$ and 200 pc for two fields, both increasing linearly with z (a) to 2×10^{-9} cm sec⁻² at $z = 100$ pc and the other (b) to 3×10^{-9} cm sec⁻². Densities are derived for two Z -velocity distributions, one comprising equal numbers of weak- and strong-line stars (which resembles F, G, and K-star velocity distributions near the galactic plane), and the other a gaussian distribution with $|\bar{Z}| = 6.5$ km/sec resembling roughly that of the A stars. It is seen that a 50% increase in the field (a) produces a 50% reduction in the A-star density at

¹⁾ This study would almost certainly require more radial velocities of faint high-latitude stars than are now available in order to ascertain whether the condition holds at the large z -distances at which fields have been computed.

TABLE 25

The sensitivity of relative densities $\Delta(z)/\Delta(o)$ near the galactic plane, of early- and late-type stars, to the field K_z .

z (pc)	A stars		F, G, K stars	
	(a)	(b)	(a)	(b)
50	.89	.84	.96	.94
100	.63	.50	.85	.78
150	.35	.21	.70	.59
200	.16	.06	.53	.41

$$(a) K_z = 2 \times 10^{-9} \frac{z}{100} \text{ cm sec}^{-2}$$

$$(b) K_z = 3 \times 10^{-9} \frac{z}{100} \text{ cm sec}^{-2},$$

where z is measured in parsecs.

$z = 200$ pc, and consequently, in the star counts compared to only about 20% with the F, G, and K stars. Thus, provided there are sufficient A stars to get reliable counts, and clustering influences can be eliminated, the A stars might give a much more sensitive verification of OORT's earlier data on the field nearer the galactic plane than dwarf F or G stars. Tentative calculations made by the author indicate that the advantage of the high-density gradient of A stars is probably offset by the comparatively meagre knowledge of the Z -velocity distribution, especially among the higher velocities. On the other hand, the densities of A stars can be studied quite thoroughly, as has been shown recently by VAN RHIJN (1955), who deduced z -densities near the galactic plane from A stars, many of which are at quite low latitudes, by taking into account absorptions measured from colour-excesses. Moreover, if from these investigations, it is possible to study the Z -velocity distribution as a function of position in the galactic plane, it may well be possible to ascertain the influence of spiral arms on K_z .

Regarding the consequences of new techniques being developed for giving accurate luminosity classifications to rather faint stars, a consequence of great importance needs stressing. In the computation above with K stars, the major part of the work involved using proper-motion distributions, via estimates of transverse-velocity distributions, to find the percentage of giants at various apparent magnitudes so that star counts for giants alone could be deduced. With directly observed spectral and luminosity classification, freed from magnitude equations, these counts may be derived without the labour of the computation described earlier and the uncertainties accompanying it. As K giants should provide a very extensive, and a potentially quite reliable field, their luminosity classification at high latitudes and to the faintest possible apparent magnitudes seems a programme of great importance for the future.

There is still a very great need for radial velocities of faint high-latitude stars — i.e., measurements similar to those of EDMONDSON. Here, to compensate for difficulties associated with measurements on faint stars, is the fact that, on the average, the velocities can be expected to be somewhat larger than those of the bright stars. The uses to which such measures may be put are at least three-fold. They may be used to study

- the validity of the well-mixed condition at large z -distances,
- the high-velocity stars, after the manner described earlier, and
- the field K_z .

Eventually, with luminosity classifications of high-latitude stars, it would be possible to study the motions of the rare high Z -velocity stars in much greater detail with proper-motion distributions.

For reasons given earlier in this section it would appear that radial velocities of faint high-latitude A and giant K stars are the most urgently needed for K_z -computations. Velocities of F stars would be valuable, but for the above purposes they would be of secondary importance compared to K stars.

G stars, on account of the predominance of dwarfs, the high average Z -velocity and the intrinsic faintness, are of little importance in high-latitude studies except possibly in studying K_z between 200 and 400 pc from the plane.

Finally, regarding K_z for $z > 1$ kpc intrinsically very bright variable stars such as long-period and RR Lyrae variables are the only source of information at

the present. Improvement of results obtainable from these stars can only be achieved with the aid of many more radial velocities of variables at distances from the galactic plane corresponding to the density distributions.

APPENDIX

The distribution of the peculiar motions of strong- and weak-line stars

The studies of the Z -motions of K giants described earlier involved a statistical conversion of observed peculiar radial velocities of these stars into Z -motions. This necessitated some consideration being given to the distribution of peculiar motions of strong- and weak-line stars which constitute the greater number of local K stars. Results of these computations are now given together with other results which, although they are of little importance to the present work, are nevertheless of considerable interest. In all this work the lists, described earlier, of space motions due to DE VOS VAN STEENWIJK have been used.

In the first place, the average peculiar speeds $\bar{u}$, $\bar{v}$, and $\bar{w}$ along the directions $l=0$, $b=0$; $l=90^\circ$, $b=0$; $b=90^\circ$, respectively, were calculated for each star group—after cluster stars had been omitted. These averages are shown in Table A1. It is seen from this table that along all three directions there are distinct differences in the average motions of the two star groups.

Following this, a preliminary enquiry was made into the direction of the vertex of peculiar motions for each star group. For simplicity, only the galactic

TABLE A1
Average peculiar velocities of strong- and weak-line stars in km sec⁻¹.

star group	$\bar{u}$	$\bar{v}$	$\bar{w}$	$\bar{a}$	$\bar{b}$	$\bar{c}$	l_v	n
strong-line	18.1 ± .7 (pe)	12.0 ± .5	8.9 ± 0.4	18.4 ± .7	11.6 ± .5	8.9 ± .4	347°	138
weak-line	22.7 ± 1.0	19.4 ± .8	14.9 ± .6	23.6 ± 1.0	18.4 ± .8	14.9 ± .6	335	157
A stars				12.6 ± 1.1	8.2 ± .8	8.5 ± 1.6	350	

longitude of the vertex direction for each star group was investigated. This has been done in two distinct ways; by studying, firstly, the galactic-longitude distribution of the peculiar-velocity vectors, and, secondly, the average planar velocity $(u^2 + v^2)^{1/2}$. In both methods the quantity concerned was computed for intervals of 20° in longitude, then the resulting longitude distribution was smoothed by averaging the results for three successive longitude intervals. Data from opposite sectors were then combined. Longitude variations in the velocity vectors, and average peculiar velocity in the galactic plane so obtained are shown in Figure A1. The longitudes about which these distributions are symmetrical, i.e.

the vertex directions which they yield, are shown in Table A2.

Table A2 shows that both methods give concordant results, indicating that the strong-line stars have a vertex deviating from the galactic centre by about 20°. On the other hand, concordance shown between the two methods seems to be accidental in the case of the weak-line stars in view of the larger error which may arise from the nearly isotropic distribution of the motions parallel to the galactic plane. Thus not much may be said regarding the vertex direction of weak-line stars.

Adopting the vertex directions given in the one but last column of Table A1, $\bar{u}$ and $\bar{v}$ may be converted

FIGURE A1

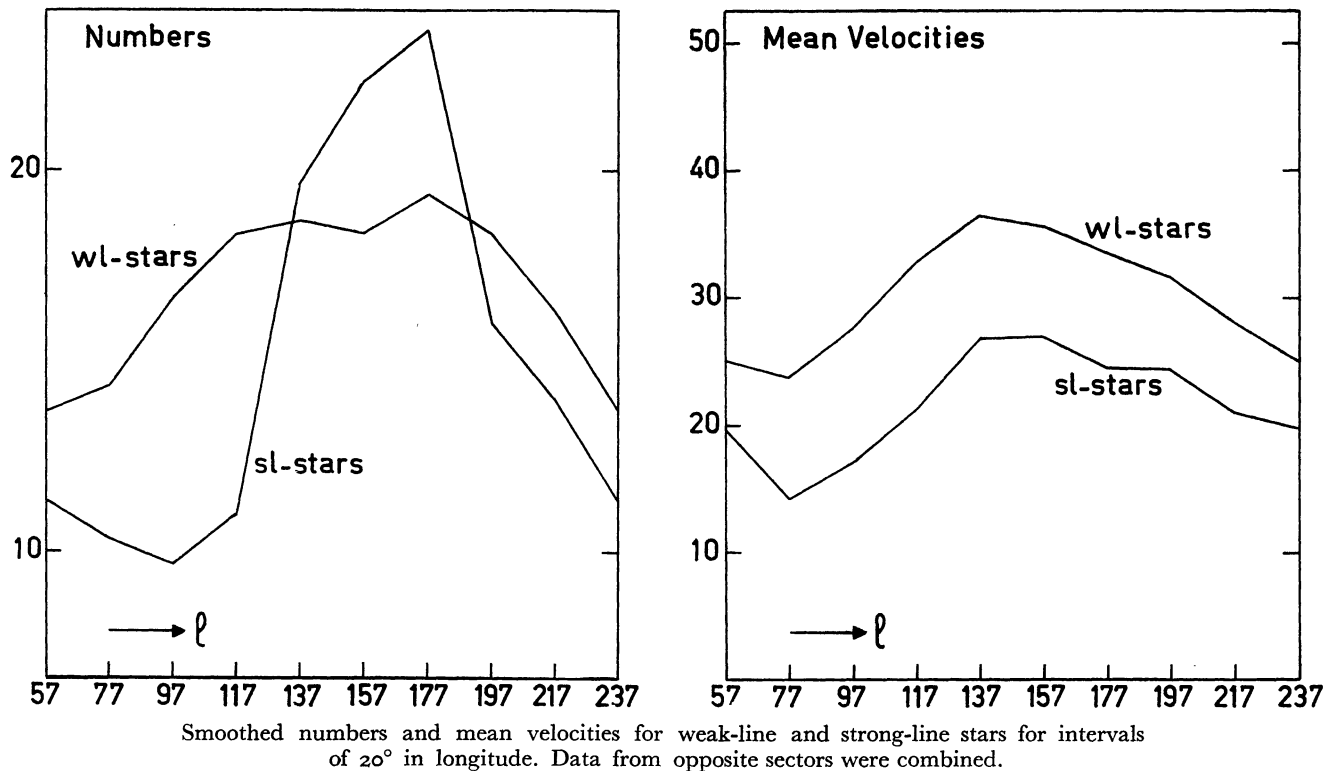


TABLE A2

Vertex directions of strong- and weak-line stars

Method	vertex direction	
	strong-line	weak-line
n_l	$l_V = 348^\circ$	336°
$\{(u^2 + v^2)^{1/2}\}_l$	346°	335°

into the corresponding average velocities, $\bar{a}$ and $\bar{b}$ along the principal axes in the galactic plane of the velocity ellipsoid. The relations between these four average velocities are

$$\bar{u} = \sqrt{\bar{a}^2 \cos^2 \alpha + \bar{b}^2 \sin^2 \alpha}, \quad \bar{v} = \sqrt{\bar{a}^2 \sin^2 \alpha + \bar{b}^2 \cos^2 \alpha},$$

where α is the angle between the u and a -axes¹⁾. Results of this conversion are given in Table A1, and indicate that the space motions of weak-line stars approach much nearer to spherical symmetry than those of the strong-line stars.

It is of interest to compare the above results with what is known about A stars. One of the reasons for

such a comparison is that there is known to be an abnormally large difference in the average peculiar motions of A and F stars; A stars have peculiar motions which are much smaller than stars of F and later spectral classes. Also A stars show a larger deviation between the direction of the vertex and that of the galactic centre than the other stars. Moreover, what vertex deviation is shown by F and later-type stars appears to be due to stars with small peculiar motions (DELHAYE 1951, cf. Figure 14). These facts suggest that A stars might be kinematically akin to strong-line stars²⁾, so that they might have comparable peculiar motions and exhibit the large vertex deviation shown by A stars.

In Table A1, the data on peculiar motions derived by MACRAE (1947) from radial velocities of seventh-magnitude A stars are shown. It is seen that, although there is not complete agreement with the space-motion results for strong-line stars (which incidentally have not been corrected for errors of observation), there is sufficient indication that there are resemblances in their kinematical properties. This impression is further strengthened when one compares the vertex direction which DELHAYE finds for A stars, i.e. $l_V \simeq 350^\circ \pm 2^\circ$

¹⁾ For the derivation of the more general relations upon which these expressions are based, see TRUMPLER and WEAVER (1953).

²⁾ Spectroscopically, strong-line stars as defined in terms of line and band strengths in spectra of F and G stars cannot be found among A stars because of the masking effects of rotational broadening in the latter.

with that given by combining the two results in Table A2 for strong-line stars, i.e. $l_v = 347^\circ$.

It seems clear that the weak-line stars must correspond to the higher-velocity stars which DELHAYE considers to show no, or little, vertex deviation. This has not been shown definitely to be the case by results in Table A2; however, results given there by no means exclude the possibility.

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