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On the origin of comets, by *A. J. J. van Woerkom.*

In the first section a critical review is given of the more important previous investigations concerning the origin of comets. Already here it appears that small perturbations of the orbits, mainly caused by Jupiter, must be of paramount importance for the distribution of the major axes a of the long-period comets.

In the second section the perturbations by Jupiter during a passage through the planetary system are calculated. If Jupiter and the comet approach each other to within 1·1 A.U. the principal part of the perturbative function gives the most important contribution. The perturbation depends on the angle under which the comet crosses the orbit of Jupiter, as well as on the shortest distance between the two bodies. Values of the perturbation are shown in Figure 5. The complementary part of the perturbative function has been integrated analytically. If the comet remains at a considerable distance from Jupiter this yields the more important contribution. The total average change of the reciprocal semi-major axis at one perihelion passage is $0{\cdot}53 \times 10^{-3}$ and $0{\cdot}71 \times 10^{-3}$ for comets with a perihelion distance of 1 A.U. and 4·5 A.U., respectively; the first value is representative for the known long-period comets.

In Section 3 an analysis is given of the way in which the small perturbations will determine the distribution of the major axes if successive perturbations are supposed to be independent of each other. The problem bears much resemblance to a problem of diffusion. The speed of the "diffusion" in $1/a$ is determined by the mean square of Jupiter's perturbations. It is found that, if the comets diffuse inward from a constant field of parabolic comets, the numbers that pass perihelion per year will in the long run become constant in constant intervals of $1/a$. For comets with perihelion distances of 1 A.U. the process will go more slowly than for those with perihelion distances of 4·5 A.U. After 10^6 years the distribution will have become constant down to $a = 100$ and $a = 25$ A.U., respectively.

Section 4 contains a discussion of the process of diffusion in case the comets would be captured from an interstellar field. With a normal interstellar velocity distribution the frequencies of $1/a$ for the comets captured would become similar to those discussed in Section 3. The complete lack of strongly hyperbolic orbits among the observed comets can only be explained if the interstellar field moves with the sun, which is highly improbable. Neither the hypothesis that the comets would have been caught during a passage of the sun through a dense interstellar cloud in the near past, nor the idea that the long-period comets move in stable orbits, nor either the supposition that they would always have belonged to the solar system and would have been acted upon solely by forces within this system, appears to lead to a satisfactory explanation of the phenomena. In the latter case the comets that can come within the zone of visibility would long ago have been diffused out of the solar system by the action of Jupiter.

In Section 5 a discussion is given of the hypothesis that the comets have originated from eruptions on one of the major planets. For the long-period and parabolic comets this would lead to a distribution of inclinations which is incompatible with the observed distribution. The possibility that the short-period comets have been captured from the parabolic and long-period orbits through large perturbations by Jupiter is next discussed. This theory gives a ready explanation of the existence of a maximum frequency of "jovicentric velocities" in the vicinity of one half of Jupiter's orbital velocity, which is one of the most characteristic features of the short-period comets. The rate at which short-period comets are captured from the parabolic field comes out slightly smaller than the probable rate at which short-period comets are lost by disintegration. Two possible explanations of this slight discrepancy are suggested.

1. *Introduction.*

A considerable fraction of the comets are moving in orbits in which they reach large distances from the sun. A number of these orbits have been shown to be elliptical. These comets return after a very long period to the sun. Therefore we call them long-period comets. Though the others have in all probability moved also in elliptical orbits toward the sun, we shall call them parabolic comets, because the eccentricity is not exactly established by the observations. Another part of the comets is moving in elliptical orbits which are completely situated within the planetary system. We call these short-period comets.

Several attempts have been made to explain the origin of comets. In the first place it has been suggested that the parabolic and long-period comets are coming from interstellar space. In that case the principal problem is whether an interstellar field of comets can give rise to a distribution of the major axes and other parameters like that which is observed. It has been shown that plausible assumptions concerning the comets in interstellar space give always rise to mainly hyperbolic comet orbits in the solar system, which is not in accordance with the observations.

The authors who have studied the hypothesis that the long-period comets have their origin in interstellar space have all overlooked an important effect. In order to be visible from the earth comets must come within a short distance from the sun, say within 2 A.U. They must therefore penetrate into the planetary system, and will suffer considerable perturbations by the planets. As a consequence of these perturbations, by which for instance hyperbolic orbits may be transformed into elliptical ones, the distribution of major axes will become quite different from that resulting directly from an interstellar field. In the present article the gradual effects of the small perturbations by Jupiter will be investigated in some detail.

The short-period comets also have their problems. Part of these comets will be dissolved by the tidal action of the sun into meteor swarms. The perturbative action of the planets will transform the orbits of a number of short-period comets into orbits in which they are no longer visible from the earth. Also the reversed action will take place. However, it has been generally assumed that the probability of such a transformation is too small to replace the vanishing comets at a sufficient rate. For this reason some investigators have looked for the origin of the short-period comets within the solar system.

In the present paper some of the difficulties of this theory will be investigated. These are shown to be serious. For this reason we have again directed our attention to the theory that the short-period comets have been captured by Jupiter from the general field of comets of longer periods.

We will first give a short survey of the most important treatises dealing with the origin of the comets.

As has already been remarked, two hypotheses have mainly been considered. The first, proposed by LAPLACE[1]), assumed that the comets are coming from interstellar space. The other, by LAGRANGE[2]), supposes that comets and meteors are formed by explosions on the larger planets, and that this process is still going on. The hypothesis had been inspired by OLBERS and HARDING having found common intersection points for Minor Planets.

LAPLACE introduced two important conceptions, viz. the sphere of action of the sun, a sphere with the sun as a centre and a radius of 10^5 A.U., outside of which the influence of the sun may be neglected, and the zone of visibility, also a sphere with the sun as a centre, with a radius of 2 A.U., inside of which the comets will be visible from the earth. (By the improvements of the instruments and observing technique this radius should, according to MOISSEIEV and VODOPIANOVA[3]), now be enlarged to 2·5 or 3 A.U.)

LAPLACE assumed the existence of a field of comets at the outside of the sphere of action. Let us denote by $y(v)dv$ the number of comets with space velocities between v and $v + dv$. LAPLACE assumes that in this outside field $y(v)$ = constant for all velocities from 0 to ∞. What is then the distribution of the major axes of the comets which come inside the zone of visibility? He found that the number of elliptic and parabolic comets must surpass by far the number of hyperbolic comets. Out of 6000 comets only one will show a pronounced hyperbolic orbit. This is just what is observed. SCHIAPARELLI, however, showed[1]) that, when we take into account the velocity of the sun with respect to the supposed interstellar field of comets, the whole picture is altered. In that case the number of hyperbolic orbits must be preponderant. With a velocity of the sun of 20 km/sec, the ratio of the number of elliptic and parabolic orbits to that of hyperbolic orbits would come out as 22 : 401[2]).

SCHIAPARELLI therefore concludes that the comets, without being a part of the solar system, form a cloud, which has accompanied the sun from its origin. It should be noted that the velocity distribution assumed by LAPLACE, and taken over by SCHIAPARELLI and VON NIESSL, is a very peculiar one. With this distribution the density in the velocity space is infinite for $v = 0$. It is very unlikely that such a velocity distribution will give a fair representation of the velocities of eventual interstellar comets. The preponderance of parabolic comets found by LAPLACE is entirely due to the singularity at $v = 0$ in the assumed velocity distribution, while this singularity likewise explains the great difference between the results found by LAPLACE on one hand, and by SCHIAPARELLI and VON NIESSL on the other hand.

A more plausible velocity distribution was used by FABRY in his thesis[3]). His starting point was the distribution function $y(v) = 4\pi v^2$, with an upper boundary of the velocity, b, so that $y(v) = 0$ for $v > b$. On page 158 of his thesis he reaches the following conclusion: "L'absence des orbites fortement hyperboliques, parmi les comètes que nous voyons passer au périhélie, est inexplicable dans la théorie qui fait venir ces astres des espaces interstellaires. Nous concluons donc finalement que cette théorie doit être rejetée et que les comètes sont des membres permanents du système solaire".

In recent time the hypothesis of LAPLACE has been most fundamentally worked out by N. MOISSEIEV in a series of publications[4]) "Über einige Grundfragen der Theorie des Ursprungs der Kometen, Meteore und des kosmischen Staubes", which has not yet been completed. He showed that the approximations intro-

[1]) *Oeuvres Complètes* **13**, 88, 1904.

[2]) *Additions à la Connaissance du Temps pour* 1814; see also TISSERAND, *B.A.* **7**, 452, 1890.

[3]) *Publ. Sternberg* **6**, 103, 1933.

[1]) "Entwurf einer Astronomischen Theorie der Sternschnuppen und Meteore", Stettin, 1871.

[2]) VON NIESSL, *A.N.* **135**, 137, 1894.

[3]) "Etudes sur la probabilité des comètes hyperboliques et l'origine des comètes", Marseille, 1893.

[4]) *Publ. de l'Inst. d'Astroph. de Moscou* **5**, 1, 1938; *Publ. Sternberg* 5_2, 1, 1933; *R.A.J.* **9**, 30-52, 1933.

FIGURE 1 *a*

Distribution of periods of comets

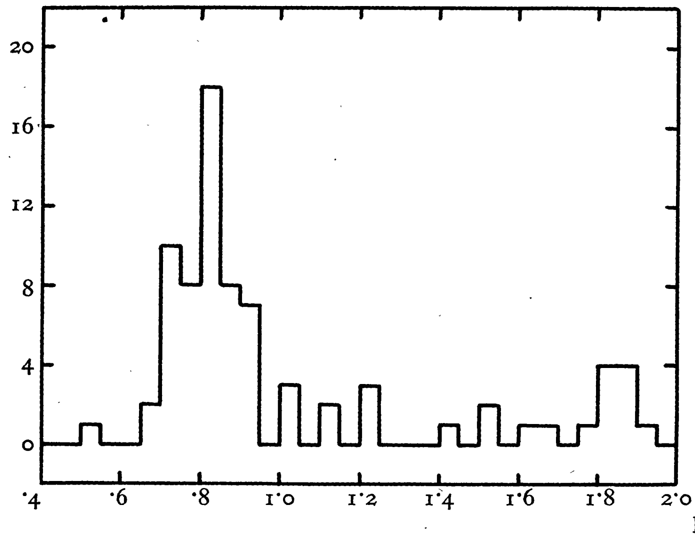
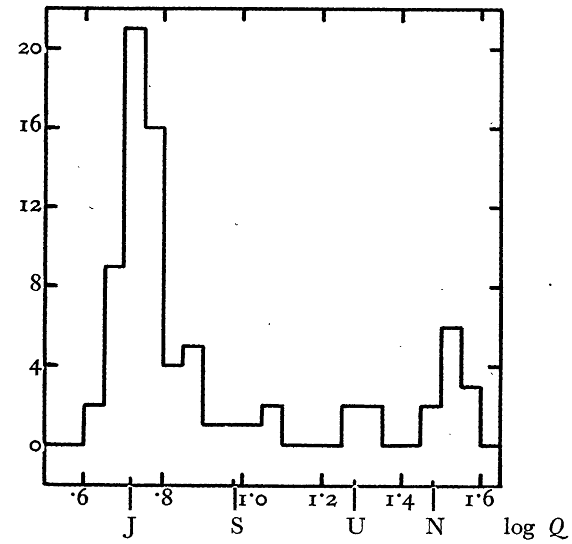


FIGURE 1 *b*

Distribution of aphelia of comets



duced by SCHIAPARELLI, FABRY and others are strictly superfluous. Both in the case of a sun fixed with respect to the cloud of comets, and in that of a moving sun, all questions about the density distribution and the distribution of the orbital elements can be solved exactly. He treats mainly two schemes, namely the "explosion scheme" and the scheme discussed by FABRY. In treating the first one he investigates the distribution of orbital elements corresponding to a given velocity distribution at the boundary of the sphere of action, without taking account of the fact that the number of comets entering this sphere per unit of time will depend upon their velocity. This is admittedly a very inadequate picture, but it simplifies the formulae, and enables to make comparisons with earlier investigators of the explosion scheme. In discussing FABRY's scheme he uses a more adequate model, and computes the numbers of comets actually penetrating into the sphere of action per unit of time. For the velocity distribution of the comets outside the sphere of action three assumptions have been made: 1) the velocity distribution of LAPLACE: $y(v)$ = constant for $0 \leqq v \leqq b$ and $y(v) = 0$ for $v > b$, b being the upper boundary of the velocities; 2) the distribution of FABRY: $y(v) = 4\pi v^2$ for $0 \leqq v \leqq b$ and $y(v) = 0$ for $v > b$; 3) the distribution of NATANSON, in which all comets are assumed to have the same velocity. The importance of these articles lies principally in the method developed. The results have little direct value, because the adopted velocity distribution and velocity of the sun are not plausible. In the fifth part[1]) a qualitative speculation is given of the space density in the sphere of action for the case of a moving sun.

1) *R.A.J.* **9**, 30-52.

Instigated by the disappearance of comet Lexell, LAPLACE was the first to give an explanation for the appearance of short-period comets[1]). This comet of 5·6 years period had been discovered in 1770 by MESSIER. The situation at the following perihelion passage, in 1777, was unfavourable for the observation. Though the situation was favourable at the passage in 1783, the comet was not seen. LAPLACE showed that the comet had passed Jupiter in 1779 at a distance of only ·01 A.U. By this it had met with such great disturbances that the orbit was completely transformed. The same thing had also happened three years before the discovery. LAPLACE supposed that in this manner short-period comets had generally been formed from long-period and parabolic comets. The problem is very complex, and an accurate theory was only given in 1889 by TISSERAND[2]). It has been further developed by H. A. NEWTON[3]). He determined the chance that from a field of parabolic comets, in which all velocity directions are equally probable, an orbit will be transformed by a single encounter with a planet into an elliptic orbit with a semi-major axis smaller than a given limit, a. If m is the mass of the disturbing planet expressed in the sun's mass as a unit, and r the radius of its orbit, this probability for $a > r$ equals $\frac{4}{3}\frac{m^2a^2}{r^2}$. A more accurate calculation showed that of 10^9 parabolic comets coming inside a sphere with a radius equal to that of Jupiter's

1) *Méc. Cél.* IV, livre 9, chap. 3, 1805.
2) *B.A.* **6**, 241, 289, 1889; *Méc. Cél.* IV, chap. 12, 1896.
3) *Mem. Nat. Acad. Sc. Washington* **6**, 1, 1893.

orbit, 126 are thrown into orbits with a period less than half that of Jupiter, 839 obtain periods less than Jupiter's period, and 2670 less than twice that of Jupiter. For the majority of the comets captured the motions will be in the same sense as that of the planet. Of the 839 comets with periods less than Jupiter's, 257 will have an inclination less than 30°, and only 51 an inclination over 150°.

It is well-known how the periods and aphelion distances of short-period comets exhibit their relation to the major planets, especially to Jupiter (see Figures 1*a* and 1*b*). The aphelia in particular appear to group themselves around the orbits of at least some of the major planets. From this it might be expected that the short-period comets might be separated into four groups, according to the planet by which they have been captured. The true criterion, however, that a comet has been captured by a certain planet, is not that its aphelion distance differs but little from the orbital radius of that planet, but that its orbit passes close to that of the planet. Upon applying this test to the calculated distances RUSSELL[1]) finds that all the comets of the Jupiter group can come very close to this planet, and that this group is thus real. Of the five comets whose aphelion distances are close to the radius of Saturn's orbit only two pass fairly near the orbit of this planet. Of the four aphelion distances near the radius of Uranus' orbit, two out of four belong to comets that come close to this planet's orbit, while of the eleven members of the "Neptune group" none can come nearer to Neptune than 3·8 A.U. All of these latter approach more closely than this to the orbits of Jupiter and Saturn. There is no reason, therefore, to ascribe any reality to this latter group. Also the connection of the Uranus and Saturn groups with these planets seems to be accidental, at least for half their members.

The periodic comets with longer periods can likewise be formed by a single encounter with Jupiter. The probability at one passage is proportional to a^2 (see above). If we should assume that these orbits are the result of one such encounter we must thus expect that the number of comets which pass perihelion in a unit of time is proportional to $\sqrt{a}$. This is, in fact, the case for the comets with periods between 100 and 2200 years[2]). Yet, we may expect that in this case there would be an excess of comets passing the orbit of Jupiter at a small distance. However[3]), the distribution of the mutual distances between the orbits of these comets and that of Jupiter proves to be perfectly haphazard. To overcome this difficulty RUSSELL suggests the four following explanations:

1. After the capture by a single encounter the orbits may have been transformed by small ordinary perturbations, so that the orbit of the disturbing planet can no longer be approached at a short distance (this idea originated from H. A. PICKERING[1]), who attributed the great perturbations to a transneptunian planet with great inclination).
2. The periods have been shortened by the cumulation of small ordinary perturbations.
3. The periods have been shortened by the resistance of an interplanetary medium.
4. The long-period comets which we observe are moving in stable orbits, and may be the stable remainder of a great field of comets, the others having been expelled by perturbations.

If the first explanation is correct, we must assume that the orbits have endured important changes, especially with regard to the orbital planes and the longitudes of perihelion. But in elongated elliptical orbits the major axis is more sensitive to small changes in the velocity (which are of course the initial effects of the disturbing forces) than the other elements. Hence, along with the other elements, the periods should also have been greatly changed. As we shall show in the next chapter, the small ordinary perturbations must in this case be of paramount importance in determining the distribution of the major axes (or periods). So the first explanation leads automatically to the second. The perturbation of a long-period comet can only take place near perihelion, and depends evidently on the mutual configuration of the planets and the comet around the time of passage through perihelion. As the changes in the period will in general be much greater than the periods of the planets, we may expect that during the next passage the configuration will be independent of that at the previous passage. In that case the perturbations of $1/a$ at successive passages are independent of each other. If at a given passage equal positive and negative perturbations of $1/a$ are equally probable, the perturbations will gradually give rise to a steady or semi-steady state, in which the number of comets for equal intervals of $1/a$ is a constant. Any given group, between definite values of $1/a$, is gaining then as many members from the groups above and below as it loses to them. The actual distribution of periods is, however, very different.

Nevertheless, it does not follow from this that the part of the small ordinary perturbations is insignificant. We shall give in the first part of this work an analysis of this process.

The third effect suggested by RUSSELL must have been of paramount importance for the comets 1843 I

[1]) *A.J.* **33**, 55, 1921.
[2]) Cf. H. N. RUSSELL, *A.J.* **33**, 49-61, 1921.
[3]) RUSSELL, *l.c.*

[1]) *H.A.* **61**, 217-220, 1908.

and 1882 II, which had small perihelion distances, ·006 and ·008 A.U., respectively. It has also been a contributing factor in the case of Encke's comet. A definitive statement about the effect cannot be made before having data about the density of the interplanetary matter.

With respect to the fourth explanation this is still practically unexplored: I know of only one investigation of this subject, which does not, however, give sufficient information for our purpose[1]). It will appear in the present work that this problem might be of importance for a real understanding of the origin of comets.

In this connection it is interesting to remark that KAMIENSKI[2]) has shown that the mean motion of comet Wolf 1 is slowing down by $4''\cdot 2 \times 10^{-7}$ per day, which cannot be attributed to either resistance or light-pressure, because their effect would be in opposite direction.

A wide variety of mechanisms operate to disperse the matter of the comets. These mechanisms are: tidal action by the sun and the planets, gradual loss of material into the tail, collisions between the particles composing the comets and other particles moving between the planets, electrical repulsion between the particles, and the Poynting-Robertson effect. For the cometary problem it is of paramount importance to know the rate of disintegration. VSESSVIATSKY finds[3]) from his investigation of the absolute magnitudes of the short-period comets a strong decrease of the brightness in the course of time. Hence he came to the conclusion that the lifetime of short-period comets cannot be longer than 500 years; for some, like the comets Brooks 2, Kopff and Wolf 1, not even more than 150 years. From a careful study of comet Wolf 1 BOBROVNIKOFF comes, however, to a completely different conclusion for this comet[4]). He points out that the estimated magnitude is strongly dependent on the aperture of the instrument used, and also on the observer. After reducing all observations as well as possible to a homogeneous scale, he finds no diminution in brightness from 1884 to 1925, whereas in the same period VSESSVIATSKY finds a decrease of 3^m (cf. p. 467 and Table 14 of the present article). It seems that VSESSVIATSKY has not given much attention to the above mentioned systematic errors. Large part of the decrease in brightness found by him will probably be due to the increase of the mean aperture of the instruments used. Hence the lifetime of short-period comets is likely to be much larger than estimated by him. Moreover, we have got an indication in this direction from an entirely different side. From the secular perturbations of comet Encke and the Taurid meteor shower WHIPPLE shows[1]) that the Taurids must have been formed 13000 years ago from Encke's comet. A more recent, accurate calculation by D. BROUWER[2]) confirms this result.

THRAEN[3]), E. STRÖMGREN[4]) and FAYET[5]) throw light on an entirely different side of the cometary problem. The eccentricity of the orbit of a long-period comet, as observed near perihelion, is by no means equal to the eccentricity of the comet orbit at great distances from the sun and the disturbing planets. By carrying out a backward calculation of the perturbations we can find the original eccentricity. FAYET[6]) found by an approximate analytical computation that only the comets 1890 II and 1898 VII remained hyperbolic, while all others are moving at great distances from the sun in elliptical orbits. After an accurate numerical calculation of the perturbations and reduction to the centre of mass of the solar system STRÖMGREN established[7]) that the difference from one of the calculated eccentricity for the two comets just mentioned is less than its mean error. Thus none of the known comets appears to have approached the sun in a distinctly hyperbolic orbit. We should, however, be careful not to conclude from this result that all comets are truly members of the solar system. For comet 1914 V (Delavan), which has been observed during 629 days, VAN BIESBROECK finds an eccentricity $e = 1\cdot 0001618 \pm \cdot 0000035$. A backward calculation of the perturbations shows that on November 8, 1908, $e = \cdot 9999781 \pm \cdot 0000035$. This corresponds to an aphelion distance of 170 000 A.U. and a period of 24×10^6 years. At such large distances from the sun there is hardly question of a connection with the solar system. A definite statement on the cosmogonical consequences of STRÖMGREN's calculations cannot be made before all orbits have been computed forward beyond perihelion. Moreover, the accuracy of the observations should be sufficient to single out cases as that of comet 1914 V.

LAGRANGE has put forward the hypothesis that the comets have been formed by explosion of a distant planet, or by eruptions from such a planet. In the first case long-period and parabolic comets would all have to pass through one point. As there is not the least indication of this, the first hypothesis has soon been discarded. TISSERAND[8]) has investigated the sec-

[1]) VON ZEIPEL, *A.N.* **183**, 346, 1910.
[2]) *M.N.* **106**, 267, 1947.
[3]) *M.N.* **90**, 706-721, 1927; *A.N.* **240**, 275, 1930.
[4]) *Pop. Astr.* **56**, 130, 1947.

[1]) *Proc. Amer. Phil. Soc.* **83**, 5, 1940; *Harv. Repr.* No. 210.
[2]) *A.J.* **52**, 190, 1947.
[3]) *A.N.* **136**, 1894.
[4]) *Lund. Acta Soc.* **7**, 1896; "Über den Ursprung der Kometen", *Kopenhagen Obs. Publ.* No. **19**, 1924.
[5]) "Recherches concernant les excentricités des comètes", *Thèse*, Paris, 1906.
[6]) *L.c.*
[7]) *L.c.*
[8]) *B.A.* **7**, 452, 1890.

ond proposal of LAGRANGE. He shows that, in order to come within the zone of visibility, a long-period or parabolic comet would have to be expelled in well-defined directions and with well-defined velocities, which appears to be quite improbable.

In order to avoid an essential difficulty in the capture theory of short-period comets VSESSVIATSKY returns in a different manner to the theory of eruption[1]). He supposes that the short-period comets originated, and still originate at the present time, by eruptions from the major planets, in particular from Jupiter. I agree with VSESSVIATSKY that in this way nearly all properties of the orbits of short-period comets can be explained. A further study shows, however, that long-period and parabolic comets cannot have originated from eruptions, as this would require a totally different distribution of orbital inclinations. If VSESSVIATSKY's hypothesis were accepted, we would thus have to give up the idea of a common origin for short- and long-period comets. This would be very unattractive in view of the continuity with which the two groups merge into each other and of their identical physical properties.

2. *Computation of the perturbations by Jupiter.*

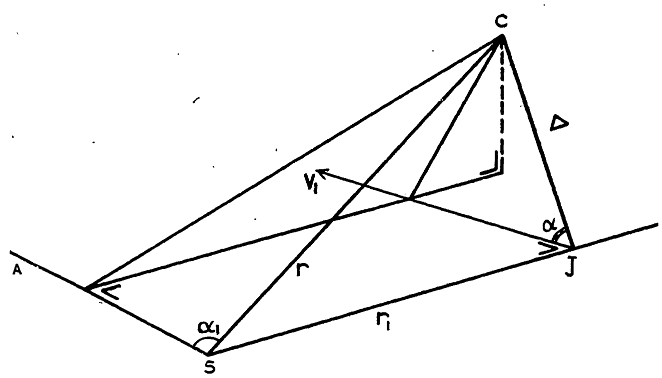
As has already been shown in the introduction, small ordinary perturbations of the orbits of the long-period and parabolic comets by the planets may be of great importance. To calculate the influence of these perturbations, we have to know them for various mutual configurations of the sun, the comet and the planets during the time the comet stays in the planetary system. Because in these perturbations Jupiter has the most important part, we shall restrict ourselves to the problem of sun, Jupiter and comet. It is not our intention to get perfectly accurate values for the perturbations, but only a proper approximation. As we want a general insight into the perturbations for various configurations, we shall mainly pay attention to the way in which these perturbations depend on the configurations.

Our principal aim is to investigate the distribution of the major axes of the orbits of the long-period comets. Therefore we only need the perturbations of this element. As the perturbations of $1/a$ and the other elements are of the same order of magnitude, the perturbations of perihelion distance, longitude of perihelion and inclination can, for a first approximation, be left out of discussion, at least for long-period comets. Since we have to calculate the perturbations for orbits of which the eccentricity differs little from 1, and because these orbits differ near perihelion little from a parabola, we need only calculate the perturbations for parabolic orbits.

During the first stage of our investigation we calculated the perturbations by numerical integration. As this led to superfluous extensiveness, we later restricted ourselves to an analytic approximation for the most important parts of the perturbations, utilizing the numerical work as a check and amplification. The method for the analytical approximation has for the greater part been derived from FAYET [1]).

Let in Figure 2 S be the sun, J Jupiter and C the comet. The co-ordinates of the comet with respect to a rectangular system of axes with the sun as origin are x, y, z; of Jupiter x_1, y_1 and z_1; and of the comet with respect to Jupiter x_0, y_0, z_0.

FIGURE 2



Then

$$x - x_1 = x_0,\quad y - y_1 = y_0,\quad z - z_1 = z_0. \tag{2,1}$$

The mass of Jupiter is m_1, with the sun's mass as a unit. Then the well-known differential equations of the comet's motion are

$$\frac{d^2x}{dt^2} + \frac{k^2x}{r^3} = fm_1\left(\frac{x_1 - x}{\Delta} - \frac{x_1}{r_1^3}\right), \tag{2,2}$$

. . . .

. . . .

in which $\Delta = \sqrt{x_0^2 + y_0^2 + z_0^2}$ is the mutual distance between the comet and Jupiter, r is the radius vector of the comet, r_1 that of Jupiter and $k^2 = f \times 1$, because the comet's mass may be neglected. Putting

$$E = V^2 - \frac{2k^2}{r} - \frac{2fm_1}{\Delta}\,, \tag{2,3}$$

we have

$$\frac{dE}{dt} = -\frac{2fm_1}{\Delta^2}\left(\frac{x_0}{\Delta}\frac{dx_1}{dt} + \ldots\right) - \frac{2fm_1}{r_1^2}\left(\frac{x_1}{r_1}\frac{dx}{dt} + \ldots\right)$$

$$= -\frac{2fm_1}{\Delta^2}\left(\frac{x_0}{\Delta}\frac{dx_1}{ds_1} + \ldots\right)\frac{ds_1}{dt} - \frac{2fm_1}{r_1^2}\left(\frac{x_1}{r_1}\frac{ds}{ds} + \ldots\right)\frac{ds}{dt} \tag{2,4}$$

[1]) *A.N.* **240**, 272, 1930; **243**, 280, 1931; *R.A.J.* **11**, 453, 1934.

[1]) "Recherches concernant les excentricités des comètes"; *Thèse*, Paris, 1906.

in which s_1 is the length of arc along the orbit of Jupiter and s that along the comet's orbit.

Hence

$$\frac{dE}{dt} = -\frac{2fm_1}{\Delta^2}\cos\alpha.\ V_1 - \frac{2fm_1}{r_1^{\,2}}\cos\beta.\ V, \qquad (2,5)$$

in which $ds_1/dt = V_1$ is Jupiter's velocity and $ds/dt = V$ that of the comet; α is the angle between the direction of the planet's velocity and JC, β the angle between the direction of the comet's velocity and SJ, $V_1 \cos\alpha$ is the projection of Jupiter's velocity on JC, and $V \cos\beta$ the projection of the comet's velocity along the radius vector of Jupiter.

If a is the semi-major axis of the comet's orbit, we have

$$V^2 - \frac{2k^2}{r} = -\frac{k^2}{a}, \qquad (2,6)$$

hence

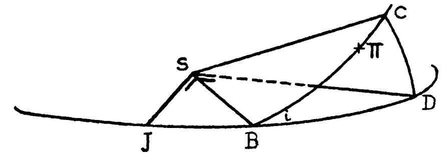
$$\frac{dE}{dt} = -k^2\frac{d\,1/a}{dt} - 2fm_1.\frac{d\,1/\Delta}{dt}. \qquad (2,7)$$

From (2,5) and (2,7) we have

$$\frac{d\,1/a}{dt} = 2m_1\frac{\Delta V_1\cos\alpha}{\Delta^3} + 2m_1\frac{V\cos\beta}{r_1^{\,2}} - 2m_1\frac{d\,1/\Delta}{dt}. \qquad (2,8)$$

From Figure 2 it is easily seen that $\Delta\cos\alpha = r\cos\alpha_1$, in which $\alpha_1 = \angle\, ASC$.

FIGURE 3



In Figure 3 BJD is Jupiter's orbital plane, $B\Pi C$ that of the comet. Let SJ be the direction of the radius vector of Jupiter and SC that of the comet. If $\angle\, JSD = 90°$, $\angle\, CSD = \alpha_1$. Let Ω be the longitude of the ascending node of the comet's orbit, ω the angular distance of perihelion from the node, v the true anomaly of the comet and $u = \omega + v$ the argument of latitude, i the mutual inclination of the two orbits, and λ_1 the longitude of Jupiter. Then $\angle\, BSJ = \Omega - \lambda_1$, and $\angle\, BSC = \omega + v = u$, hence $\cos\alpha_1 =$ $\cos u\cos(90°+\lambda_1-\Omega) + \sin u\sin(90°+\lambda_1-\Omega)\cos i =$ $-\cos u\sin(\lambda_1-\Omega) + \sin u\cos(\lambda_1-\Omega)\cos i.$ (2,9)

In order to express $V\cos\beta$ in the other elements, we first resolve the velocity of the comet in the direction of perihelion and in that perpendicular to it. Let ξ, η be the co-ordinates of the comet in the directions mentioned, then

$$\xi = r\cos v,\ \eta = r\sin v,$$

$$\begin{aligned}\frac{d\xi}{dt} &= \frac{dr}{dt}\cos v - r\sin v\frac{dv}{dt} = \frac{r^2}{p}\sin v\cos v\frac{dv}{dt} - r\sin v\frac{dv}{dt} = -\frac{k}{\sqrt{p}}\sin v\\ \frac{d\eta}{dt} &= \frac{dr}{dt}\sin v + r\cos v\frac{dv}{dt} = \frac{r^2}{p}\sin^2 v\frac{dv}{dt} + r\cos v\frac{dv}{dt} = \frac{p}{r}\frac{k}{\sqrt{p}},\end{aligned} \qquad (2,10)$$

because $r^2\dfrac{dv}{dt} = k\sqrt{p}$, where p is the parameter.

After projection of these quantities on the radius vector of Jupiter we find

$$\begin{aligned}V\cos\beta = &-\frac{k}{\sqrt{p}}\left\{\cos(\lambda_1-\Omega)\cos\omega + \sin(\lambda_1-\Omega)\sin\omega\cos i\right\}\sin v -\\ &\frac{k}{\sqrt{p}}\left\{\cos(\lambda_1-\Omega)\sin\omega - \sin(\lambda_1-\Omega)\cos\omega\cos i\right\}\frac{p}{r},\end{aligned} \qquad (2,11)$$

hence

$$\frac{d\,1/a}{dt} = 2m_1\frac{n_1 r r_1\left\{-\cos u\sin(\lambda_1-\Omega) + \sin u\cos(\lambda_1-\Omega)\cos i\right\}}{\left[r^2 + r_1^{\,2} - 2rr_1\left\{\cos u\cos(\lambda_1-\Omega) + \sin u\sin(\lambda_1-\Omega)\cos i\right\}\right]^{3/2}} + \qquad (2,12^a)$$

$$\frac{2km_1}{r_1^{\,2}\sqrt{p}}\left[-\left\{\cos(\lambda_1-\Omega)\cos\omega + \sin(\lambda_1-\Omega)\sin\omega\cos i\right\}\sin v + \left\{-\cos(\lambda_1-\Omega)\sin\omega + \sin(\lambda_1-\Omega)\cos\omega\cos i\right\}\frac{p}{r}\right] \qquad (2,12^b)$$

$$-2m_1\frac{d\,1/\Delta}{dt}, \qquad (2,12^c)$$

in which n_1 is the mean angular motion of Jupiter. Here $(2,12^b)$ is the same as formula (6) on page 11 of FAYET [1]). This can therefore be integrated in the same manner as was done by FAYET. As we want to know the perturbation for a complete passage through the planetary system, we have to integrate over t from $-\infty$ to $+\infty$. According to FAYET, page 15, formulae (12), the integrands of $\Phi_0(\mu)$ and $\Psi_1(\mu)$ are even functions, and those of $\Psi_0(\mu)$ and $\Phi_1(\mu)$ odd functions of t, so the last will not appear in formula (13), page 17, after integration from $-\infty$ to $+\infty$, whereas the first two functions occur with a factor 2. As we do not take into account the eccentricity of the orbit of Jupiter, the dashed terms in FAYET's for-

1) *L.c.*

mula (13) will be left out. We then get the following expression for the integral of ($2,12^a$) (we call $\delta_1(1/a)$ the integral of ($2,12^b$) and $\delta_2(1/a)$ that of ($2,12^a$))

$$\delta_1(1/a) = -2\vartheta C_0 \Psi_1(\mu) - 2\nu B_0 \Phi_0(\mu), \qquad (2,13)$$

in which
$$\mu = \frac{d}{r_1}\sqrt[3]{6(1+m_1)},\quad \vartheta = \frac{12\,q m_1}{\mu r_1^2},$$
$$\nu = \frac{-4qm_1}{r_1^2}\sqrt{\frac{3}{\mu}}, \qquad (2,14)$$
$$B_0 = -\cos z_0 \sin\omega + \sin z_0 \cos\omega \cos i,$$
$$C_0 = -\sin z_0 \cos\omega + \cos z_0 \sin\omega \cos i.$$

$z_0 = (\lambda_1)_0 - \Omega$ is the angular distance of Jupiter from the ascending node of the comet's orbit at the time of perihelion passage of the comet. $\Psi_1(\mu)$ and $\Phi_0(\mu)$ are the functions (12) of FAYET, page 15. The integral of ($2,12^c$)
$$-2m_1\int_{-\infty}^{+\infty}\frac{d\,1/\Delta}{dt}\,dt = -\frac{2m_1}{\Delta}\Bigg|_{-\infty}^{+\infty} = 0.$$

The remaining part, ($2,12^a$), the principal part of the perturbation function, is first integrated numerically. We take the true anomaly of the comet as the independent variable. As in the most important parts of the interval of integration the comet is moving fast, the interval of time will be small. So we can use for the entire integration equal intervals of the variable. After the transformation mentioned above we get for the integral of ($2,12^a$)

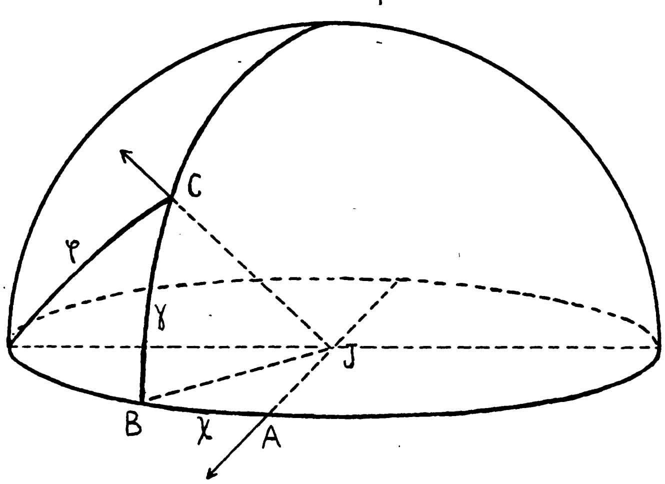
$$\delta_2(1/a) = m_1\sqrt{\frac{2}{qr_1}}\int_{-\pi}^{\pi}\frac{\left(\frac{r}{r_1}\right)^3\left[\cos^2\frac{i}{2}\sin(v - v_1 + \omega - z_0) - \sin^2\frac{i}{2}\sin(v + v_1 + \omega + z_0)\right]dv}{\left[1 + \left(\frac{r}{r_1}\right)^2 - 2\frac{r}{r_1}\left\{\cos^2\frac{i}{2}\cos(v - v_1 + \omega - z_0) + \sin^2\frac{i}{2}\cos(v + v_1 + \omega + z_0)\right\}\right]^{3/2}}, \qquad (2,15)$$

in which v_1 is the mean anomaly of Jupiter.

As the numerical evaluation would be a rather lengthy process, we have further calculated (2,15) by analytical approximation, in which we restrict ourselves to that part of the orbit in which the comet comes near Jupiter.

When crossing the orbit of Jupiter the velocity of a parabolic comet is $k\sqrt{2/r_1}$. The perturbation in the reciprocal semi-major axis $1/a$, corresponding to the principal part of the perturbative function depends mainly on the shortest distance at which the two bodies approach each other, and on the angle between the directions of Jupiter's velocity and the planetocentric velocity of the comet. We have considered

FIGURE 4



first the case in which the orbit of the comet and that of Jupiter intersect. For this case we made the following scheme. Let in Figure 4 J be the point of Jupiter's orbit where the comet is at time t_0. JA is the tangent to Jupiter's orbit at J and JC that to the comet's orbit. JB is the projection of JC on the orbital plane of Jupiter. $\angle BJA = \chi$, and $\angle BJC = \gamma$. The angle χ is counted from $+180°$ to $-180°$, γ from $+90°$ to $-90°$, φ is the angle between the direction of the comet's velocity and the prolonged radius vector of J. Then

$$\begin{aligned} \operatorname{cotg} i &= \cos\chi \operatorname{cotg}\gamma \\ \cos\varphi &= \sin\chi\cos\gamma \\ v_0 &= 180° - 2\varphi \\ q &= r\sin^2\varphi, \end{aligned} \qquad (2,16)$$

where v_0 is the true anomaly of the comet at time t_0.

The approximate perturbations have been calculated for the following values of χ and γ:

$\chi =$	$\gamma =$
0°	30°, 45°, 60° and 90°
± 30	0° and 75°
± 60	0°, 30° and 60°
± 90	75°
± 120	30° and 60°
± 150	75°
180	45°

Because we restrict ourselves to that part of the orbit in which the comet is near Jupiter, we write for ($2,12^a$) parabolic expressions for the numerator and for the bracketed expression in the denominator in the vicinity of $t = t_0$, where $r = r_1$. The parabolic expressions are expressed in powers of $n_1 t = \theta$, in which n_1 is the mean angular velocity of Jupiter. In analogy to the results of FAYET [1]) we get

[1]) *L.c.* page 31, formula 30.

$$\delta_2(1/a) = 2m_1 \int \frac{rr_1\{-\cos u \sin(\lambda_1 - \Omega) + \sin u \cos(\lambda_1 - \Omega)\cos i\}}{[r^2 + r_1^2 - 2rr_1\{\cos u \cos(\lambda_1 - \Omega) + \sin u \sin(\lambda_1 - \Omega)\cos i\}]^{3/2}} d\theta =$$

$$= \frac{2m_1}{r_1} \int_{\theta_1}^{\theta_2} \frac{A + B\theta + C\theta^2}{(a + b\theta + c\theta^2)^{2/3}} d\theta = \frac{2m_1}{r_1 c(ac - b^2/4)} \left| \frac{(Ac - aC)\left(c\theta + \frac{b}{2}\right) - (Bc - bC)\left(a + \frac{b\theta}{2}\right)}{\sqrt{a + b\theta + c\theta^2}} \right|_{\theta_1}^{\theta_2} +$$

$$+ \frac{2m_1 CM}{r_1 c^{2/3}} \log \frac{\sqrt{\frac{a + b\theta_1 + c\theta_1^2}{c}} + \frac{c\theta_1 + b/2}{c}}{\sqrt{\frac{a + b\theta_2 + c\theta_2^2}{c}} + \frac{c\theta_2 + b/2}{c}}, \qquad (2,17)$$

in which

$$\begin{aligned}
A &= -H \\
B &= -(Q\sqrt{2} - G) \\
C &= H + P\sqrt{2} \\
a &= 2(1 - G) \\
b &= -2\left(\sqrt{2}\sin\frac{\omega}{2} + C\right) \\
c &= 1 + 2B \\
G &= \cos\lambda_0 \\
H &= -\sin\lambda_0 \\
P &= -\cos\lambda_0 \sin\frac{\omega}{2} + \sin\lambda_0 \cos i \cos\frac{\omega}{2} \\
Q &= \sin\lambda_0 \sin\frac{\omega}{2} + \cos\lambda_0 \cos i \cos\frac{\omega}{2} \\
M &= {}^e\log 10.
\end{aligned}$$

In these expressions λ_0 has been put for $(\lambda_1)_0 - \Omega$, where $(\lambda_1)_0$ is λ_1 at $t = t_0$, and ω is the angular distance from the ascending node to perihelion. In the calculations we have for each case adopted for λ_0 the values $5°$, $9°$, $13°$, $-5°$, $-9°$ and $-13°$.

The interval of integration $\theta_1 - \theta_2$ is the arc over which Jupiter is moving during the perturbation. In the choice of θ_1 and θ_2 two things must be kept in mind. On the one hand the interval must be kept as large as possible, in order to bridge over the greater part of the most important part of $\delta_2(1/a)$. On the other hand the parabolic expressions must be sufficiently exact for the interval. If we choose for θ_1 and θ_2 respectively $+ \cdot 2$ and $- \cdot 2$, both conditions are satisfied as well as possible. Moreover the minimum value of the denominator is then satisfactorily near the middle of the interval, as $b/2a < {}^1/_5$. If we develop Δ^2 in a power series of θ the coefficient of θ^3 is found to be $\leqq 3/4$. At the limits of the interval the corresponding term is smaller than $3/4 \times 8 \times 10^{-3} = 6 \times 10^{-3}$. This confirms the accuracy of the parabolic expressions.

The calculations have also been made for cases in which the comet cuts the orbital plane of Jupiter at $r = 5{\cdot}4528$, that is at a distance of $\cdot 2500$ A.U. from the orbit of Jupiter. In this case the perturbations have been calculated for the following values of χ and γ:

$\chi = \quad 0°$	$\gamma = 30°$ and $60°$
$\pm\ 60$	30 and 60
$\pm\ 120$	30 and 60
180	30

After small changes in the parabolic expressions we get

$$\begin{aligned}
A &= -H \\
B &= -(1{\cdot}3364\, Q - G) \\
C &= 0{\cdot}9465\, H + 1{\cdot}3364\, P \\
a &= 2{\cdot}0784 - 2{\cdot}0769\, G \\
b &= -2{\cdot}8823 \sin\frac{\omega}{2} - 2{\cdot}7756\, P - 2{\cdot}0769\, H \\
c &= 0{\cdot}9630 + 1{\cdot}9659\, G - 2{\cdot}7756\, Q,
\end{aligned}$$

in which G, H, P and Q have the same significance as above. For λ_0 the values $0°$, $4°$, $8°$, $-4°$ and $-8°$ have been adopted.

From Table 1 it is easily seen that the perturbations, as was expected, are mainly determined by the angle ψ, under which the comet crosses the orbit of Jupiter, and by the smallest distance Δ_m at which they approach each other. The values found for the perturbations for $\lambda_0 = 0$ are not surprising, as in these cases the comet is accelerated in one part of the orbit and slowed down in the other. This domain of transition turns out to have a width of only $4°$ or $5°$ in λ_0, it does not influence the further computation. From Table 1, and after a careful consideration of the numerical results, we have compiled the graphs in Figure 5.

We shall consider the term $\delta_1(1/a)$, which results from the complementary part of the perturbation function, separately for the two cases: $q = 1$ A.U. and $q = 4{\cdot}5$ A.U. First we shall write the term in another form, and substitute at the same time the numerical values for $\Phi_0(\mu)$, $\Psi_1(\mu)$, ϑ and ν.

We thus find

TABLE I

The values of $\delta_2(1/a)$ are expressed in units of 10^{-3}, the values of Δm in A.U.

$\chi = 0°$

λ_0 \ γ	30°		45°		60°		90°	
	$\delta_2(1/a)$	Δm	$\delta_2(1/a)$	Δm	$\delta_2(1/a)$	Δm	$\delta_2(1/a)$	Δm
5°	− 10·38	·42	− 7·69	·44	− 6·50	·42	− 4·76	·36
9	− 4·84	·75	− 3·67	·79	− 3·41	·77	− 2·73	·64
13	− 2·71	1·08	− 2·13	1·13	− 2·17	1·10	− 1·85	·93
− 5	+ 10·38	·42	+ 7·69	·44	+ 6·50	·42	+ 4·76	·36
− 9	+ 4·84	·75	+ 3·67	·79	+ 3·41	·77	+ 2·73	·64
− 13	+ 2·71	1·08	+ 2·13	1·13	+ 2·17	1·10	+ 1·85	·93

$\chi = 30°$

λ_0 \ γ	0°		75°	
	$\delta_2(1/a)$	Δm	$\delta_2(1/a)$	Δm
5°	− 9·38	·42	− 5·26	·40
9	− 4·94	·70	− 2·79	·70
13	− 2·62	1·20	− 1·79	1·05
− 5	+ 11·74	·38	+ 5·56	·40
− 9	+ 4·36	·77	+ 3·04	·70
− 13	+ 2·24	1·20	+ 1·99	1·06

$\chi = 90°$

λ_0 \ γ		
	$\delta_2(1/a)$	Δm
5°	− 4·56	·36
9	− 2·54	·64
13	− 1·69	·92
− 5	+ 5·12	·36
− 9	+ 2·92	·65
− 13	+ 2·02	·93

$\chi = 60°$

λ_0 \ γ	0°		30°		60°	
	$\delta_2(1/a)$	Δm	$\delta_2(1/a)$	Δm	$\delta_2(1/a)$	Δm
5°	− 6·38	·43	− 5·51	·42	− 5·67	·40
9	− 2·24	·71	− 2·67	·74	− 2·81	·70
13	− 2·10	1·03	− 1·60	1·07	− 1·70	1·01
− 5	+ 7·35	·42	+ 6·79	·42	+ 6·59	·40
− 9	+ 3·15	·72	+ 3·79	·76	+ 3·53	·71
− 13	+ 1·91	1·08	+ 2·48	1·07	+ 2·43	1·04

$\chi = 120°$

λ_0 \ γ	30°		60°	
	$\delta_2(1/a)$	Δm	$\delta_2(1/a)$	Δm
5°	− 3·74	·27	− 4·62	·30
9	− 2·07	·47	− 2·27	·55
13	− 1·45	·68	− 1·60	·79
− 5	+ 4·69	·27	+ 4·78	·31
− 9	+ 2·92	·50	+ 2·85	·55
− 13	+ 2·25	·72	+ 2·08	·81

$\chi = 150°$

λ_0 \ γ	75°	
	$\delta_2(1/a)$	Δm
5°	− 4·40	·32
9	− 2·48	·57
13	− 1·72	·81
− 5	+ 4·59	·32
− 9	+ 2·66	·57
− 13	+ 1·90	·86

$\chi = 180°$

λ_0 \ γ	45°	
	$\delta_2(1/a)$	Δm
5°	− 3·88	·20
9	− 2·33	·35
13	− 1·79	·50
− 5	+ 3·88	·20
− 9	+ 2·33	·35
− 13	+ 1·79	·50

$\chi = 0°$

λ_0 \ γ	30°		60°	
	$\delta_2(1/a)$	Δm	$\delta_2(1/a)$	Δm
0°	0	·19	0	·19
4	− 9·32	·40	− 5·94	·40
8	− 4·53	·72	− 3·35	·71
− 4	+ 9·32	·40	+ 5·94	·40
− 8	+ 4·53	·72	+ 3·35	·71

$\chi = 60°$

λ_0 \ γ	0°		60°	
	$\delta_2(1/a)$	Δm	$\delta_2(1/a)$	Δm
0°	+ 9·83	·12	+ 3·09	·18
4	− 6·55	·30	− 5·04	·33
8	− 4·70	·63	− 2·83	·62
− 4	+ 6·88	·40	+ 5·28	·40
− 8	+ 4·13	·74	+ 3·28	·70

$\chi = 120°$

λ_0 \ γ	30°		60°	
	$\delta_2(1/a)$	Δm	$\delta_2(1/a)$	Δm
0°	+ 5·24	·17	+ 2·64	·19
4	− 4·37	·14	− 3·61	·26
8	− 2·91	·32	− 2·52	·47
− 4	+ 3·31	·36	+ 3·38	·36
− 8	+ 2·26	·58	+ 2·26	·59

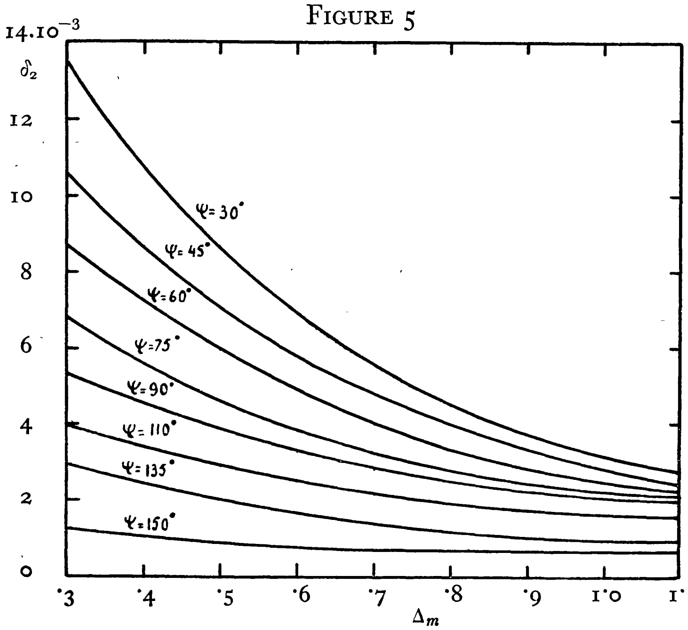
$\chi = 180°$

λ_0 \ γ	30°	
	$\delta_2(1/a)$	Δm
0°	0	·19
4	− 1·16	·23
8	− 1·43	·29
− 4	+ 1·16	·23
− 8	+ 1·43	·29

$$\text{for } q = 4{\cdot}5 \text{ A.U.:} \quad \delta_1(1/a) = \left[-0{\cdot}099 \sin(z_0 + \omega) \sin^2\frac{i}{2} + 0{\cdot}895 \sin(z_0 - \omega) \cos^2\frac{i}{2} \right] \times 10^{-3},$$

$$\text{for } q = 1 \text{ A.U.:} \quad \delta_1(1/a) = \left[-0{\cdot}389 \sin(z_0 + \omega) \sin^2\frac{i}{2} + 1{\cdot}445 \sin(z_0 - \omega) \cos^2\frac{i}{2} \right] \times 10^{-3}. \qquad (2{,}18)$$

FIGURE 5



We first consider the case $q = 4{\cdot}5$ A.U. The comet and Jupiter can approach each other at a small distance if the point of intersection of the comet's orbit and the orbital plane of Jupiter is near the orbit of Jupiter. There $\dfrac{2q}{1 + \cos\omega} \sim r_1$. For $q = 4{\cdot}5$, ω must be about 45° or 135°. Moreover, if in this case $(z_0 - \omega)$ is near 30° or 150°, respectively, $\delta_2(1/a)$ will be large. In both cases $(z_0 - \omega)$ is small and according to $(2{,}12^a)$ $\delta_1(1/a)$ will be small. Hence, if $\delta_2(1/a)$ is large, we need not take account of $\delta_1(1/a)$. If $i \leqq 60°$, the sign and the value of $\delta_2(1/a)$ (see (2,15)) are mainly determined by $\cos^2\dfrac{i}{2}\sin(v - v_1 + \omega - z_0)$, if, at least $(\omega - z_0)$ is not too small, as $v - v_1 \sim 0$. Comparing with $(2{,}12^a)$ we see that $\delta_1(1/a)$ and $\delta_2(1/a)$ have different signs. In spite of that, $\delta_1(1/a) + \delta_2(1/a)$ will have the sign of $\delta_2(1/a)$, if Δ_m (the minimum distance between the comet and Jupiter) lies between 1 and 2 A.U.

In the denominator of $\delta_2(1/a)$, $\cos(v - v_1 + \omega - z_0)$ is only slightly variable, and as r/r_1 also changes only little, the same will be true of Δ. If Δ_m lies between 1 and 2 A.U., Δ will remain practically constant for a considerable part of the orbit. It is then easy to see that $\delta_2(1/a)$ will be larger in absolute value than $\delta_1(1/a)$. From the numerical integrations and approximate calculations the perturbation for Δ_m between 1 and 2 A.U. turns out to be proportional to $1/\Delta_m^2$, while the average value of the perturbation for $\Delta_m = 1\cdot1$ A.U. equals 2×10^{-3}. For $\Delta_m > 2\cdot1$ A.U. the value will be mainly determined by $\delta_1(1/a)$.

A similar consideration can be given for $q = 1$ A.U. In this case $\delta_2(1/a)$ is large, if ω is 50° or 130°, and z_0 respectively 145° or 35°. Then $(z_0 - \omega)$ is about 90° or $-$ 90°, and consequently $\delta_1(1/a)$ is large with the same sign as $\delta_2(1/a)$. Unfortunately this spoils the symmetry. However, the term is important only for close approaches, in which the orbits are radically changed. It has no significance in the cases of small perturbations with which we shall principally be concerned when studying the distribution of the long-period comets. The value of $\delta_2(1/a)$ for Δ_m between 1 and 2 A.U. will be small, as for $v < 130°$, $r << r_1$, thus $r/r_1 << 1$, and for $v < 140°$, $r \sim \Delta$, thus $(r/\Delta)^3 \sim 1$. The perturbations for $q = 1$ and $\Delta_m > 1\cdot1$ A.U. are determined mainly by $\delta_1(1/a)$.

For $i > 90°$ the value of $\delta_2(1/a)$ is small for both cases.

3. *Influence of Jupiter's perturbations on the distribution of the major axes.*

In this section we shall investigate in what manner the small perturbations caused by Jupiter act upon the distribution of the major axes of the long-period comets. For that purpose we shall start from an original field of parabolic comets.

If the mean perturbation in $1/a$ is 10^{-3} and if a is larger than 100 A.U., the perturbation ΔT in the period will be larger than 150 years. If a is smaller than 40 A.U., ΔT will be smaller than 15 years.

For the first case we may assume that the perturbations at two successive passages of perihelion are independent of each other. For orbits with a smaller than 40 A.U. we may no longer expect this, as in that case ΔT is of the same order as the period of Jupiter. For the domain between 40 and 100 A.U. there may be stable orbits, though this does not seem very probable, because of the perturbations of the other planets.

If the perturbations during successive periods are independent of each other, the value of $1/a$ for a given comet can be changed considerably after some periods[1]). At the epoch t let $\nu(1/a, t)d1/a$ represent the number of comets passing through perihelion per year and having reciprocal semi-major axes between $1/a$ and $1/a + d\,1/a$. If $N(1/a, t)$ is the total number of comets with the same values of $1/a$, then

$$N(1/a, t) = a^{3/2}\,\nu(1/a, t). \qquad (3,1)$$

Let us denote by $\varphi(\delta)$ the probability that a comet will suffer at a passage of its perihelion a perturbation δ in $1/a$. We shall call this function the transition function, because the entire process shows much similarity with a diffusion process. This transition function will be independent of $1/a$, because all orbits considered can be sufficiently approximated by parabolas.

During a year the number of comets with a definite value of $1/a$ will increase by

$$\int_{-\infty}^{+\infty} \nu(1/a - \delta, t)\,\varphi(\delta)\,d\delta, \qquad (3,2)$$

while those which originally had the value of $1/a$ considered will be removed to other orbits. The total number of comets with reciprocal semi-major axes $1/a$ will therefore increase by

$$\frac{\partial N}{\partial t} = \int_{-\infty}^{+\infty} \nu(1/a - \delta, t)\,\varphi(\delta)\,d\delta - \nu(1/a, t). \quad (3,3)$$

Developing $\nu(1/a - \delta, t)$ in ascending powers of δ we obtain

$$\frac{\partial N}{\partial t} = \int_{-\infty}^{+\infty} \left\{ \nu(1/a, t) + \frac{\partial \nu}{\partial \delta}\delta + \frac{\partial^2 \nu}{\partial \delta^2}\frac{\delta^2}{2} + \ldots \right\} \varphi(\delta)\,d\delta - \nu(1/a, t). \qquad (3,4)$$

If $\varphi(\delta)$ is symmetrical, so that $\int_{-\infty}^{+\infty} \varphi(\delta)\,d\delta = 0$, we may write

$$\frac{\partial N}{\partial t} = \tfrac{1}{2}\Phi^2 \frac{\partial^2 \nu}{\partial \delta^2}, \qquad (3,5)$$

in which

$$\Phi^2 = \int_{-\infty}^{+\infty} \varphi(\delta)\,\delta^2\,d\delta. \qquad (3,6)$$

These formulae evidently hold only for intervals of time that are long compared with the duration of the passage of a comet through the planetary system.

We shall search for a solution of the partial differential equation (3,5) such that $\nu(0, t) = 1$.

Denoting $\frac{1}{2}\Phi^2$ by $1/D$ and $1/a$ by z, we have, as $\frac{\partial \nu}{\partial \delta} = -\frac{\partial \nu}{\partial 1/a}$,

1) Russell, *A.J.* 33, 60.

$$\frac{\partial N}{\partial t} = \frac{1}{D}\frac{\partial^2}{\partial z^2}(Nz^{3/2}). \qquad (3,7)$$

Let us suppose that $Nz^{3/2}$ can be written as $ye^{\lambda t}$, in which y and λ are independent of t; then, for all values of λ,

$$\frac{d^2y}{dz^2} - \lambda D y z^{-3/2} = 0. \qquad (3,8)$$

We need a solution satisfying the condition $z = 0$, $y = 1$. Introducing $x = \sqrt{z}$, (3,8) may be written

$$x\frac{d^2y}{dx^2} - \frac{dy}{dx} - 4\lambda D y = 0. \qquad (3,9)$$

A solution satisfying the boundary condition is

$$y(\lambda, x) = (4\lambda D)^2 \int_0^\infty s e^{-\left(\frac{x}{s} + 4\lambda D s\right)} ds. \qquad (3,10)$$

Put $u = s\sqrt{\frac{4\lambda D}{x}}$, then

$$y(\lambda,x) = 4\lambda D x \int^\infty e^{-\sqrt{4\lambda D x}\,(u + 1/u)} u\, du. \qquad (3,11)$$

Now,

$$Na^{-3/2} = \frac{1}{2\pi i}\int_{\sigma - i\infty}^{\sigma + i\infty} \frac{e^{\lambda t}}{\lambda} y(\lambda, x)\, d\lambda = \frac{x}{2\pi i}\int_{\sigma - i\infty}^{\sigma + i\infty} \frac{e^{\lambda t}}{\lambda}\left(\int_0^\infty 4\lambda D e^{-\sqrt{4\lambda D x}\,(u + 1/u)} u\, du\right) d\lambda, \qquad (3,12)$$

in which $\Re(\sigma) > 0$ [1]).

We put $4Dx = A^2$ and introduce the transformation $\lambda = s^2$. Then we first carry out the integration for λ. We get

$$\int_{\sigma - i\infty}^{\sigma + i\infty} e^{\lambda t - A\sqrt{\lambda}(u + 1/u)} d\lambda = \int_{\varkappa - i\infty}^{\varkappa + i\infty} e^{s^2 t - As(u + 1/u)} 2s ds, \qquad (3,13)$$

in which $\Re(\varkappa) > 0$.

In (3,13) we introduce the transformation $s = \varkappa + ir$, then (3,13) becomes

$$2i\, e^{-\frac{A^2}{4t}(u + 1/u)^2} \int_{-\infty}^{+\infty} e^{-tr^2}\left\{ ir + \frac{A}{2t}(u + 1/u)\right\} dr =$$

$$= 2i\, e^{-\frac{A^2}{4t}(u + 1/u)^2}\left[\int_{-\infty}^{0} e^{-tr^2}\left\{ ir + \frac{A}{2t}(u + 1/u)\right\} dr + \int_0^\infty e^{-tr^2}\left\{ ir + \frac{A}{2t}(u + 1/u)\right\} dr\right]. \qquad (3,14)$$

In the second integral of the right-hand member of (3,14), we replace r by $-r$, then (3,14) becomes

$$2i\, e^{-\frac{A^2}{4t}(u + 1/u)^2} \int_0^\infty e^{-tr^2}\frac{A}{t}(u + 1/u)\, dr. \qquad (3,15)$$

Carrying out the integration we obtain

$$\frac{iA\sqrt{\pi}}{t\sqrt{t}} e^{-\frac{A^2}{4t}(u + 1/u)^2}(u + 1/u). \qquad (3,16)$$

Then, according to (3,12) and (3,16)

$$Na^{-3/2} = \frac{1}{2\sqrt{\pi}}\frac{A^3}{t\sqrt{t}}\int_0^\infty e^{-\frac{A^2}{4t}(u + 1/u)^2}(u + 1/u)\, u du. \qquad (3,17)$$

Putting $u^2 + 1/u^2 = s^2 + 2$, (3,17) is transformed into

$$\frac{1}{2\sqrt{\pi}}\frac{A^3}{t\sqrt{t}}\int_{-\infty}^{+\infty} e^{-\frac{A^2}{4t}(s^2 + 4)}(s\sqrt{s^2 + 4} + s^2 + 2)\, ds =$$

$$= \frac{1}{2\sqrt{\pi}}\frac{A^3}{t\sqrt{t}}\int_{-\infty}^{+\infty} e^{-\frac{A^2}{4t}(s^2 + 4)}(s^2 + 2)\, ds, \qquad (3,18)$$

as $\int_{-\infty}^{+\infty} e^{-\frac{A^2}{4t}(s^2 + 4)} s\sqrt{s^2 + 4}\, ds = 0$.

Hence

$$Na^{-3/2} = e^{-\frac{A^2}{t}}\left(1 + \frac{A^2}{t}\right). \qquad (3,19)$$

As

$$A^2 = 4D\sqrt{z} = 4D\sqrt{1/a}$$

(3,19) becomes

$$Na^{-3/2} = \nu(1/a, t) = e^{-\frac{4D\sqrt{1/a}}{t}}\left(1 + \frac{4D\sqrt{1/a}}{t}\right). \qquad (3,20)$$

From this formula we see that in the course of time the number of comets passing perihelion per year will become constant for equal intervals of $1/a$, because for $t \to \infty$ $\nu(1/a, t) \to 1$.

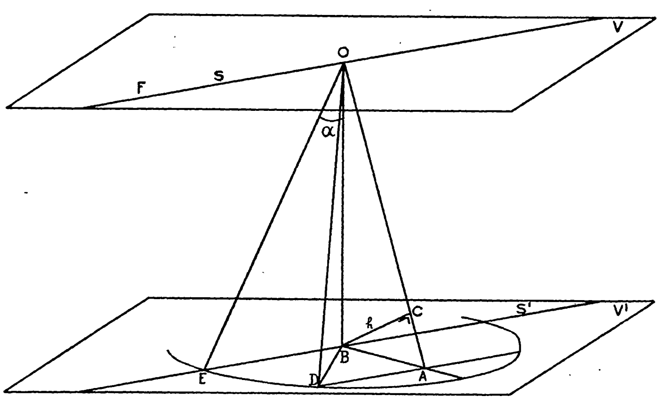
At $t = 0$, $\nu = 0$ for all values of $1/a$, except for $1/a = 0$: we have started from a distribution in which all comets had parabolic orbits. Formula (3,20) indicates the manner in which these comets are gradually captured into the solar system by small perturbations.

The constant D has to be determined by the values

[1]) See COURANT HILBERT: "Methoden der Mathematischen Physik", 2, page 211, 1937.

of the small ordinary perturbations found in the preceding section. We have there derived the values of the perturbations as a function of the angle at which the comet crosses the orbit of Jupiter, and of the shortest distance to which the comet and Jupiter approach each other. In order to compute D we must know the probability that a comet with a given perihelion distance q crosses the orbit of Jupiter at an angle ψ, and approaches Jupiter at as small a distance as Δ_m. For that purpose we first calculate the probability that a parabolic orbit with given perihelion distance crosses the orbit of Jupiter at a distance less than h. A parabolic orbit with $q = 4{\cdot}5$ A.U. intersects the sphere with a radius equal to that of Jupiter's orbit at an angle of 70° with the radius vector of the point of intersection. For a given point of the sphere the orbits of this kind are thus tangent to a cone with the radius vector from that point as axis, and an angle of 140°. The orbits for which $q = 1$ A.U. are similarly tangent to a cone with an angle of 50°. In Figure 6 V represents a plane tangent to the sphere in a point on Jupiter's orbit. O is a point of that orbit and s its tangent in O. The cone with OB as axis and with aperture $2\,\alpha$ is the cone just mentioned. V_1 is an auxiliary plane parallel to V. If $\angle\, ABD = \varphi$, then the number of comet orbits in this part of the cone is proportional to φ.

FIGURE 6



Let OD be an arbitrary orbit on the cone. The perpendicular BC from B on a plane DOA through OD and s is the distance between OD and s'. If the radius BE is taken as unit we have $AB = \cos\varphi$,

$$OB = \text{cotg}\,\alpha, \text{ and } \sin OAB = \frac{1}{\sqrt{1 + \text{tg}^2\,\alpha \cos^2\varphi}}.$$

How far can we shift OB parallel to itself in a direction perpendicular to s, so that the distance between OD and s remains smaller than a given distance h? It is at once evident from Figure 6 that to a shift BA corresponds a distance between the two lines amounting to $BA \sin OAB$. Putting this equal to h we obtain $BA = h\sqrt{1 + \text{tg}^2\,\alpha \cos^2\varphi}$. Let us consider values of BA which are small compared to the radius a_j of Jupiter's orbit, so that the plane V may sufficiently approximate the sphere. For any given φ a fraction $\frac{2BA \times 2\pi a_j}{4\pi a^2_j}$ of the sphere is contained between two parallel circles at distances BA from Jupiter's orbit. On the other hand a fraction $\frac{d\varphi}{2\pi}$ will have values of φ between φ and $\varphi + d\varphi$. The total fraction of the comets penetrating the sphere under an angle α with the normal and crossing Jupiter's orbit at a distance less than h is, therefore,

$$W(h) = \frac{h}{2\pi a_j}\int_0^{2\pi}\sqrt{1 + \text{tg}^2\,\alpha\cos^2\varphi}\,d\varphi, \quad (3{,}21)$$

in which the expression found above for BA has been inserted.

For $q = 1, \alpha = 25°$. As $\text{tg}\,25° \sim {}^1/_2$, $\sqrt{1 + \text{tg}^2\,25 \cos^2\varphi}$ can be developed in powers of $\text{tg}^2\,25 \cos^2\varphi$. After carrying out the integrations we obtain

$$W(h) = \frac{h}{a_j}\,(1 + {}^1/_4\,\text{tg}^2\,25 - {}^1/_8 \cdot {}^3/_8\,\text{tg}^4\,25 + \ldots) =$$
$$= \frac{h}{a_j}\,(1 + {}^1/_{16} - {}^3/_{1024} + \ldots) \sim \frac{h}{a_j} \times 1{\cdot}06. \quad (3{,}22)$$

For $q = 4{\cdot}5$, $\alpha = 70°$, we thus get

$$\sqrt{1 + \text{tg}^2\,70\cos^2\varphi} = \sqrt{1 + \text{tg}^2\,70}\,\sqrt{1 - c\sin^2\varphi},$$

in which $c = \dfrac{\text{tg}^2\,70}{1 + \text{tg}^2\,70} = {\cdot}88.$

As

$$\int_0^{\frac{\pi}{2}}\sqrt{1 - c\sin^2\varphi}\,d\varphi =$$
$$= \frac{\pi}{2}\,(1 - {}^1/_4\,c - {}^3/_8 \cdot {}^1/_8\,c^2 - {}^1/_{16} \cdot {}^{15}/_{48}\,c^3 \ldots) = \frac{\pi}{2} \times {\cdot}72,$$

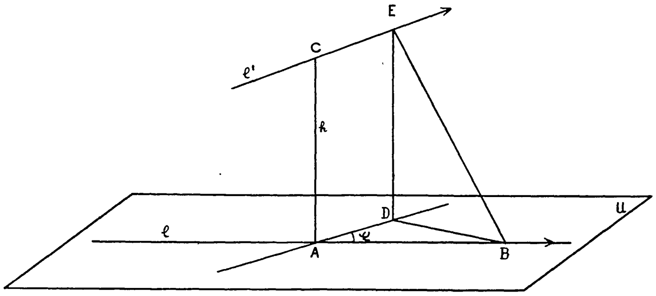
$$W(h) = \frac{h}{a_j} \times 2{\cdot}10. \quad (3{,}23)$$

The approximation will be sufficient for $h \leq {}^2/_3$, as then the extreme intersection points of the cones are not more than 20° distant from the orbit. Such a part of the sphere may be approximated by a plane. For larger values of h the axis of the cone for the extreme points will be no longer perpendicular to V. As compared with the idealized picture considered above, one part of the orbits will then be nearer to Jupiter's orbit, the other part, however, further away. The two effects will nearly counterbalance each other, and we may still consider the above results as a fair approximation.

We must now calculate the probability that an orbit with definite perihelion distance q, which crosses Jupiter's orbit under an angle ψ, approaches

Jupiter at a distance between Δ and $\Delta + d\Delta$.

FIGURE 7



In Figure 7 l is the orbit of Jupiter and l' the comet's orbit. Both are supposed to be straight. U is a plane through l and parallel to l'. Let CA represent the shortest distance h between the two orbits. Suppose that, when the comet is in C, Jupiter is at a distance g from A. Let the velocity of Jupiter be V_1; the comet's velocity is then $V_1\sqrt{2}$.

If at time t the comet is in E, and Jupiter in B, then
$AB = g + V_1 t$,
$CE = V_1 t\sqrt{2} = AD$, if AD is the projection of CE on the plane U, and $\angle DAB = \psi$.
$EB^2 = V_1^2 t^2 (3 - 2\sqrt{2}\cos\psi) + V_1 t(2g - 2g\sqrt{2}\cos\psi) + g^2 + h^2$.

The minimum value Δ^2_m for EB^2 equals

$$\Delta^2_m = \frac{2g^2 \sin^2\psi}{3 - 2\sqrt{2}\cos\psi} + h^2. \qquad (3,24)$$

For constant values of ψ and Δ_m the expression (3,24) is an ellipse in the (h, g)-plane.

The number of orbits $W(\psi, \Delta_m)\, d\Delta_m$, for $q = 4{\cdot}5$, for which ψ is a constant and Δ_m lies between Δ_m and $\Delta_m + d\Delta_m$, equals, according to (3,23),

$$\iint_G \frac{2{\cdot}10}{a_j}\, dh \times \frac{dg}{2\pi a_j}, \qquad (3,25)$$

in which the domain G is half the surface between the two ellipses (3,24) for which Δ_m equals Δ_m and $\Delta_m + d\Delta_m$. We must have half the surface, because $h \geqq 0$. Therefore

$$W(\psi, \Delta_m) = \frac{\Delta_m}{2} \frac{\sqrt{3 - 2\sqrt{2}\cos\psi}}{\sqrt{2}\sin\psi} \times \frac{2{\cdot}1}{a_j^2}. \qquad (3,26)$$

Let us suppose that in Figure 6 OF is the direction of Jupiter's motion, and OD that of the comet. Then $\angle DOF = \psi$ and

$$\cos\psi = \sin\varphi \sin\alpha. \qquad (3,27)$$

The number of comets which have a value of ψ between certain limits can be easily derived from this relation. We have computed the values of D with the aid of the function $W(\psi, \Delta_m)$, the relation (3,27), the graphs in Figure 5 and the values of the perturbations for $\Delta_m > 1{\cdot}1$ A.U. given on page 455. Moreover, we have taken into account the fact that the orbits of the comets which come within a sphere with a radius equal to Jupiter's orbital radius intersect this sphere two times, in consequence of which $W(\psi, \Delta_m)$ has to be multiplied by 2. The computations have been carried out for $q = 1$ A.U. and $q = 4{\cdot}5$ A.U. For the first case we also derived the values of D separately for direct and retrograde comets.

We shall not give here all details of the computation, but communicate at once the results. So we have

$$\text{for} \quad q = 4{\cdot}5 \text{ A.U.},\ 4D = 4{\cdot}9 \times 10^6,\ \nu(1/a, t) = e^{-\frac{4{\cdot}9 \times 10^6 \times \sqrt{1/a}}{t}} \left(1 + \frac{4{\cdot}9 \times 10^6 \times \sqrt{1/a}}{t}\right), \qquad (3,28^a)$$

$$\text{for} \quad q = 1 \text{ A.U.},\ 4D = 13{\cdot}3 \times 10^6,\ \nu(1/a, t) = e^{-\frac{13{\cdot}3 \times 10^6 \times \sqrt{1/a}}{t}} \left(1 + \frac{13{\cdot}3 \times 10^6 \times \sqrt{1/a}}{t}\right). \qquad (3,28^b)$$

For the case $q = 1$ A.U. the value of $4D$ for direct motion is $9{\cdot}5 \times 10^6$ and for retrograde motion $22{\cdot}0 \times 10^6$.

Because the perturbations have been approximated in a comparatively rough way the value of D may have an error of 20%. We do not think that the error will be larger.

We see from (3,28^a) and (3,28^b) that in about 10^6 years the distribution over $1/a$ for the comets captured from a constant parabolic field must be nearly constant down to $a = 25$ and 200 A.U. respectively. Thus in about 10^6 years a parabolic comet will have changed into an elliptic one with a period of some 400 years.

For $q = 4{\cdot}5$ A.U. the process is going three times more rapidly than for $q = 1$ A.U. If the process is still in an initial stage there will be for $q = 1$ A.U. and $q = 4{\cdot}5$ A.U. a difference in the total number of comets which pass their perihelion per year, the ratio of both numbers being the inverse square of the ratio of the coefficients D. In that case nine times more comets will pass their perihelion per year with perihelion distances between 4 and 5 A.U. than between 1 and 2 A.U. In an advanced stage of the process, however, both numbers will become equal.

So far we have only considered the diffusion of comets starting from a field of parabolic comets. Our next step must be to investigate in what way this process takes place if we start from more general assumptions about the field of comets in interstellar space.

4. *The origin of the present distribution of major axes.*

This section deals with problems connected with the capture of comets from interstellar space, and with attempts to explain the present distribution of the major axes of long-period and parabolic comets.

Let us denote by W_s the velocity of the sun with respect to the average field of the comets which are beyond the sun's sphere of action. In the following we shall call these the "distant" comets. W_s will be of the same order of magnitude as the velocity of the sun with respect to the neighbouring stars. The distribution of the velocities of the distant comets presumably can be represented approximately by an exponential function. We want to specify first which comets will come within a distance q from the sun, and to find the distribution of their orbital elements; the perihelion distances and the semi-major axes being of particular interest. In order that a comet starting from a certain point on the sphere of action with a certain heliocentric velocity V might come within a distance q from the sun, the heliocentric velocity vector should lie within a hyperboloid of revolution with the radius vector of that point as its axis. The equation of this hyperboloid is [1])

$$V^2\left(\sin^2\vartheta - \frac{q^2}{\rho^2}\right) = \frac{2q}{\rho^2}\left(1 - \frac{q}{\rho}\right), \qquad (4,1)$$

where ρ is the radius of the sphere of action and ϑ the angle between the velocity and the radius vector.

We shall derive the heliocentric velocity distribution of the distant comets from the exponential velocity distribution and the velocity W_s. Suppose $\varphi(W)\,dW$ is the number of comets with velocities between W and $W + dW$ with respect to their centre of gravity. Then the space density of the vector points W is $\varphi(W)/4\pi W^2$. The number of comets with heliocentric velocities between V and $V + dV$ is

$$\mu(V)\,dV = \int_0^\pi 2\pi V^2 \sin A\,dA\,dV \frac{\varphi(W)}{4\pi W^2}, \qquad (4,2)$$

in which A is the angle between the vectors V and W_s. Inserting $W^2 = W_s^2 + V^2 - 2VW_s\cos A$

$$\mu(V)dV = \int_0^\pi 2\pi V^2 \sin A\,dA\,dV \frac{\varphi\left(\sqrt{W_s^2 + V^2 - 2VW_s\cos A}\right)}{4\pi\left(W_s^2 + V^2 - 2VW_s\cos A\right)}. \qquad (4,3)$$

If

$$\varphi(W) = 2W^2 \frac{n}{\sigma^3\sqrt{2\pi}} e^{-\frac{W^2}{2\sigma^2}}, \qquad (4,4)$$

in which n is the number of comets per (A.U.)^3 at the boundary of the sphere of action,

$$\mu(V) = \frac{2nV}{W_s\sigma\sqrt{2\pi}} e^{-\frac{W_s^2}{2\sigma^2}} e^{-\frac{V^2}{2\sigma^2}} \sinh\frac{W_sV}{\sigma^2}. \qquad (4,5)$$

Let $P(q)$ be the number of comets entering per unit of time into the sphere of action and having perihelion distances smaller than q. Whatever distribution of the velocities of the distant comets we start with, the result will always be that, counted for the whole boundary of the sphere of action, the distribution of the directions of the heliocentric velocities with respect to the radius vector is uniform. In order to calculate the number of comets $P(q)$ we therefore start from a velocity distribution $\mu(V)$ in which all directions of a certain velocity V are equally probable. The number of comets with a velocity V, flowing through a surface element of the sphere of action, and moving at an angle between ϑ and $\vartheta + d\vartheta$ with the radius vector, is then $\mu(V)\frac{\sin\vartheta\,d\vartheta}{2}V\cos\vartheta$. To find $P(q)$ the expression must be integrated over all velocity vectors V within the hyperboloid (4,1). If $d(q)$ represents the real axis of the hyperboloid, then all angles ϑ between 0 and $\pi/2$ are possible for $V \leqq d(q)$. For $V > d(q)$, however, $\vartheta \leqq \vartheta(V)$ if $\vartheta(V)$ is the smallest root of the equation

$$V^2\left(\sin^2\vartheta - \frac{q^2}{\rho^2}\right) = \frac{2q}{\rho^2}\left(1 - \frac{q}{\rho}\right). \qquad (4,6)$$

Then

$$P(q) = \int_0^\infty 4\pi\rho^2\mu(V)\,dV\int_0^{\vartheta(V)}\frac{\sin\vartheta\,d\vartheta}{2}V\cos\vartheta =$$

$$\pi\rho^2\left[\int_0^{d(q)} V\mu(V)\,dV + \int_{d(q)}^\infty V\mu(V)\sin^2\vartheta(V)\,dV\right], \qquad (4,7)$$

in which

$$\sin^2\vartheta(V) = \frac{q^2}{\rho^2} + \frac{2q}{\rho^2}\left(1 - \frac{q}{\rho}\right)\frac{1}{V^2} \qquad (4,8)$$

and

$$d(q) = \sqrt{\frac{2q}{\rho^2\left(1 + \frac{q}{\rho}\right)}}. \qquad (4,9)$$

The number of comets with perihelion distances between q and $q + dq$ will be denoted by $Q(q)\,dq$.

1) See *Encycl. Math. Wiss.* **6**$_2$, 18, page 926.

$$Q(q) = \frac{\partial P}{\partial q} = \pi \rho^2 \int_{d(q)}^{\infty} V\mu(V)\, dV \left\{ \frac{2q}{\rho^2} + \frac{2}{\rho^2}\left(1 - \frac{2q}{\rho}\right)\frac{1}{V^2} \right\} = \pi \int_{d(q)}^{\infty} \left\{ 2q + 2\left(1 - \frac{2q}{\rho}\right)\frac{1}{V^2} \right\} V\mu(V)\, dV. \quad (4{,}10)$$

For the calculation of the number of comets captured by small perturbations the distribution of the reciprocal major axes of the comets represented by $Q(q)$ must be known. We therefore replace the variable V by $1/a$. We have evidently

$$1/a = \frac{2 - \rho V^2}{\rho}, \quad V = \sqrt{2/\rho - 1/a}$$

and

$$dV = \frac{d(1/a)}{2\sqrt{2/\rho - 1/a}}, \quad (4{,}11)$$

so that (4,10) becomes

$$Q(q) = \pi \int \left\{ q + \left(1 - \frac{2q}{\rho}\right)\frac{1}{2/\rho - 1/a} \right\} \mu\left(\sqrt{2/\rho - 1/a}\right) d(1/a). \quad (4{,}12)$$

The distribution $f\,(1/a)$ of the reciprocal major axes is given by the integrand

$$f_q\,(1/a) = \pi \left\{ q + \left(1 - \frac{2q}{\rho}\right)\frac{1}{2/\rho - 1/a} \right\} \mu\left(\sqrt{2/\rho - 1/a}\right). \quad (4{,}13)$$

By integration we get

$$P(q) = \frac{n\sqrt{\pi}}{W_s}\left[3\sqrt{2}\,\sigma W_s q^2 e^{-\frac{W_s^2}{2\sigma^2}} + \left\{ 2q^2(\sigma^2 + W_s^2) + 4q\left(1 - \frac{q}{\rho}\right) \right\} \operatorname{erf}\frac{W_s}{\sigma\sqrt{2}} \right], \quad (4{,}14)$$

$$Q(q) = \frac{n\sqrt{\pi}}{W_s}\left[6\sqrt{2}\,\sigma W_s q\, e^{-\frac{W_s^2}{2\sigma^2}} + \left\{ 4q(\sigma^2 + W_s^2) + 4\left(1 - \frac{2q}{\rho}\right) \right\} \operatorname{erf}\frac{W_s}{\sigma\sqrt{2}} \right], \quad (4{,}15)$$

whereas

$$f_q\,(1/a) = \frac{n\sqrt{\pi}}{W_s \sigma\sqrt{2}}\, e^{-\frac{W_s^2}{2\sigma^2}}\, e^{-\frac{2/\rho - 1/a}{2\sigma^2}} \sqrt{2/\rho - 1/a}\left[2q + 2\left(1 - \frac{2q}{\rho}\right)\frac{1}{2/\rho - 1/a} \right] \sinh\frac{W_s\sqrt{2/\rho - 1/a}}{\sigma^2}. \quad (4{,}16)$$

The unit of velocity in these formulae is 1 A.U. per year. With $\rho = 10^5$ A.U., $\sqrt{2/\rho}$ corresponds with 0·133 km/sec.

We have computed $P(q)$, $Q(q)$ and $f_q(1/a)$ for various combinations of W_s and σ, as shown in Tables 2, 3 and 4. The velocities are expressed in km/sec. The most probable velocities of interstellar comets are of the order of 10 km/sec. In the following such velocities will be referred to as normal velocities.

TABLE 2

The number of comets $Q(q)$, passing per year through a perihelion between $q - 1/2$ and $q + 1/2$ A.U. if the density of "distant" comets outside the sphere of action is assumed to be one per cubic A.U. W_s and σ are expressed in km/sec.

q	1 A.U.			2 A.U.			3 A.U.			5 A.U.		
σ \ W_s	1·3	6·7	13·3	1·3	6·7	13·3	1·3	6·7	13·3	1·3	6·7	13·3
·94	119	29	15	119	31	17	120	32	18	121	35	21
4·7	31	26	16	32	29	18	33	32	19	35	37	23
9·4	21	20	17	28	26	22	34	32	28	43	36	41

TABLE 3

The function $P(q) = \int_0^q Q(q)\, dq$.

q	2 A.U.					5 A.U.			10 A.U.		
σ \ W_s	·27	·53	1·3	6·7	13·3	1·3	6·7	13·3	1·3	6·7	13·3
·19	1184	700	280								
·49			238	59	31	596	149	88	1201	316	211
4·7			162	54	33	165	140	91	327	325	229
9·4			46	44	37	148	133	103	432	386	294

TABLE 4
The function $f_q(1/a)$.

σ	·94			9·4			·19		
$1/a$ \ W_s	1·3	6·7	13·3	1·3	6·7	13·3	·27	·53	1·3
-1	0.10^3	0	0	·5	·4	·2	0.10^5	0.10^4	0
-10^{-1}	0 „	74	25.10^{-2}	48	37	18	0 „	0 „	0
-10^{-2}	2 „	18	12.10^{-24}	78	62	29	0 „	0 „	0
-10^{-3}	24 „	11.10^{-5}	22.10^{-35}	78	62	29	0 „	4 „	89.10^2
-10^{-4}	28 „	23.10^{-6}	38.10^{-39}	78	62	29	10 „	28 „	13.10^{-2}
-10^{-5}	28 „	13 „	69.10^{-40}	78	62	29	16 „	15 „	18.10^{-4}
-10^{-6}	28 „	13 „	53 „	78	62	29	16 „	13 „	81.10^{-6}
0	28 „	13 „	53 „	78	62	29	17 „	13 „	79 „
$+10^{-6}$	28 „	12 „	52 „	78	62	29	17 „	13 „	73 „
$+10^{-5}$	28 „	12 „	40 „	78	62	29	17 „	11 „	29 „
2.10^{-5}	0 „	0 „	0 „	0	0	0	0 „	0 „	0 „

We want to study the way in which a diffusion of comets into the solar system by small perturbations out of the general field has taken place. From Table 4 we see that the function $f_q(1/a)$ is constant from $1/a \sim -10^{-2}$ till beyond the parabolic boundary $1/a = 0$, except for $\sigma = {}$·94, $W_s = 6$·7 and 13·3, respectively. As soon as the function $\nu(1/a, t)$, which for the captured comets represents the same as $f(1/a)$ for the interstellar ones, has become constant for $1/a$ from 0 to about 10^{-3}, there is no longer much difference with the situation where the comets are captured from a constant parabolic field supplied from the outside, as described in the preceding chapter. We are therefore able to describe by means of formula (3,28) the course of the process of diffusion of the comets into the solar system from an early stage, soon after the beginning of the diffusion. What has happened comes practically to a shift of the boundary of the described process of diffusion from $1/a = 0$ to $1/a = 10^{-3}$. The perturbation in $1/a$ being of the order of 10^{-3}, the small number of comets already approaching the sun in elliptic orbits without perturbations will be spread, and will not affect the process. Thus, from an early stage, $\nu(0, t)$ will be equal to $f_q(0)$, and we must expect a constant distribution of $1/a$ after a relatively short period of time.

It is evident from the observations that the distribution of the comets over $1/a$ is entirely different from the expected distribution (see Table 5). This may be due to the following four causes:

a) A great number of comets with large $1/a$ has been removed as a consequence of strong perturbations by the larger planets.

b) Comets with large values of $1/a$ disintegrate more rapidly.

c) The process of diffusion is still in its initial stage.

d) The diffusion process does not influence the long-period comets, as they move in stable orbits.

TABLE 5
Observed numbers of comets from 1850—1936
in equal intervals of $1/a$.

$1/a$	Period	Number	$1/a$	Period	Number
					1
0.10^{-3}	∞		14.10^{-3}	598	
		177			1½
2	11180		16	501	
		10			1½
4	3953		18	419	
		8			4
6	2158		20	353	
		7			2
8	1398		22	302	
		2½			0
10	1000		24	272	
		6½			
12	756				

The first cause will be discussed in the next section. A rapid disintegration of comets with large $1/a$ does not seem likely. We have estimated the time elapsed in the course of the transition from a parabolic orbit into an elliptic one with a period of 1000 years to be about 10^6 years, and the total number of perihelion passages during this time to be about 100. If in the course of this time an appreciable fraction of the comets would be disintegrated, we should expect the time of disintegration of short-period comets with periods of about 7 years on the average, to be at most of the order of 500 years. It has been explained in the introduction (p. 449) that the observations do not support such a rapid disintegration. In order to decide whether the process of diffusion is still in its initial stage, much more accurate data for the comets with long periods are required. The observed distribution of the values $1/a$ does not exclude this possibility. We notice, however, that, if the process would have started only recently, the field of distant comets should supply at least twice as many direct as retrograde comets, and this is not in accordance with the observations.

What will be the proportion of hyperbolic and elliptic comets for the various values of W_s and σ? Assuming the distribution in $1/a$ to be constant up to $1/a \sim 2.10^{-3}$ we find by means of Table 4 for these proportions the values given in Table 6. Only for very small values of σ and W_s a majority of elliptic comets can be expected. In that case there would exist a cloud of comets with small internal velocities moving with the sun. It seems doubtful whether it would not be disrupted by neighbouring stars, or by the differential galactic rotation.

TABLE 6
Ratio of the number of hyperbolic comets to the number of elliptical ones.

W_s \ σ	·94	9·4	W_s \ σ	·19
1·3	1	10^3	·27	$^1/_{30}$
6·7	10^6	2.10^2	·53	$^1/_5$
13·3	10^{28}	10^2	1·3	10

One might ask whether the absence of hyperbolic comets can be due to the fact that the comets observed at present have been captured during a past passage of the sun through an interstellar cloud, by the process of diffusion. This process would ultimately have led to equal numbers of captured comets for equal intervals of $1/a$, but this constant frequency would have been realized for the small values of $1/a$ considerably earlier than for the larger ones. The distribution of $1/a$ actually reached when the sun left the cloud will since have changed mainly for the nearly parabolic comets, i.e. for the smallest values of $1/a$, because these are most sensitive to disturbing forces, and may easily "evaporate" into interstellar space. Let us assume that the value of $1/a$ above which the distribution had not yet reached its final form was the same as that which at present marks the beginning of the decrease of the frequency with increasing $1/a$, say, $1/a = 2.10^{-3}$. Even if we assume the density of the cloud to be as high as one comet of average mass 10^{17} grams [1]) per 2 cubic A.U., corresponding to 0·2 solar masses per cubic parsec, the present number of perihelion passages per year of long-period and parabolic comets is much higher than can be explained on the basis of normal velocities in the cloud. This follows from the estimated numbers of perihelion passages for comets captured from the cloud (see Table 7). The computations are based on the data of Tables 3 and 4. Moreover the number of parabolic comets at the time when the sun left the cloud must, for the reason just mentioned, have been still higher than the present number.

TABLE 7
Number of comets captured per year from an interstellar cloud of comets.

W_s \ σ	·94	9·4	W_s \ σ	·19
1·3	238	5.10^{-2}	·27	3.10^4
6·7	6.10^{-5}	2.10^{-1}	·53	4.10^3
13·3	4.10^{-27}	4.10^{-1}	1·3	28

Perhaps the described process of diffusion is incorrect. We might suppose that the observed comets move in stable orbits. This is a rather improbable assumption however, because the majority of the observed orbits are close to the parabolic limit, and it can hardly be assumed that the perturbations by the planets during the successive passages through perihelion are dependent on each other in such a way as to make these orbits stable. But even if this were possible we could not explain the actual number of elliptic comets as having been captured from interstellar space. From the fact that no truly hyperbolic comet has ever been observed we may conclude that probably less than one hyperbolic comet passes through the observable region per century. Supposing the average perturbation in $1/a$ to be about 10^{-3}, and assuming a normal distribution of the velocities of the distant comets, we find from Table 4 that we would get less than 10^{-2} parabolic or elliptic comets per century, or 10^5 in 10^9 years. For a equal to 10^3 A.U. the period is about 30.000 years, and hence the number of perihelion passages of these 10^5 comets would be less than 3 per year. The actual number, however, would be very much smaller, because only a quite small fraction of the approaching hyperbolic comets could possibly be captured into stable orbits, and part of these may get lost again. Perhaps the number of hyperbolic comets may have been larger in the past than is observed at present, due to a greater density of the interstellar field.

We shall now examine whether the comets may have belonged to the solar system since its origin, and have been influenced solely by forces within this system. In that case those comets which occasionally enter into the planetary system should have evaporated from the solar system after 10^6 or 10^7 years by the process of diffusion. For the remaining ones a decrease of $\nu(1/a, t)$ is to be expected near the parabolic limit. There will remain, however, a supply of

[1]) The nucleus of Halley's comet is estimated by VORONTSOV-VELYAMINOV to be about 10^{19} gm (*Ap.J.* **104**, 232, 1946). From the total mass of the Perseids and the Leonids the masses of the nuclei of the comets 1861 I and 1866 I are estimated to have been 10^{18} and 10^{16} gm respectively. BALDET showed (*C.R.* **185**, 39, 1927) from the brightness of the nucleus that the diameter of comet Pons-Winnecke can be at most 400 m. This would give a mass of only 2×10^{14} gm.

those comets for which the process of diffusion takes more than 10^9 years, corresponding with perturbations in $1/a$ of the order of 10^{-5}. The nearest boundary of the region of these comets should lie somewhat further than Saturn's orbit. We wonder whether, perhaps, the field of visible comets which is thinning out as a consequence of the diffusion, can be replenished from this supply. This would require a strong shortening of perihelion distances for the latter comets which, however, would be accompanied by a change of the values of $1/a$ of the same order as the change in $1/q$. These comets then must take part in the process of diffusion, and will have disappeared long before an appreciable change in q can have taken place, and therefore also long before they enter into the zone of visibility.

We thus find that all the suppositions worked out, except *possibly* that of the stable orbits, lead to a negative answer to the question how to explain the observed distribution of $1/a$. We can say the same about the hypothesis that the comets would have originated by the breaking up of a planet. For, as the comets show no longer any sign of passing near one point, they would have had to undergo considerable perturbations since their birth, but these perturbations would at the same time have expelled practically all comets coming within reach of the major planets.

Although the marked concentration of orbits near the parabolic limit appears at first sight to indicate that the comets come from interstellar space, there seem to be unsurmountable objections to such an origin. Another difficulty, encountered by all theories, is the equality of the observed numbers of direct and retrograde comets; the predicted number of direct comets is always smaller than the number of retrograde ones in the three possibilities last mentioned, because direct comets are more liable to perturbations by the planets than retrograde ones. Only in the case of a cloud moving permanently with the sun this effect works in the other direction; here, however, it may have been smoothed out at the time when a steady state has been reached.

Further research along the lines of E. STRÖMGREN's work seems very desirable, especially if attention be paid to the problem which orbits are followed by the comets receding from the sun, in order to decide whether any of the "parabolic" comets are really being lost, as we would expect on the basis of the considerations given in the present article.

5. *The short-period comets.*

Thus far we did not consider the effect of the large perturbations. During a single encounter with a major planet, a large perturbation may change a parabolic orbit into an elliptical one with short period, while, of course, the reverse may also happen. H. A. NEWTON [1]) has computed the probability of such encounters, assuming a field of parabolic comets in which all directions of the velocities are equally probable. Of those parabolic comets which approach the sun within distances smaller than that of a certain planet with orbital radius r, single passages through the planetary system will bring the fraction

$$\frac{1}{4}\frac{m^2}{r^2}\int\left\{4a^2-\left(\frac{a-r-a\,s^2}{s}\right)^2\right\}ds \qquad (5,1)$$

into elliptic orbits with semi-major axes smaller than a. In this formula m is the mass of the planet expressed in the sun's mass as a unit, and s the ratio of the planetocentric velocity of the comet to the heliocentric velocity of the planet [2]). For $a > r$ the integral can be approximated with sufficient accuracy by the expression $\frac{4}{3}\frac{m^2\,a^2}{r^2}$ [3]). For smaller values of a the coefficient should be taken smaller than $^4/_3$. For Jupiter, the fraction of the parabolic orbits changed into elliptical ones with periods smaller than that of the planet amounts to about 10^{-6} per perihelion passage. The number of comets with periods in excess of 350 years, and discovered between the years 1850 and 1930, is 203, or $2\frac{1}{2}$ per year. Since in a field of parabolic comets, like that adopted by NEWTON, the number of comets with perihelion distances less than q is proportional to q [4]), and since the radius of the zone of visibility is $^2/_5$ of Jupiter's orbital radius, we may expect $2{\cdot}5 \times 2{\cdot}5 \sim 6$ comets per year to come within the distance of Jupiter from the sun. Hence the number of comets captured into the Jupiter group will be 6×10^{-6} per year. The number of newly discovered members of the Jupiter group between the years 1850 and 1930 is 37, corresponding to ·46 per year. This does not agree with the estimate just made. Perhaps, the observed members of the Jupiter group have been captured a long time ago; the time required for these captures would then be at least about 10^7 years, and the age of these comets would also have to be of this order. As was already remarked in the introduction, VSESSVIATSKY believes that a strong decrease of the brightness of the comets of the Jupiter group can be shown to exist, and that the decrease is

[1]) *Mem. Nat. Ac. Washington*, **6**, 19, 1893.

[2]) Integrated over all values of $s > \sqrt{2} - 1$ which make the integrand > 0.

[3]) Cf. H. N. RUSSELL, *l.c.* page 50.

[4]) If we denote by ϑ the angle between the velocity vector of the comet and the radius vector, and by V its velocity, the number of comets entering a sphere with radius r and with a value of ϑ between ϑ and $\vartheta + d\vartheta$ is proportional to $dP \sim V\cos\vartheta\sin\vartheta\,d\vartheta$. On the other hand, in a parabolic orbit $q = r\sin^2\vartheta$, so that $dq = 2r\sin\vartheta\cos\vartheta\,d\vartheta$. Hence $dP \sim dq$.

correlated to the minimum distance to which the comets approach Jupiter's orbit (cf. Table 8, where d is this minimum distance and ΔH the decrease of brightness per 50 years). From this VSESSVIATSKY infers that the mean lifetime for members of the Jupiter group is 500 years, which would exclude the capture theory. We have pointed out in Section 1 that it is unlikely that the strong decrease in brightness supposed by this author is real. The short lifetime found led VSESSVIATSKY to look for another explanation of the short-period comets, and to return to the old eruption hypothesis of LAGRANGE.

TABLE 8

Decrease in brightness of comets per 50 years according to VSESSVIATSKY.

comet	d	ΔH	comet	d	ΔH
	A.U.	m			m
Encke	·92	·5	Kopff	·04	5·0
Tempel 2	·63	1·1	Finlay	·01	4·4
Brorsen	·14	2 – 3	Biela	·42	·5
Tempel-Swift	·58	1·8	Wolf 1	·05	3·7
Pons-Winnecke	·13	·9	Holmes	·36	?
de Vico-Swift	·22	2·0	Borelly	·46	3·0
Perrine	·09	4 – 9	Brooks	·02	4·5
Giacobini-Zinner	·20	·0	Faye	·10	3·4
d'Arrest	·11	·4	Schaumasse	·34	·5 – 1

The eruption theory [1]).

We shall now describe this theory, which in certain respects bears resemblance to the theory of captures by large perturbations.

The planet is supposed to move in a circular orbit with radius a_j. We introduce a system of co-ordinate axes with the sun in the origin, the x-axis co-inciding with the radius vector through the planet and being counted positive in the direction of this radius vector, the positive y-axis being in the direction of motion of the planet.

For the determination of the heliocentric orbit of the erupted comet we only need to know the velocity at the boundary of the planet's sphere of action [1]). For convenience we shall neglect the radius of this sphere. The jovicentric velocity at the surface of the sphere of action will be denoted by s, with components s_x, s_y and s_z, and the co-ordinates of the comet by x, y, z, respectively. Then we have at the time $t = 0$, that is, when the comet is leaving the sphere of action of Jupiter,

$$\begin{aligned} x_0 &= a_j, & y_0 &= 0, & z_0 &= 0, \\ \dot{x}_0 &= s_x, & \dot{y}_0 &= s_y + V_1, & \dot{z}_0 &= s_z, \end{aligned} \qquad (5,2)$$

in which $V_1 = 1/\sqrt{a_j}$ is the heliocentric velocity of Jupiter.

We introduce spherical co-ordinates: the angle α in the orbital plane of the planet is counted from the prolonged radius vector; it is 90° in the direction of motion of Jupiter. The angular distance from this orbital plane is denoted by β. We have

$$\begin{aligned} s_x &= s \cos\alpha \cos\beta, \\ s_y &= s \sin\alpha \cos\beta, \\ s_z &= s \sin\beta. \end{aligned} \qquad (5,3)$$

s and β can be found as functions of χ, the latitude of the point of eruption on Jupiter, and n, the velocity of eruption at the surface of Jupiter, which is supposed to have the direction of the normal to Jupiter's surface. We get

$$s^2 = n^2 + {\cdot}694 \cos^2\chi - 3{\cdot}996, \qquad (5,4)$$

$$\sin\beta = \frac{-{\cdot}367\cos^2\chi + 1 + {\cdot}183\, n \cos^2\chi \sqrt{n^2 + {\cdot}694\cos^2\chi - 3{\cdot}996}}{1 + {\cdot}174\,(n^2 + {\cdot}694\cos^2\chi - 3{\cdot}996)\cos^2\chi} \sin\chi. \qquad (5,5)$$

For the eruptions all values of α are assumed to be equally probable. Denoting by p the parameter $a(1 - e^2)$, and by i and Ω the inclination and the longitude of the ascending node of the comet's heliocentric orbit, we have

$$\begin{aligned} \sqrt{p} \sin i \sin\Omega &= y_0\dot{z}_0 - z_0\dot{y}_0 = 0, \\ \sqrt{p} \sin i \cos\Omega &= x_0\dot{z}_0 - z_0\dot{x}_0 = a_j s_z = a_j s \sin\beta, \\ \sqrt{p} \cos i \quad &= x_0\dot{y}_0 - y_0\dot{x}_0 = \\ a_j (V_1 + s_y) &= a_j (V_1 + s \sin\alpha \cos\beta). \end{aligned} \qquad (5,6)$$

From the first two equations of (5,6) we infer that $\sin\Omega = 0$, hence $\Omega = 0°$ or $180°$. The heliocentric velocity of the comet at time $t = 0$ is

$$V^2 = V_1^2 + s^2 + 2\, V_1 s \cos\beta \sin\alpha. \qquad (5,7)$$

The semi-major axis of the comet's orbit, the inclination and the parameter are determined by

$$1/a = 1/a_j - s^2 - 2\, s\, V_1 \cos\beta \sin\alpha, \qquad (5,8)$$

$$\begin{aligned} \operatorname{tg} i &= \frac{|s_z|}{V_1 + s_y} = \frac{s\,|\sin\beta|}{V_1 + s\sin\alpha\cos\beta} = \\ &= |\operatorname{tg}\beta|\, \frac{1}{V_1/s \cos\beta + \sin\alpha}, \end{aligned} \qquad (5,9)$$

$$p = a_j^2 \{s^2 \sin^2\beta + (V_1 + s\sin\alpha\cos\beta)^2\}. \qquad (5,10)$$

We shall now try to define, for various α and β, limiting values of s, between which the comets will enter the zone of visibility. As a criterion of visibility we adopt $p \leqq 4$. This leads to simple computations and, as we consider only elliptic and parabolic comets,

[1]) Cf. TISSERAND, *B.A.* **7**, 452, 1890; VSESSVIATSKY, *R.A.J.* **10**, 18–41, 1934.

[1]) The sphere of action of a planet is generally defined as the locus of points for which the ratio between the sun's central force and the planet's perturbating force exerted on a body with heliocentric motion equals the ratio between the planet's central force and the sun's perturbating force exerted on a body with planetocentric motion. For Jupiter the radius of this sphere is ·322 A.U.

for which $e \leqq 1$, it is to be preferred to the criterion $q \leqq 2$ A.U. It involves an extension of the radius of the zone of visibility of elliptic comets, while for parabolic ones this remains exactly 2 A.U. Table 9 shows the limiting values for $\beta = 0°, 30°, 60°$ and $90°$ and for α between 180° and 360°. For $0° < \alpha < 180°$ not a single comet enters the zone of visibility. The values of s in Table 9 are expressed in the velocity V_1 as a unit. We next determined for various values of α and β those limiting values of s for which the resulting orbits have semi-major axes smaller than 5, 10, 100 and ∞ A.U. These values are given in Table 10.

TABLE 9

Limits of s for which comets will enter the zone of visibility (unit of s is the velocity of the planet).

α	180° 360°	200 340	220 320	240 300	260 280
$\beta = 0°$ s	∞ ∞	5·41 ·27	2·88 ·14	2·13 ·11	1·88 ·09
$\beta = 30°$ s		1·14 ·56	1·73 ·20	1·65 ·14	1·57 ·12
$\beta = 60°$ s					·54 ·39
$\beta = 90°$					

TABLE 10

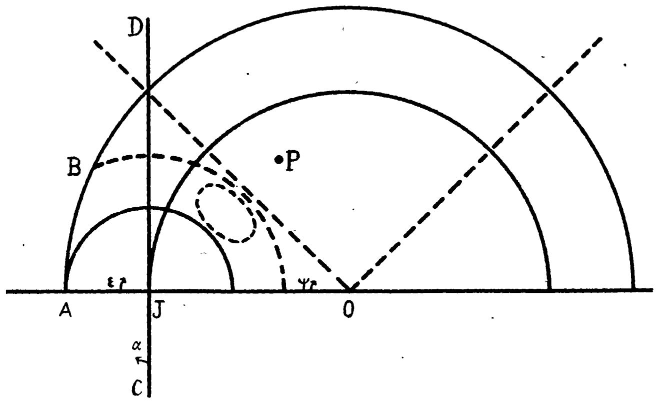
Values of s corresponding to orbits with semi-major axes a A.U.

α	0° 180°	20 160	40 140	60 120	80 100	200 340	220 320	240 300	260 280
$\beta = 0°$									
$a =$ 5						·67	1·30	1·73	1·96
10	·70	·43	·30	·24	·21	1·13	1·62	2·03	2·24
100	·98	·68	·52	·43	·39	1·39	1·84	2·22	2·39
∞	1·00	·71	·54	·45	·42	1·41	1·86	2·24	2·41
$\beta = 30°$									
$a =$ 5						·54	1·11	1·52	1·73
10	·70	·45	·33	·27	·24	1·06	1·47	1·81	2·09
100	·98	·72	·56	·47	·43	1·33	1·70	2·02	2·19
∞	1·00	·74	·58	·49	·45	1·35	1·73	2·04	2·21

α	0° 180°	20 160	40 140	60 120	80 100	200 340	220 320	240 300	260 280
$\beta = 60°$									
$a =$ 5							·62	·85	·98
10	·70	·54	·44	·38	·36		1·10	1·27	1·37
100	·98	·81	·70	·63	·59		1·36	1·52	1·61
∞	1·00	·84	·72	·65	·62		1·38	1·54	1·63
$\beta = 90°$									
$a =$ 5	0								
10	·70								
100	·98								
∞	1·00								

The vector diagram Figure 8 may serve to obtain some insight into the aspects of the eruption theory. OJ represents the velocity of Jupiter with respect to the sun. For comets with periods equal to that of Jupiter, the vector points will lie on the circle (O, OJ), and for parabolic comets on the circle (O, OA), where $OA = OJ.\sqrt{2}$. If OP is the velocity of a comet with respect to the sun, then JP is the jovicentric

FIGURE 8



velocity, i.e. the "eruption velocity", s, at the boundary of Jupiter's sphere of action. The angle between the heliocentric velocities of Jupiter and of the comet will be denoted by ψ. If we imagine the figure to be rotated about OA, we can also draw the angles α and β; α being reckoned from JC in clockwise direction. Evidently, for $0° < \alpha < 180°$, periods shorter than Jupiter's period cannot be formed. The diagram can also serve to find the values given in Table 10. For small eruption velocities, for instance equal to one half of Jupiter's orbital velocity, ψ is below 30°, and the inclination cannot exceed this value. All orbits will be direct in that case, and have small inclinations. We notice in particular that for short-period comets with orbital elements as observed, the values of s must be small.

With the aid of the relation

$$1/a + 2/a_j^{3/2} \sqrt{p} \cos i = 3/a_j - s^2, \qquad (5,11)$$

which can be derived from (5,6), (5,8) and (5,10), we have calculated the values of s for all known short-period comets. Table 11 gives the distribution of these values for the comets with period smaller than 20 years. Values of Jacobi's constant

$$1/a + 2/a_j^{3/2} \sqrt{p} \cos i$$

were taken from T. VODOPIANOVA[1]). The values of s so obtained will be independent of Jupiter's perturbing influence, because Jacobi's constant is an integral for the restricted problem of the sun, Jupiter

[1]) *Publ. Sternberg* 9_2, 388, 1940.

and the comet, where Jupiter is supposed to move in a circular orbit. Secular variations of this constant may be caused only by the eccentricity of Jupiter's orbit, by perturbations due to the other planets, or by a resisting medium. These secular variations will

TABLE 11
Distribution of velocities s expressed in Jupiter's orbital velocity as unit.

velocity	number
·00 — ·12	0
·12 — ·24	3
·24 — ·36	9
·36 — ·48	11
·48 — ·61	13
·61 — ·73	9
·73 — ·80	3
·80 — ·97	3

be of little importance for short-period comets, except in cases like Encke's comet, where the period is shortened by a resisting medium. The almost complete absence of comets with small values of s in Table 11 may be due partly to observational selection. For these cases represent nearly circular orbits with the same time of revolution as Jupiter, and such comets will be invisible from the earth. However, such circular orbits are rather unstable, and may be expected to change within a short time, of the order of a few revolutions, by perturbations by Jupiter, after which they can no more cross Jupiter's orbit. Just like the asteroids, which avoid the region within 1 A.U. from Jupiter's orbit (with the exception of the Trojan group and a few others), we may expect these comets also to be removed from this region by strong perturbations, resulting in a shortening or a lengthening of their periods. The eccentricities will also be changed, and therefore part of these comets will become visible from the earth. Of the three comets with s between ·12 and ·24 in Table 11, comet Schwassmann-Wachmann 1 may well have originated from such a circular orbit. But its large perihelion distance (6 A.U.) would render a long lifetime for this comet possible, and hence considerable changes in Jacobi's constant may have occurred. For the other two comets, Tempel 1 and Tempel 2, it does not seem probable that they have originated from a circular orbit, and other causes may have led to the small values of s. We conclude that, although a selection in the material of observation must be present, the absence of small values of s is a real feature. We notice that there is a maximum frequency for values of s about ·5, the occurrence of which must be considered as the most characteristic property of the short-period comets. It must be connected with other peculiarities in the distribution of orbital elements; for instance, the observed statistical relation between the parameter p and the inclination i, shown in Table 12 [1]), is readily explained by the relation (5,11). Any theory of the origin of short-period comets should give an explanation of this peculiar distribution of the values of s.

TABLE 12
Statistical relation between i and p.

i	p	n
0° — 8°	2·01	15
8° — 16°	2·17	20
> 16°	2·26	14

It is not clear how it would fit in the eruption theory. We would rather expect a maximum of the distribution at $s = 0$. CORLIN [2]) shows that the eruption velocities at Jupiter's surface would have to be of the order of 600 km/sec, about 10 times the velocity of escape, because of the resistance by Jupiter's atmosphere. However, CORLIN has not taken into account that the comet's mass is probably only a fraction of the total originally erupted, nor has he considered that the behaviour of gases under the extreme pressure of 10^9 atmospheres may be much different from that assumed by him. This latter point has been investigated by WILDT [3]) and by PECK [4]), who find that at a pressure of about 1400 atmospheres the gases of the atmosphere would reach the density of the solid state, and beyond that are almost incompressible. According to SIMON [5]) H_2 and He become liquid at pressures of 32500 and 65000 atmospheres, respectively, if the temperature is 200°. Therefore, 10^4 atmospheres should be a maximum for the pressure at the bottom of Jupiter's atmosphere. CORLIN's theory then leads to an initial velocity equal to ten times the velocity of escape if the erupted body has a radius of 5 km, and about equal to the velocity of escape for a body with a radius of 50 km.

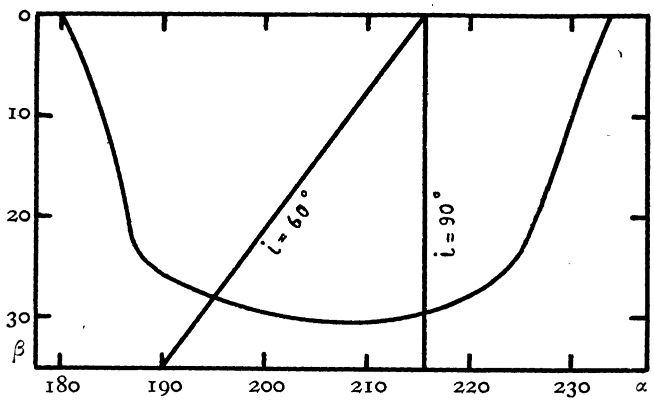
Parabolic comets can be formed with small velocity of eruption only if $0° < \alpha < 180°$, but will then remain outside the zone of visibility. Long-period and parabolic comets entering into this zone must have been formed with high velocities in the intervals of α and β indicated in Figure 9. In this diagram the curves marked $i = 60°$ and $i = 90°$ have been derived from the relation

$$\operatorname{tg} i = |\operatorname{tg} \beta| \frac{1}{1/s \cos \beta + \sin \alpha}.$$

[1]) From VSESSVIATSKY, *R.A.J.* **11**, 443, 1934.
[2]) *Zs.f. Ap.* **15**, 239, 1938; *Festschrift Östen Bergstrand*, p. 109, 1938.
[3]) *Veröff. Gött.* **40**, 1934.
[4]) *M.N.* **97**, 574, 1937.
[5]) *Zs.f. phys. Chemie*, B 6, 331, 1939.

For $s \cos \beta$ a fixed value 1·71 was adopted. For $\beta < 10°$ the curves $i = 60°$ and $i = 90°$ will be shifted slightly to the right, and for large values of β to the left. We conclude from Figure 9 that the observed distribution of inclinations is not in accordance with that expected on the basis of the eruption theory. For, from Figure 9 we would expect that the number of comets with inclinations between 0° and 60° is at least three

FIGURE 9
Limits of α and β for which parabolic comets enter the zone of visibility.



times larger than the number with inclinations between 60° and 90°, and, moreover, that the number of direct comets is three times larger than the number of retrograde ones. If we take into account the fact that the velocity of eruption must increase with increasing α, both arguments are strengthened.

Finally, the orbits of comets having their origin on Jupiter will first cross Jupiter's orbit, and may later on be more or less removed from it as a consequence of perturbations. But even then we should still expect them to approach it at short distances. To investigate this we have determined the points of intersection of the orbits of the long-period and parabolic comets, observed after 1800, with the sphere which has the sun as centre and a radius equal to the orbital radius

TABLE 13

b	O	C
0°– 10°	93	99
10 – 20	104	95
20 – 30	96	89
30 – 40	91	81
40 – 50	65	70
50 – 60	63	57
60 – 70	32	42
70 – 80	24	25
80 – 90	18	8

of Jupiter. In Table 13 is given the distribution of the angular distances b of these points from Jupiter's orbit. For comparison the last column shows the numbers if the points of intersection were equally distributed over the sphere. We see that there is no tendency for the comets to approach Jupiter's orbit at short distances.

The theory of capture.

On account of the arguments mentioned it seems impossible to us to explain the phenomena with the eruption theory. For that reason we shall once more consider whether the capture theory can explain the observed numbers and the distribution of short-period comets. To start with, we mention a few points which require a closer investigation:

α) The decrease in brightness pointed out by VSESSVIATSKY may have been considerably exaggerated by systematic errors in the absolute magnitudes.

β) It is possible that the short-period comets have not been captured directly from parabolic or long-period orbits, but that the periods have been shortened by successive small perturbations.

γ) The number of parabolic and long-period comets actually present may be much larger than the observed number, because the parabolic comets with smaller absolute brightness escape observation, while the elliptic comets will be discovered to much fainter limits.

α) In the introduction we have already remarked that VSESSVIATSKY pays too little attention to a careful reduction of the estimates of brightness, which is necessary in order to avoid the systematic errors resulting from differences in aperture of the instruments used and from personal errors. In an instrument with small aperture the comet is estimated systematically too bright. Consequently, the absolute magnitudes derived from the older estimates are systematically too high. Table 14, in which we have compared the

TABLE 14
Comparison of the magnitudes of comet Wolf 1 given by BOBROVNIKOFF (B) and by VSESSVIATSKY (V)

Time	B	V
	m	m
1884	10·9	7·6
1891	10·9	7·6
1898	10·7	8·0
1912	10·3	9·1
1918	11·0	9·9
1925	11·2	10·6

values given for comet Wolf 1 by VSESSVIATSKY with the carefully reduced values of BOBROVNIKOFF, gives a clear impression of the importance of this effect. In the absolute magnitudes of the other short-period comets in VSESSVIATSKY's tables[1]) we often notice sud-

[1]) *M.N.* **90**, 706, 1927.

den decreases of brightness around 1900 and 1920, which may be ascribed to the fact that at these times larger instruments began to be used, and the photographic photometry was started. For the brighter comets this systematic error will have little influence, as even now these are chiefly observed with smaller instruments. A real decrease of brightness remains, however, indicated for several comets. As BOBROVNIKOFF has pointed out[1]) this decrease should not be ascribed to an actual disintegration, but to a decrease of the power of the comet to develop gases. Therefore it is preferable to speak of time of visibility, rather than of time of existence. Although practically all short-period comets whose orbits intersect the earth's orbit are accompanied by a stream of meteors, there is no question for most of them of a rapid decrease in brightness. This constancy was evident, for instance, for Biela's comet before its division, or for Halley's comet. A real disintegration happens rather abruptly, as e.g. with Biela's comet, which split up in 1842 during a close approach to Jupiter, comet 1916 I (Taylor), which was divided while being in perihelion, and the parabolic comet Ensor, which was completely dissolved when it passed near the sun. On the other hand, there is no question of a *general* rapid disintegration such as assumed by VSESSVIATSKY.

It may be possible that comets which have become invisible without being completely disintegrated vary in brightness for some reason, and thus give rise to the appearance of short-period comets observed during one period only.

β) Since 1767 eleven short-period comets have been discovered shortly after a close approach to Jupiter (see Table 15), during which the orbits of these comets had been considerably changed. For six of these comets the perturbations have been accurately determined. It was found that their orbits had originally larger perihelion distances and longer periods (see Table 16). For three of these the period has, after some decades, during another close approach to Jupiter, been increased again. If we want to study thoroughly the adventures of the comets of the Jupiter group we must extend this group to include also all invisible comets captured by Jupiter and having a semi-major axis smaller than 50 A.U. (or a period less than 350 years), i.e. to that value of a for which the small perturbations can no longer be supposed to be independent of each other (see p. 455).

γ) BOURGEOIS and COX[1]) have estimated the number of comets which, though they enter the zone of visibility, yet remain undiscovered. For this purpose they compute for different positions of the orbital plane and different positions in the orbit relative to the earth the probability of discovery for comets of various absolute magnitudes. For the comets entering

TABLE 15
Comets discovered shortly after a close passage by Jupiter.

comet	near Jupiter	discovered	i	ω	Q	d	ψ	s	P
			°	°	A.U.	A.U.	°		y
Lexell	1767	1770	1·6	224	5·6	·00	35	·62	5·5
Brorsen	1842	1846	29·4	15	5·6	·14	45	·73	5·5
Wolf 1	1875	1884	27·3	161	5·8	·05	36	·55	8·3
Brooks 2	1886	1889	5·5	195	5·4	·02	16	·33	6·8
Faye	1841	1843	10·6	200	5·9	·10	28	·49	7·4
Finlay	1862	1886	3·4	321	6·2	·01	39	·62	6·9
Perrine	1888	1896	15·7	167	5·8	·09	34	·58	6·6
Swift	1886	1895	3·0	297	5·1	·08	3	·57	5·9
Whipple	1922	1933	10·2	190	5·2	·26	10	·24	7·5
Schwassmann-Wachmann 2	1921	1929	3·7	358	4·8	·30	4	..	6·5
Comas Sola	1912	1936	13·7	39	6·6			·57	8·5

TABLE 16
Comets with well-determined changes of orbits[2]).

comet	period	a	q	P
		A.U.	A.U.	y
Lexell	before 1767	5·06	2·96	11·4
	1770	3·15	·67	5·6
	after 1779	6·37	3·33	16·2
Brooks 2	before 1889	9·0	5·44	27·0
	1889—1921	3·59	1·95	6·8
	after 1921	3·64	1·86	6·95
Wolf 1	before 1875	4·18	2·54	8·54
	1875—1922	3·59	1·59	6·80
	after 1922	4·07	2·36	8·20
Comas Sola	before 1912	4·46	2·15	9·43
	after 1912	4·17	1·77	8·52
Schwassmann-Wachmann 2	before 1921	4·43	3·55	9·30
	after 1921	3·46	2·09	6·42
Whipple	before 1922	4·74	3·90	10·30
	after 1922	3·83	2·50	7·50

[1]) *Pop. Astr.* **56**, 130, 1947.
[2]) Cf. WATSON, „Between the Planets", page 57, Philadelphia, 1941.

[1]) *B.A.* **9**, 349, 1934.

the zone of visibility the real distribution of the absolute magnitudes is derived from the observed distribution, which has been taken from VSESSVIATSKY[1]). In the observed distribution of magnitudes given by VSESSVIATSKY, which covers all observations from about 1600, the effect mentioned under α) will have caused a displacement towards smaller magnitudes. The chief result probably is that the number for the 7th magnitude is too large and that for the 9th magnitude too small. For the comparison we have assumed that the comets of absolute magnitude brighter than 4 have been completely discovered. Table 17 gives

TABLE 17
Observed and true numbers of comets of different absolute magnitudes.

m	0	2	4	6	8	10	12
O	6	22	71	84	34	14	
C	4	24	160	480	560	560	

then the calculated numbers of fainter comets, according to Cox and BOURGEOIS.

As explained under α) the disappearances from the Jupiter family are not due to a gradual rapid disintegration. As no definite rule can be given, we will consider the disappearances in somewhat greater detail. We shall start from the four following possibilities:

a) The orbit has been changed by a perturbation. If the modified period is smaller than 350 years, the comet still belongs to the family.

b) The comet has disintegrated, and has been lost to the family.

c) The comet existed before its discovery, as a faint, invisible comet which had brightened temporarily, but has faded again. In this case nothing changed in the family.

d) The comet's power of developing gases is being exhausted, and hence it will become invisible. This will be a slow process. These cases may be easily distinguished.

[1]) *R.A.J.* **10**, 327, 1933.

As there are evidently but little data about the disappearances of comets of the Jupiter group that have periods longer than 12 years, we will investigate only the disappearances for the group with periods shorter than 12 years. We divide these comets into two groups. Group I: comets which have been observed during more than one period; group II: comets which have been observed during one period only. Up to 1940, 26 comets have been observed in group I, 30 in group II, of which latter, one, viz. 1889 VI (Swift), however, is very dubious.

If we want to consider the disappearances from the two groups we should take account only of comets disappeared before 1915, because a comet which has disappeared after that time may still be rediscovered. Then 6 comets of group I and 20 of group II have disappeared.

The best criterion for case *a*) is the shortest distance at which the comet can approach Jupiter's orbit. We shall assume case *a*) to be probable if this distance is smaller than ·15 A.U. According to RUSSELL's Table 1, this occurs for 10 comets of group II. We call this group II[a]. For one of these comets (1770 I, Lexell) it has been shown that the orbit has actually been changed by a large perturbation into one with longer period and greater perihelion distance (see Table 16). Another comet of group II[a] (1916 I, Taylor) has been broken up and disintegrated in 1916 while being in perihelion.

Case *b*) can only be said to have occurred when a division or rapid dissolution has actually been observed. As mentioned above, no general rule can be given for this process. In case *c*) the comet may be expected to be faint. As the mean absolute magnitude of group II is 10, this possibility is certainly not excluded.

For those comets which pass Jupiter's orbit at a distance larger than ·15 A.U. (group II[b]), cases *b*) and *c*) are most probable, preference being given to *c*).

For the comets of group I which have disappeared

TABLE 18
Comets disappeared after having been observed during more than one period (only comets with periods less than 12 years have been tabulated).

comet	discovered	last observed	number of periods observed	*P*	*Q*	*d*	brightness
				y	A.U.	A.U.	m
Brorsen	1846	1879	5	5·5	5·6	·14	10
de Vico-Swift	1814	1894	3	5·9	5·1	·22	10·5
Tempel 1	1867	1879	3	6·0	4·8	·16	10·4
Biela	1772	1852	6	6·6	6·2	·42	8·1
Holmes	1892	1906	3	6·9	5·1	·36	11·0
Tempel-Swift	1869	1908	4	5·7	5·2	·58	15

(see Table 18), case *a*) may have occurred for Brorsen and Tempel 1. These two comets have been observed during several revolutions; hence case *c*) is not probable here, as a sudden increase of brightness of a short-period comet lasts only a short time (cf. comet Schwassmann-Wachmann 1). Comet Biela is known with certainty to have disintegrated. For Holmes case *b*) probably applies, considering the large, and certainly real, decrease in brightness during the short time of its visibility. For the remaining comets in Table 18 we have to choose between *b*) and *c*). There is no reason for any of the observed disappearances to be ascribed to case *d*).

In order to determine the total number of comets actually being lost we may distinguish the following four possibilities:

b) is considered only when division has actually been observed. In this case only two comets would have been lost since 1770.

The possibility *c*) is admitted only for disappearance from group II^b. On this assumption 5 comets would have been lost since 1770.

Case *c*) is excluded. The disappearances in group II^a are ascribed to *a*) and those in group II^b to *b*). Then the comets lost number 15.

All disappearances are ascribed to *b*), which yields a total loss of 26 comets.

The second supposition, corresponding to a loss of 3 comets per century, seems the most probable to us.

These losses from the Jupiter family would have to be replenished by captures from the general field of parabolic and long-period comets. For the members of this general field we must take into account only the comets brighter than the 10th magnitude, in order to be sure that they will be visible as short-period comets. According to Table 17 the total number of these comets entering the zone of visibility is $4^1/_2$ times larger than the number discovered. As this last number is in the mean $2^1/_2$ per year, we find that about 11 parabolic comets brighter than the 10th magnitude enter the zone of visibility per year. We assume all directions of velocities to be equally probable in the field of parabolic comets. In such a field the number of comets with a perihelion distance smaller than q is proportional to q. As the radius of the zone of visibility is 2 A.U., the number of comets entering per year the sphere with radius equal to that of Jupiter's orbit is $2^1/_2 \times 11 = 28$. According to H. A. Newton the fraction of these comets of which, by a single encounter with Jupiter, the semi-major axis a is changed to one smaller than 50 A.U. equals $^4/_3 \times 10^{-4}$. Hence, each year $^4/_3 \times 10^{-4} \times 28 \sim 37 \times 10^{-4}$ comets will be captured into the Jupiter family. As this fraction is proportional to a^2, only 1% will get directly a period less than 12 years. Of course the opposite process occurs too; it causes losses to the Jupiter family practically only from the group with periods between 12 and 350 years.

With the aid of Figure 8 we shall now first consider more closely the process of the large perturbations itself. Upon entering and leaving the sphere of action of Jupiter before and after the large perturbation the comet has equal jovicentric velocities. As a result of the perturbation the extremity of the vector representing the jovicentric velocity will be displaced along a circle centred at *J*. H. A. Newton[1]) has shown that the perturbation in $1/a$ will be larger than δ only if the comet cuts the plane containing the tangent to Jupiter's orbit and the common perpendicular of the tangents to Jupiter's orbit and the unperturbed orbit of the comet within an ellipse with surface

$$\frac{4\,\pi\, m_1^2\, \delta^{-2}}{s^2 \sin \varepsilon} \left\{ 1 - (\cos \varepsilon - \frac{a_J\, \delta}{2\, s})^2 \right\}.$$

Here ε is the angle between the comet's jovicentric velocity and Jupiter's heliocentric velocity. For $\cos \varepsilon > 0$, or $\varepsilon < 90°$, this surface is larger for $\delta = b$ than for $\delta = -b$, b being a positive number, whereas the opposite holds for $\cos \varepsilon < 0$, or $\varepsilon > 90°$. Hence for $\varepsilon < 90°$ the chance of the period being shortened is greater than that of a lengthening of the period, whereas for $\varepsilon > 90°$ the reverse is true. For the latter comets, with vector points to the right of the line *JC*, the chance of a lengthening of the period increases continuously

Table 19

Relative probabilities for a change of 0·1 in $1/a$ at a perihelion passage of a parabolic comet. The plus sign refers to a change of + 0·1, the minus sign to one of − 0·1. ε is the angle between the comet's jovicentric velocity and Jupiter's orbital velocity at the time of their closest approach.

ε		ε		ε	
32°	+ 8·09	111°	+ 0·37	150°	+ 0·01
	− 0		− 0·51		− 0·20
55	+ 3·52	118	+ 0·27	156	+ 0
	− 0		− 0·32		− 0·17
72	+ 1·91	125	+ 0·19	162	+ 0
	− 1·08		− 0·38		− 0·18
84	+ 1·17	131	+ 0·13	168	+ 0
	− 1·02		− 0·28		− 0·21
94	+ 0·76	138	+ 0·09	174	+ 0
	− 0·83		− 0·24		− 0·32
103	+ 0·53	144	+ 0·05		
	− 0·65		− 0·21		

[1]) *Mem. Nat. Ac. Washington*, **6**, 12, 1893.

with increasing ε, as compared to the chance of shortening. In order to show the proportions we give in Table 19 the values of the surface of the ellipse for $\delta = {}^1/_{10}$ and for values of ε from 0° — 180°. For comets with their vector points on a definite line *JP*, that is, for a definite value of ε, the probability of a certain change is inversely proportional to s^2. If ψ is the angle between the comet's and Jupiter's heliocentric velocities, it is at once obvious from Figure 8 that for the group of comets for which ψ<45° after capture, a strong preference exists for shortening of the periods, for the probability of capture is proportional to a^2. We assume that no comets of this group will be thrown back into the general parabolic field. H. A. NEWTON has computed[1]) the distribution of ψ for the comets having after capture a period less than 12 years (see

TABLE 20

ψ	n	ψ	n
0° — 15°	6	90° — 105°	67
15 — 30	91	105 — 120	53
30 — 45	170	120 — 135	37
45 — 60	152	135 — 150	23
60 — 75	124	150 — 165	11
75 — 90	101	165 — 180	4

Table 20, which is the same as NEWTON's Table III). If we consider all comets captured in the Jupiter family, the distribution will be about the same. Hence, for 40% of the total number of comets captured, or for 0·4 × 37 × 10⁻⁴ ~ 15 × 10⁻⁴ per year, ψ lies between 0° and 45°. These are the only ones which would remain in the Jupiter family.

For the group of comets for which the value of ψ after capture exceeds 45°, the chance of lengthening of the period will be greater than that of shortening; for these comets it increases with decreasing period. Very few of the comets of this group with periods exceeding 12 years will be able to penetrate to orbits with periods less than 12 years. Therefore, this group will suffer hardly any disintegration, and it will be practically in equilibrium with the general parabolic field. So, for all comets captured in this group an equal number returns to the general field. It is now clear that nearly all short-period comets must originate from the group with ψ<45°. For the comets of this group with ε<90°, the number passing perihelion per year increases with decreasing period. If we take into account all properties of the perturbations in encounters between a comet and Jupiter, such as the decreasing preference for a shortening of the period with increasing ε, which changes for values of ε>90° into a preference for lengthening, the decreasing chance of perturbation for increasing *s*, and the circumstance that the comets must originally come from parabolic orbits (say between *A* and *B* in Figure 8, so that the vector points must remain between the circles (*J*, *JA*) and (*J*, *JB*)) we may infer from Figure 8 that there will be a preference for jovicentric velocities equal to about half of Jupiter's velocity. The vector points of the short-period comets must lie for the greater part within the dotted ellipse in Figure 8. Contrary to the eruption theory the theory of capture by large perturbations supplies a simple explanation of the fact that the values of *s* cluster around 0·5, and moreover of the rareness of values of *s*<0·4.

Of the comets with periods longer than 12 years, those for which α and β lie in the intervals shown in Figure 9, will enter the zone of visibility. These will number about one third of all comets for which 45°<ψ<135°, as β must be smaller than 30°. Comets of these periods with other values of ψ will not become visible. As we have seen above, this group must be in equilibrium with the general field. Does this fact enable us to estimate empirically the number of comets captured into the Jupiter family? For this purpose we must know the number of comets thrown out of this group into hyperbolic orbits.

In order to obtain an estimate of this we first try to estimate the chance of large perturbations for the comets already captured in short-period orbits. We use the following alternative assumptions:

1. The comets of group II^a have disappeared through a large perturbation (mechanism *a*);

2. The disappearance of these comets was due to some other cause.

In both cases we include among the comets taking an active part in the process only those which can approach Jupiter's orbit within a short distance (say <·15 A.U.) For the interval from 1770—1935 this number is 24. Under the first assumption 11 comets would have suffered a large perturbation during this time; with the second assumption this number would be 3. The mean period being 6·7 years we have 165/6·7 × 24 = 576 perihelion passages. Hence, in the first case the probability of a large perturbation is ·019, in the second case ·005 per perihelion passage. We may expect the visible comets with periods between 20 and 350 years to belong, with only a few exceptions, to the group of comets for which 45°<ψ <135°. Their number is 19. Part of them cannot approach sufficiently close to Jupiter's orbit on account of perturbations by the other planets. A rough estimate shows this to be about half the total number. We have seen above that about one third of these comets enter the zone of visibility. The total number must therefore be three times larger. Moreover we have to correct for the incompleteness of the observations, the correction factor being $4^1/_2$ (see p. 470).

[1]) *L.c.* page 21

Accordingly, $\frac{1}{2} \times 3 \times 4\frac{1}{2} \times 19 \sim 100$ comets, with periods between 20 and 350 years, passing perihelion in 85 years, approach Jupiter's orbit within a short distance, or 120 per 100 years. For the comets of this group s is about twice as large as for the comets with periods less than 12 years, for which we have just computed the chance of a large perturbation (see Figure 8); the probability is thus 4 times smaller, or ·005 and ·0011, respectively. The first assumption thus leads to one comet in 200 years, the second to one in 700 years. The total number of "Jupiter comets" with periods between 20 and 350 years which are being lost is one in 200, or 700 years, respectively. Let us compare this to the number captured from the general field of parabolic comets. On page 471 this was found to be 15×10^{-4} per year, or one per 650 years. This is in remarkable agreement with the numbers of losses by perturbation which we have just estimated. It does not, however, seem to suffice entirely to replenish the losses from the Jupiter family by disintegration, which we found to be almost 3 per century (cf. page 470).

Nearly all comets which are really captured into the Jupiter family have before capture perihelion distances between 4 and 5·2 A.U. It is possible that the number of perihelion passages per year of the long-period and parabolic comets which have their perihelia within this interval, is larger than for those with perihelia in the zone of visibility. The number of captured comets would then be enlarged in the same ratio. A second possibility is that the comets during their encounter with Jupiter, or in their perihelia, are divided into several parts, and that in this way the number of short-period comets is enlarged. FAYET[1]) has pointed out that several groups of short-period comets must have a common origin. His criterion was that the points of nearest approach of the orbits were in the neighbourhood of the orbit of Jupiter. The real criterion, however, must be that the comets have been at the same time near Jupiter. As long as this has not yet been investigated we must consider this possibility as a mere hypothesis.

In order to get a real and definitive explanation of the origin of the short-period comets one should investigate the history of each member of this group. With modern mechanical equipment this comes within the range of possibilities.

I am indebted to Prof. HERTZSPRUNG, who suggested the problem of this article. It is a great pleasure to me to express my gratitude to Prof. OORT, for his valuable criticism and advice during the last stage of the work. I am very grateful to Prof. OORT, Miss KLUYVER and Dr BLAAUW for much help and advice during the preparation of the manuscript. I wish to express my thanks to Mr SWAAK, who carried out part of the computation and to Mr DE ROOY, who prepared the drawings.

[1]) *B.A.* **28**, 168, 1911.

Principal Notations

$f = k^2$	gravitational constant, the mass of the sun being taken as unit;
m_1	mass of Jupiter (unit sun's mass);
a	semi-major axis of the comet's orbit;
r	radius vector of the comet;
r_1, a_J	radius vector of Jupiter;
Δ	distance between comet and Jupiter;
Δ_m	minimum distance between comet and Jupiter;
e	eccentricity of the comet's orbit;
i	mutual inclination between orbital planes of comet and Jupiter;
$p = a(1-e^2)$;	
q, Q	perihelion, resp. aphelion distance of comet;
P	period of comet;
V, V_1	heliocentric velocity of comet, resp. Jupiter;
ω	angular distance from ascending node to perihelion;
Ω	longitude of ascending node;
v, v_1	true anomaly of comet, resp. Jupiter;
$u = \omega + v$;	
λ_1	mean longitude of Jupiter;
n_1	mean motion of Jupiter; $\theta = n_1 t$
δ_1, δ_2	contributions to perturbations from complementary, resp. principal part of perturbative function;
ψ	angle between the tangents at the comet's and Jupiter's orbit in the points which are nearest to each other;
$\nu(1/a, t)$	distribution of $1/a$ for the comets passing per year through perihelion;
$N(1/a, t)$	distribution of $1/a$ for total number of comets;
$\varphi(\delta)$	transition function giving the probability that a perturbation δ of $1/a$ takes place;
$D = 2/\int \varphi(\delta)\,\delta^2 d\delta$;	
ρ	radius of the sphere of solar action;
W_s	sun's velocity with respect to the interstellar field of comets;
W	comet's velocity with respect to the centre of gravity of the interstellar field;
σ^2	mean square of the velocities W;
ϑ	angle between the comet's heliocentric velocity and the radius vector;
$Q(q)$	number of distant comets passing per year through a perihelion between $q-1/2$ and $q+1/2$ A.U.;
$P(q) = \int_0^q Q(q)\,dq$;	
$f_q(1/a)$	distribution of $1/a$ for the comets $Q(q)$;
d	shortest distance between the comet and Jupiter's orbit;
s	jovicentric velocity of the comet;
ε	angle between the jovicentric velocity of the comet and Jupiter's heliocentric velocity.