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On the diffusion of monochromatic radiation through a medium consisting of plan-parallel layers in relative motion, by *J. Woltjer Jr.*

The drift theory of the Ca^+ chromosphere [1]) requires an analysis of the diffusion of the incident photospheric radiation in passing outwards. The corresponding problem for matter at rest: the determination of the intensity of the emerging radiation, if a system of plan-parallel layers only scatters the incident stream of quanta, has been extensively treated by several authors. However the fact that the layers of Ca^+ atoms are supposed to be moving outwards with increasing velocity radically changes the aspect of this problem and gives rise to transparency largely in excess of that obtained for matter at rest.

Consider radiation emitted by the photosphere in a direction perpendicular to the plan-parallel layers of Ca^+, and having a frequency somewhat in excess of that corresponding to the *K*-line. The velocity of each atom consists of the drift-velocity and the "thermal" velocity. Only those atoms which have a total velocity equal to the value required for reducing the frequency of the incident radiation to the *K*-line frequency, take part in the scattering of the incident quanta. At lower levels the atoms with outward thermal velocity are effective, at higher levels only those with inward thermal velocity. Hence at each level the atoms with inward thermal velocity are shielded from the photosphere by a column of atoms scattering exactly that radiation, which they themselves are able to scatter. However, this shielding is partly made ineffective by the fact that at the same level are present atoms with outward thermal velocity, hence exposed to the nearly unreduced photospheric radiation, that, though scattering a different frequency of photospheric radiation, still may excite these atoms, if only the scattering takes place in the appropriate direction.

[1]) Cf. *B.A.N.* 167, 180, 182, 213 and *Nature* 129, p. 580.

These considerations have been analysed mathematically in the following sections. I start from a value of the emission coefficient for each group of atoms obtaining in the case of matter at rest. The next step consists in the determination of the corresponding intensity distribution. Hence a new value of the emission coefficient has been derived. This value has been used in the computation of the intensity of the emerging radiation. The result and some considerations on its astrophysical implications may be found in the last section of this note.

1. Consider a system of coordinates x, y, z and matter, arranged in plan-parallel layers normal to the axis of z, moving with a drift-velocity W in the direction of the positive z-axis; W is supposed to increase with increasing z. The components of random velocity of each atom are u, v, w.

Select a group of atoms at P with velocitycomponents within specified narrow limits. These atoms can be excited by radiation emitted by atoms at P' only if the relative velocity along $P'P$ is sufficiently close to zero.

Suppose each atom capable of scattering radiation of frequency between $\nu_0 - \delta\nu$ and $\nu_0 + \delta\nu$, if referred to the atom in motion; let δ^* be the velocity-equivalent of $\delta\nu$. The components of the vector (u, v, w) along $P'P$ and two directions forming with $P'P$ an orthogonal system of coordinates are denoted by ζ, ξ and η. Then only those atoms at P' interact with the selected group at P, that satisfy the relation:

$$W' \cos\vartheta + \zeta' = W \cos\vartheta + \zeta + \varepsilon\delta^*$$
$$-1 \leq \varepsilon \leq +1;$$

ϑ is the angle between $P'P$ and the positive z-axis; accented quantities refer to P', those without accent to P.

Suppose the velocity distribution to be given by the proportionality to $e^{-\gamma\,(u^2+v^2+w^2)}$. The fraction of the total number per unit volume at P' that interact is[1]):

$$\left(\frac{\gamma}{\pi}\right)^{\frac{1}{2}} e^{-\gamma\zeta'^2}(2\delta^*)\ ,\ \zeta' = (W-W')\cos\vartheta + \zeta.$$

Hence the contribution to the optical depth of those atoms corresponding to radiation travelling along $P'P$ and capable of being scattered by the selected group of atoms at P is:

$$d\tau_s = \varkappa_\nu\rho\left(\frac{\gamma}{\pi}\right)^{\frac{1}{2}} e^{-\gamma\zeta'^2}(2\delta^*)\ ds;$$

$\varkappa_\nu$ is the mass-coefficient of scattering, ρ the density, ds an element of length measured along $P'P$. The optical depth from P' to P equals

$$\tau_s = \int_{P'}^{P} \varkappa_\nu\rho\left(\frac{\gamma}{\pi}\right)^{\frac{1}{2}} e^{-\gamma\zeta''^2}(2\delta^*)\ ds,$$

the double accent referring to the points intermediate between P' and P. As $ds = \frac{ds}{dW''}dW''$, $dW'' = -d\zeta''\sec\vartheta$, this value may be written:

$$\tau_s = \int_{\zeta}^{\zeta'} \varkappa_\nu\rho\left(\frac{\gamma}{\pi}\right)^{\frac{1}{2}}(2\delta^*)\ e^{-\gamma\zeta''^2}\frac{ds}{dW''}\sec\vartheta\, d\zeta''.$$

It is unnecessary to take account of the variations of $\varkappa_\nu\rho$ and $\frac{ds}{dW''}$. The quantity $\frac{ds}{dW''}$ is equal to $\frac{dz}{dW}\sec\vartheta$. The quantity $2\delta^*\varkappa_\nu$ equals $\frac{c}{\nu_0}\int\varkappa_\nu d\nu$.

Hence:

$$\tau_s = \frac{dz}{dW}\sec^2\vartheta\frac{c}{\nu_0}\left(\int\varkappa_\nu d\nu\right)\rho\left(\frac{\gamma}{\pi}\right)^{\frac{1}{2}}\int_{\zeta}^{\zeta'} e^{-\gamma\omega^2}\,d\omega.$$

The optical depth T_s results if the integration is extended to infinity.

Hence, collecting results we have:

$$\tau_s = q_0\sec^2\vartheta\int_{\zeta}^{\zeta'} e^{-\gamma\omega^2}\,d\omega$$

$$T_s = q_0\sec^2\vartheta\int_{\zeta}^{\infty} e^{-\gamma\omega^2}\,d\omega$$

$$q_0 = \frac{dz}{dW}\frac{c}{\nu_0}\rho\left(\frac{\gamma}{\pi}\right)^{\frac{1}{2}}\int\varkappa_\nu d\nu.$$

1) The fraction of atoms excited has been neglected in this respect.

2. Consider the atoms with a given w-component at a certain level. The atoms at any other level that are capable of interaction along the z-axis have the velocity component along the z-axis:

$$W - W' + w + \varepsilon\delta^*.$$

I consider all these atoms together and compute the emission coefficient obtaining if these atoms were really independent of the rest. The emission coefficient depends on the optical depth τ measured inwards in the direction of z decreasing. According to the results of the preceding sections:

$$\tau = q_0\int_{-w}^{\infty} e^{-\gamma\omega^2}\,d\omega = q_0\int_{-\infty}^{w} e^{-\gamma\omega^2}\,d\omega.$$

Using the well-known approximations, the emission coefficient $(E_\nu)_p$ results from the relation:

$$\left(\frac{E_\nu}{\varkappa_\nu\rho}\right)_p = I_{ph}\frac{1+\tau}{2+T};$$

I_{ph} is the intensity of the incident radiation; T is the value of τ, if $w = +\infty$, the total optical depth; the index p is added to denote the fact that $\varkappa_\nu\rho$ only refers to the particular group of w-atoms selected. These atoms form a fraction

$$\left(\frac{\gamma}{\pi}\right)^{\frac{1}{2}} e^{-\gamma w^2}(2\delta^*)$$

from all the atoms present at the level. Hence:

$$(\varkappa_\nu\rho)_p = \left(\frac{\gamma}{\pi}\right)^{\frac{1}{2}}\varkappa_\nu\rho\ e^{-\gamma w^2}(2\delta^*).$$

It is to be noted that according to the usual approximation involved in the analysis of the diffusion of radiation through a system of plan-parallel layers at rest, I_{ph} represents the value of the incident photospheric radiation averaged over all directions.

3. The value of the coefficient of emission deduced in the previous section forms the starting point of the computations of this note. The next step consists in the determination of the radiation incident at P on a selected (u, v, w) group of atoms. To know the intensity of this radiation it is necessary to determine the amount contributed in a specified direction by all elements of volume.

Consider a group of atoms at P', with velocity components ξ, η, ζ within specified limits, so chosen that interaction with the group at P is possible. Take the axis of ξ in the plane through $P'P$ parallel to the axis of z, so that $+\xi$ makes an angle $\frac{1}{2}\pi + \vartheta$ with $+z$.

The number of the P'-group is equal to a fraction of the total number present at P', given by

$$\left(\frac{\gamma}{\pi}\right)^{\frac{3}{2}} e^{-\gamma(\xi'^2+\eta'^2+\zeta'^2)} d\xi' d\eta' (2\delta^*).$$

The contribution of this group to the emission coefficient depends on the value of w', which equals

$$\zeta' \cos\vartheta - \xi' \sin\vartheta.$$

Hence the emission coefficient at P' of those atoms that are capable of interaction with the selected (u, v, w) group at P is equal to

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} I_{ph} \frac{1+\left(\frac{\gamma}{\pi}\right)^{\frac{1}{2}} T \int_{-\infty}^{w'} e^{-\gamma\omega^2} d\omega}{2+T} \left(\frac{\gamma}{\pi}\right)^{\frac{3}{2}} x_\nu \rho (2\delta^*)\, e^{-\gamma(\xi'^2+\eta'^2+\zeta'^2)} d\xi' d\eta'.$$

The integration with respect to η' is trivial; that with respect to ξ' is performed by a transformation of variables. Hence: [1])

$$\int_{-\infty}^{+\infty} e^{-\gamma\xi'^2} d\xi' \int_{-\infty}^{w'} e^{-\gamma\omega^2} d\omega = \left(\frac{\pi}{\gamma}\right)^{\frac{1}{2}} \int_{-\infty}^{\frac{\zeta'\cos\vartheta}{(1+\sin^2\vartheta)^{\frac{1}{2}}}} e^{-\gamma\omega^2} d\omega,$$

and the required emission coefficient equals:

$$\frac{I_{ph}}{2+T}\left\{1+\left(\frac{\gamma}{\pi}\right)^{\frac{1}{2}} T \int_{-\infty}^{\frac{\zeta'\cos\vartheta}{(1+\sin^2\vartheta)^{\frac{1}{2}}}} e^{-\gamma\omega^2} d\omega\right\} \left(\frac{\gamma}{\pi}\right)^{\frac{1}{2}} e^{-\gamma\zeta'^2} x_\nu \rho\, (2\delta^*).$$

As the mass coefficient of scattering is equal to $\left(\frac{\gamma}{\pi}\right)^{\frac{1}{2}} e^{-\gamma\zeta'^2} x_\nu \rho (2\delta^*)$, the ratio of emission coefficient to the product of density and mass coefficient of scattering of those atoms at P' that interact with the selected (u, v, w) group at P equals

$$\frac{I_{ph}}{2+T}\left\{1+\left(\frac{\gamma}{\pi}\right)^{\frac{1}{2}} T \int_{-\infty}^{\frac{\zeta'\cos\vartheta}{(1+\sin^2\vartheta)^{\frac{1}{2}}}} e^{-\gamma\omega^2} d\omega\right\}.$$

Hence the intensity of the specified radiation incident at P in a direction that makes an angle ϑ with the z-axis is:

$$\frac{I_{ph}}{2+T}\int_0^{T_s} e^{-\tau_s} d\tau_s \left\{1+\left(\frac{\gamma}{\pi}\right)^{\frac{1}{2}} T \int_{-\infty}^{\frac{\zeta'\cos\vartheta}{(1+\sin^2\vartheta)^{\frac{1}{2}}}} e^{-\gamma\omega^2} d\omega\right\} + \text{the contribution by the incident photospheric radiation.}$$

The second term may be transformed by partial integration:

$$\int_0^{T_s} e^{-\tau_s} d\tau_s \int_{-\infty}^{\frac{\zeta'\cos\vartheta}{(1+\sin^2\vartheta)^{\frac{1}{2}}}} e^{-\gamma\omega^2} d\omega = - e^{-\tau_s} \int_{-\infty}^{\frac{\zeta'\cos\vartheta}{(1+\sin^2\vartheta)^{\frac{1}{2}}}} e^{-\gamma\omega^2} d\omega \Bigg|_{\zeta'=\zeta}^{\zeta'=+\infty} + \frac{\cos\vartheta}{(1+\sin^2\vartheta)^{\frac{1}{2}}} \int_0^{T_s} e^{-\tau_s} e^{-\gamma\zeta'^2 \frac{\cos^2\vartheta}{1+\sin^2\vartheta}} \frac{d\zeta'}{d\tau_s} d\tau_s.$$

The upper limit of the first term of the right-hand member is zero if $\cos\vartheta$ is negative, $-\left(\frac{\pi}{\gamma}\right)^{\frac{1}{2}} e^{-T_s}$ if $\cos\vartheta$ is positive; so, the intensity of the specified radiation incident at P, omitting the contribution from the photospheric radiation, equals:

$$\frac{I_{ph}}{2+T}\left\{1+\left(\frac{\gamma}{\pi}\right)^{\frac{1}{2}} T \int_{-\infty}^{\frac{\zeta\cos\vartheta}{(1+\sin^2\vartheta)^{\frac{1}{2}}}} e^{-\gamma\omega^2} d\omega + R_s\right\};$$

$$R_s = - e^{-T_s} \underset{(\text{only if } \cos\vartheta \text{ is } +)}{- T e^{-T_s}} + \frac{\cos\vartheta}{(1+\sin^2\vartheta)^{\frac{1}{2}}} \left(\frac{\gamma}{\pi}\right)^{\frac{1}{2}} T \int_0^{T_s} e^{-\tau_s} e^{-\gamma\zeta'^2 \frac{\cos^2\vartheta}{1+\sin^2\vartheta}} \frac{d\zeta'}{d\tau_s} d\tau_s.$$

4. The radiation incident at P, as computed in the preceding section, is scattered in all directions. The amount of energy scattered by the selected (u, v, w)

[1]) It is to be noted that the symbol ω is used without a definite meaning, only as a designation of the variable of the integration.

group of atoms at P results from the value of the intensity of radiation by integrating over all directions and multiplying with the product of the density and the mass scattering coefficient of these atoms. As these atoms form a fraction

$$\left(\frac{\gamma}{\pi}\right)^{\frac{3}{2}} e^{-\gamma(u^2+v^2+w^2)}\, du\, dv\, dw$$

of those present at P, the energy scattered per unit of volume and time and frequency interval is equal to

$$x_\nu \rho \frac{I_{ph}}{2+T}\left(\frac{\gamma}{\pi}\right)^{\frac{3}{2}} e^{-\gamma(u^2+v^2+w^2)}\, du\, dv\, dw \int_0^{2\pi} d\varphi \int_0^{\pi} \sin\vartheta\, d\vartheta \left\{ 1 + \left(\frac{\gamma}{\pi}\right)^{\frac{1}{2}} T \int_{-\infty}^{\frac{\zeta\cos\vartheta}{(1+\sin^2\vartheta)^{\frac{1}{2}}}} e^{-\gamma\omega^2} d\omega + R_s \right\};$$

φ is the angle between the plane x-z and the plane s–z; the direct radiation from the photosphere has not been taken into account. The value of ζ is equal to

$$u\sin\vartheta\cos\varphi + v\sin\vartheta\sin\varphi + w\cos\vartheta.$$

The corresponding value of $\left(\frac{E_\nu}{x_\nu \rho}\right)_p$ referring to this selected group of atoms at P depends on $(u^2 + v^2)^{\frac{1}{2}}$ and w, contrary to the dependency on w only, involved in the use that has been made in section 3 of the starting value of the coefficient of emission.

As I intend to compute the radiation emerging in the direction $+ z$ from the scattering layers, I derive the value of $\left(\frac{E_\nu}{x_\nu \rho}\right)_p$ referring to all atoms at P having a certain value of w. So I integrate the foregoing expression for the amount of energy scattered by the selected (u, v, w) group at P over all values of u and v. The resulting value of the energy scattered per unit volume per unit frequency interval per unit time by those atoms at P that have w-component velocities between w and $w + dw$ equals:

$$2\pi\, x_\nu \rho \frac{I_{ph}}{2+T}\left(\frac{\gamma}{\pi}\right) e^{-\gamma w^2} dw \int_0^{\pi} \sin\vartheta\, d\vartheta \int_{-\infty}^{+\infty} e^{-\gamma v^2} dv \left\{ 1 + \left(\frac{\gamma}{\pi}\right)^{\frac{1}{2}} T \int_{-\infty}^{\frac{\zeta\cos\vartheta}{(1+\sin^2\vartheta)^{\frac{1}{2}}}} e^{-\gamma\omega^2} d\omega + R_s \right\},$$

$$\zeta = v\sin\vartheta + w\cos\vartheta.$$

The corresponding value of $\left(\frac{E_\nu}{x_\nu \rho}\right)_p$ referring to these atoms is:

$$\left(\frac{E_\nu}{x_\nu \rho}\right)_p = \tfrac{1}{2}\frac{I_{ph}}{2+T}\left(\frac{\gamma}{\pi}\right)^{\frac{1}{2}} \int_0^{\pi} \sin\vartheta\, d\vartheta \int_{-\infty}^{+\infty} e^{-\gamma v^2} dv \left\{ 1 + \left(\frac{\gamma}{\pi}\right)^{\frac{1}{2}} T \int_{-\infty}^{\frac{\zeta\cos\vartheta}{(1+\sin^2\vartheta)^{\frac{1}{2}}}} e^{-\gamma\omega^2} d\omega + R_s \right\}.$$

Transformation of the integrals reduces this relation to:

$$\left(\frac{E_\nu}{x_\nu \rho}\right)_p = \frac{I_{ph}}{2+T}\left\{ 1 + \tfrac{1}{2}\left(\frac{\gamma}{\pi}\right)^{\frac{1}{2}} T \int_0^{\pi} \sin\vartheta\, d\vartheta \int_{-\infty}^{\frac{w\cos^2\vartheta}{(2-\cos^4\vartheta)^{\frac{1}{2}}}} e^{-\gamma\omega^2} d\omega + \tfrac{1}{2}\left(\frac{\gamma}{\pi}\right)^{\frac{1}{2}} \int_0^{\pi} \sin\vartheta\, d\vartheta \int_{-\infty}^{+\infty} e^{-\gamma v^2} dv\, R_s \right\}.$$

The reduction of the second term in the right hand member is as follows:

$$\int_0^{\pi} \sin\vartheta\, d\vartheta \int_{-\infty}^{\frac{w\cos^2\vartheta}{(2-\cos^4\vartheta)^{\frac{1}{2}}}} e^{-\gamma\omega^2} d\omega = -\cos\vartheta \int_{-\infty}^{\frac{w\cos^2\vartheta}{(2-\cos^4\vartheta)^{\frac{1}{2}}}} e^{-\gamma\omega^2} d\omega \Bigg|_{\vartheta=0}^{\vartheta=\pi} + w\int_0^{\pi} \cos\vartheta\, e^{-\gamma w^2 \frac{\cos^4\vartheta}{2-\cos^4\vartheta}} \frac{d\frac{\cos^2\vartheta}{(2-\cos^4\vartheta)^{\frac{1}{2}}}}{d\vartheta} d\vartheta.$$

Introduce the new variable η by the relation

$$\eta^4 = \frac{\cos^4\vartheta}{2-\cos^4\vartheta},\ \cos\vartheta = 2^{\frac{1}{4}}\frac{\eta}{(1+\eta^4)^{\frac{1}{4}}};$$

then the right hand member of the preceding relation reduces to:

$$2\int_{-\infty}^{w} e^{-\gamma\omega^2} d\omega - 2^{\frac{9}{4}}\, w \int_0^{1} \frac{\eta^2}{(1+\eta^4)^{\frac{1}{4}}} e^{-\gamma w^2 \eta^4} d\eta.$$

So the relation results:

$$\left(\frac{E_\nu}{x_\nu \rho}\right)_p = \frac{I_{ph}}{2+T}\left\{ 1 + \tau - \frac{2^{\frac{5}{4}}}{\sqrt{\pi}} T w \gamma^{\frac{1}{2}} \int_0^{1} \frac{\eta^2}{(1+\eta^4)^{\frac{1}{4}}} e^{-\gamma w^2 \eta^4} d\eta + \tfrac{1}{2}\left(\frac{\gamma}{\pi}\right)^{\frac{1}{2}} \int_0^{\pi} \sin\vartheta\, d\vartheta \int_{-\infty}^{+\infty} e^{-\gamma v^2} dv\, R_s \right\}.$$

It is to be remembered that the index p denotes the selection of those atoms, that have a z-component of random velocity equal to w; also that the direct radiation of the photosphere has not been included.

The intensity of the radiation emerging from the scattering layers in a direction parallel to the z-axis is given by the equation:

$$I_\nu = \int_0^T \left(\frac{E_\nu}{x_\nu \rho}\right)_p e^{-\tau} d\tau.$$

As it is my intention to consider the value of I_ν only if T is large, the influence of the direct photospheric radiation is probably irrelevant and certainly of a positive amount. The evaluation of I_ν according to the value of $\left(\frac{E_\nu}{x_\nu \rho}\right)_p$ derived is preceded in the next section by a computation of the possible effect of the rest-term R_s.

5. I consider the contribution of R_s to the value of I_ν in the interval $0 \leq \tau \leq \frac{T}{2}$. Hence I introduce

$$R = \tfrac{1}{2}\left(\frac{\gamma}{\pi}\right)^{\frac{1}{2}} \frac{I_{ph}}{2+T} \int_0^{\frac{T}{2}} e^{-\tau} d\tau \int_0^{\pi} \sin \vartheta \, d\vartheta \int_{-\infty}^{+\infty} e^{-\gamma v^2} dv \, R_s$$

$$\zeta = v \sin \vartheta + w \cos \vartheta$$

$$R_s = -e^{-T_s} - T e^{-T_s} \text{ (only if } \cos\vartheta \text{ is } +) + \frac{\cos \vartheta}{(1+\sin^2\vartheta)^{\frac{1}{2}}} \left(\frac{\gamma}{\pi}\right)^{\frac{1}{2}} T \int_0^{T_s} e^{-\tau_s} e^{-\gamma \zeta'^2 \frac{\cos^2\vartheta}{1+\sin^2\vartheta}} \frac{d\zeta'}{d\tau_s} d\tau_s.$$

Denoting with index 1 the value of R relating to the first term of R_s only we have

$$R_1 \geq -\tfrac{1}{2}\left(\frac{\gamma}{\pi}\right)^{\frac{1}{2}} \frac{I_{ph}}{2+T} \int_0^{\frac{T}{2}} e^{-\tau} d\tau \int_0^{\pi} \sin \vartheta \, d\vartheta \int_{-\infty}^{+\infty} e^{-\gamma v^2} dv$$

$$R_1 \geq -\frac{I_{ph}}{2+T}.$$

The remaining part R_2 however is more difficult to analyse. Introducing ζ as variable of the integration we have:

$$R_2 = \tfrac{1}{2}\left(\frac{\gamma}{\pi}\right)^{\frac{1}{2}} \frac{I_{ph}}{2+T} \int_0^{\pi} d\vartheta \int_0^{\frac{T}{2}} e^{-\tau} d\tau \int_{-\infty}^{+\infty} e^{-\gamma \frac{(\zeta - w\cos\vartheta)^2}{\sin^2\vartheta}} d\zeta \, (R_s)_2 .$$

I consider separately the four terms arising by division of the range of the variables ϑ and ζ each in two parts, and remember the restriction of the w values to the range $-\infty \leq w \leq 0$.

$w-$; $\zeta+$; $\cos\vartheta+$.

As $\zeta - w \cos\vartheta \geq \zeta \geq 0$ we have the contribution to R_2 bounded at the negative side by

$$-\tfrac{1}{2}\left(\frac{\gamma}{\pi}\right)^{\frac{1}{2}} \frac{I_{ph}}{2+T} \int_0^{\frac{\pi}{2}} d\vartheta \int_0^{\infty} e^{-\gamma\zeta^2} d\zeta \, T e^{-T_s},$$

hence, as $e^{-\gamma\zeta^2} d\zeta = -\frac{dT_s}{q_0 \sec^2\vartheta}$, by

$$-\tfrac{1}{2}\left(\frac{\gamma}{\pi}\right)^{\frac{1}{2}} \frac{I_{ph}}{2+T} \int_0^{\frac{\pi}{2}} \cos^2\vartheta \, d\vartheta \int_0^{T_s(0)} \frac{T}{q_0} e^{-T_s} dT_s,$$

$T_s(0)$ denoting the value of T_s if $\zeta = 0$.

Hence: $R_2(w-; \zeta+; \cos\vartheta+) \geq -\frac{\pi}{8} \frac{I_{ph}}{2+T}$.

$w-$; $\zeta-$; $\cos\vartheta+$ and $-$. As $e^{-\frac{\gamma(\zeta - w\cos\vartheta)^2}{\sin^2\vartheta}}$, if $\cos\vartheta$ is positive exceeds the value corresponding to $\cos\vartheta$ negative with the same absolute value, the last integral in R_s contributes a positive amount in the interval considered to R_2. As $T_s \geq T_s(0) \geq \frac{1}{2} T$ we have:

$$R_2(w-; \zeta-; \cos\vartheta+ \text{ and } -) \geq -\frac{I_{ph}}{2+T} \frac{T}{2} e^{-\frac{T}{2}}.$$

$w-$; $\zeta+$; $\cos\vartheta-$. The contribution to R_2 is equal to

$$-\tfrac{1}{2}\left(\frac{\gamma}{\pi}\right) \frac{I_{ph}}{2+T} T \int_{\frac{\pi}{2}}^{\pi} d\vartheta \int_0^{\frac{T}{2}} e^{-\tau} d\tau \int_0^{\infty} d\zeta \, e^{-\gamma \frac{(\zeta - w\cos\vartheta)^2}{\sin^2\vartheta}} \lambda \int_0^{T_s} e^{-\tau_s} e^{-\gamma\zeta'^2\lambda^2} \frac{d\zeta'}{d\tau_s} d\tau_s$$

$$\lambda = \frac{|\cos\vartheta|}{(1+\sin^2\vartheta)^{\frac{1}{2}}}.$$

6. The evaluation of the last integral of the preceding section involves some difficulty as the relation between ζ' and τ_s is not simple. We have:

$$\frac{d\zeta'}{d\tau_s} = \frac{1}{q_0 \sec^2 \vartheta} e^{\gamma\zeta'^2};$$

hence the integral equals:

$$\frac{1}{q_0} \cos^2 \vartheta \int_0^{T_s} e^{-\tau_s} e^{\gamma\zeta'^2(1-\lambda^2)} d\tau_s.$$

As

$$e^{\gamma\zeta'^2} \int_{\zeta'}^{\infty} e^{-\gamma\omega^2} d\omega \leq \tfrac{1}{2} \left(\frac{\pi}{\gamma}\right)^{\frac{1}{2}},$$

hence

$$e^{\gamma\zeta'^2} \leq \frac{T_s(0)}{T_s - \tau_s},$$

and $1 - \lambda^2 \geq 0$, we have the inequality

$$e^{\gamma\zeta'^2(1-\lambda^2)} \leq \left\{\frac{T_s(0)}{T_s}\right\}^{1-\lambda^2} \left(1 - \frac{\tau_s}{T_s}\right)^{\lambda^2 - 1}.$$

So:

$$\int_0^{T_s} e^{-\tau_s} e^{-\gamma\zeta'^2\lambda^2} \frac{d\zeta'}{d\tau_s} d\tau_s \leq \frac{1}{q_0 \sec^2 \vartheta} \left\{\frac{T_s(0)}{T_s}\right\}^{1-\lambda^2} T_s \int_0^1 e^{-\xi T_s} (1-\xi)^{\lambda^2 - 1} d\xi.$$

The integral may be reduced as follows:

$$\int_0^1 e^{-\xi T_s} (1-\xi)^{\lambda^2-1} d\xi = -e^{-\xi T_s} \frac{(1-\xi)^{\lambda^2}}{\lambda^2}\Bigg|_0^1 - T_s \int_0^1 \frac{(1-\xi)^{\lambda^2}}{\lambda^2} e^{-\xi T_s} d\xi$$

$$= \frac{1}{\lambda^2} \left\{ 1 - T_s e^{-T_s} \int_0^1 \eta^{\lambda^2} e^{\eta T_s} d\eta \right\}.$$

The integral

$$\int_0^1 \eta^{\lambda^2} e^{\eta T_s} d\eta$$

decreases monotonously if λ^2 increases from 0 to 1 and T_s is kept constant. At the same time the derivative with respect to λ^2 is negative and increases monotonously. Hence its value is bounded at the lower side by the value for $\lambda^2 = 0$ augmented by the product of λ^2 and this derivative at $\lambda^2 = 0$.

As the value for $\lambda^2 = 0$ is $\frac{e^{T_s} - 1}{T_s}$ and the derivative at $\lambda^2 = 0$ is

$$\int_0^1 \log \eta \, e^{\eta T_s} d\eta$$

the following relation holds:

$$\int_0^1 e^{-\xi T_s} (1-\xi)^{\lambda^2-1} d\xi \leq \frac{1}{\lambda^2} \left\{ e^{-T_s} - T_s e^{-T_s} \lambda^2 \int_0^1 \log \eta \, e^{\eta T_s} d\eta \right\}.$$

The integral in the last term is reduced in the following way:

$$-T_s e^{-T_s} \int_0^1 \log \eta \, e^{\eta T_s} d\eta = -e^{-T_s} \sum_0^\infty \frac{T_s^{n+1}}{n!} \int_0^1 \eta^n \log \eta \, d\eta = e^{-T_s} \sum_0^\infty \frac{T_s^{n+1}}{(n+1)!(n+1)}.$$

The infinite series may be related to the exponental series by the equations:

$$\sum_0^\infty \frac{T_s^{n+1}}{(n+1)!(n+1)} = \frac{1}{T_s}\left\{T_s^2 + \frac{T_s^3}{2.2!} + \frac{T_s^4}{3.3!} + \ldots\right\} \leq \frac{1}{T_s}\left\{T_s^2 + \frac{3}{2}\left(\frac{T_s^3}{3!} + \frac{T_s^4}{4!} + \frac{T_s^5}{5!} + \ldots\right)\right\}$$

$$\text{i. e. } \frac{1}{T_s}\left\{T_s^2 + \frac{3}{2}\left(e^{T_s} - \frac{T_s^2}{2} - T_s - 1\right)\right\}.$$

Hence:

$$\int_0^1 e^{-\xi T_s} (1-\xi)^{\lambda^2-1} d\xi \leq \frac{1}{\lambda^2} e^{-T_s} + \frac{e^{-T_s}}{T_s} \left\{\frac{3}{2} e^{T_s} + \frac{1}{4} T_s^2\right\}.$$

As the maximum value of $T_s^2 e^{-T_s}$ is $4e^{-2}$ we have:

$$\int_0^1 e^{-\xi T_s}(1-\xi)^{\lambda^2-1} d\xi \leq \frac{1}{\lambda^2} e^{-T_s} + \frac{\frac{3}{2}+\frac{1}{e^2}}{T_s} \leq \frac{1}{\lambda^2} e^{-T_s} + \frac{3}{2}\left(1+\frac{1}{e^2}\right)\frac{1}{T_s} \text{ [1]}.$$

Though valid if $T_s \geq 0$, the relation is only of use if $T_s \geq 1$. In the interval $0 \leq T_s \leq 1$ I use the relations (also valid in the former interval):

$$\sum_0^\infty \frac{T_s^{\,n+1}}{(n+1)!\,(n+1)} \leq \frac{2}{T_s}\left(e^{T_s} - T_s - 1\right),$$

$$\int_0^1 e^{-\xi T_s}(1-\xi)^{\lambda^2-1} d\xi \leq \frac{1}{\lambda^2} e^{-T_s} + \frac{2}{T_s}\left\{1-(1+T_s)\,e^{-T_s}\right\}.$$

I introduce the discontinuous function $\psi(T_s)$ defined by the relations:

$$0 \leq T_s < 1 \quad \psi(T_s) = T_s \qquad\qquad 1 \leq T_s \leq \infty\,,\ \ \psi(T_s) = \frac{3}{2}\frac{\left(1+\frac{1}{e^2}\right)}{T_s}.$$

The resulting lower limit of the contribution ($\zeta +$; $\cos\vartheta -$) to R_2 is equal to

$$-\frac{1}{4}\left(\frac{\gamma}{\pi}\right)^{\frac{1}{2}} \frac{I_{ph}}{2+T}\, T \int_{\frac{\pi}{2}}^{\pi} d\vartheta \int_0^{\frac{T}{2}} e^{-\tau} d\tau \int_0^\infty d\zeta\, e^{-\gamma\frac{(\zeta - w\cos\vartheta)^2}{\sin^2\vartheta}} \left\{\frac{T_s}{T_s(0)}\right\}^{\lambda^2} \left\{\frac{e^{-T_s}}{\lambda} + \lambda\psi(T_s)\right\}.$$

I consider separately the contribution with factor λ^{-1} and that with factor λ.

a. The terms factored by λ^{-1}.

The exponent of e is equal to

$$-\gamma\frac{(\zeta - w\cos\vartheta)^2}{\sin^2\vartheta} = -\gamma\,\frac{\zeta^2(1+\cos\vartheta) - (\zeta+w)^2\cos\vartheta + w^2(\cos^2\vartheta+\cos\vartheta)}{\sin^2\vartheta}$$

$$= -\gamma\zeta^2 - \gamma(\zeta^2+w^2)\frac{\cos\vartheta+\cos^2\vartheta}{\sin^2\vartheta} + \gamma(\zeta+w)^2\frac{\cos\vartheta}{\sin^2\vartheta}.$$

As $\cos\vartheta$ is negative, the second term has a positive coefficient of (ζ^2+w^2) and the third term is always negative. Hence the exponential function has the upper limit:

$$e^{-\gamma\zeta^2-\gamma(\zeta^2+w^2)\frac{\cos\vartheta+\cos^2\vartheta}{\sin^2\vartheta}}.$$

As $e^{\gamma\zeta^2} \leq \frac{T_s(0)}{T_s}$ and $e^{\gamma w^2} \leq \frac{\frac{1}{2}T}{\tau}$, a new upper limit is:

$$e^{-\gamma\zeta^2}\left\{\frac{T_s(0)}{T_s}\,\frac{\frac{1}{2}T}{\tau}\right\}^{-\frac{\cos\vartheta+\cos^2\vartheta}{\sin^2\vartheta}};$$

it is to be remembered that the exponent of the second factor really is positive or zero.

Hence the "term factored by λ^{-1}" that is being considered has the lower limit:

$$(a)\quad -\frac{1}{4}\frac{I_{ph}}{2+T}\int_{\frac{\pi}{2}}^{\pi} d\vartheta\,\lambda^{-1}\left(\frac{T}{2}\right)^{1-\frac{\cos\vartheta+\cos^2\vartheta}{\sin^2\vartheta}} T_s(0)^{-\lambda^2-\frac{\cos\vartheta+\cos^2\vartheta}{\sin^2\vartheta}-1} \int_0^\infty T_s^{\lambda^2+\frac{\cos\vartheta+\cos^2\vartheta}{\sin^2\vartheta}} e^{-T_s} dT_s \int_0^\infty \tau^{\frac{\cos\vartheta+\cos^2\vartheta}{\sin^2\vartheta}} e^{-\tau} d\tau.$$

It is to be noted that, as $\frac{\cos\vartheta+\cos^2\vartheta}{\sin^2\vartheta} = \frac{\cos\vartheta}{2\sin^2\frac{1}{2}\vartheta}$, the exponents have no singularity in the interval $\frac{\pi}{2} \leq \vartheta \leq \pi$.

The two last integrals are of the general type $\int_0^\infty e^{-x} x^s\,dx$. In the interval $-0{\cdot}5 \leq s \leq +0{\cdot}5$ we have:

$$\int_0^\infty e^{-x} x^s dx \leq \int_0^1 e^{-x} x^s dx + \int_1^\infty e^{-x} x^{\frac{1}{2}} dx \leq \frac{1}{s+1} - \frac{1}{s+2} + \frac{1}{2(s+3)} + \frac{1}{e} + \int_1^\infty e^{-\eta^2} d\eta.$$

[1]) The last term in the inequality has been introduced to conform to some computations performed with this limit.

With this upper limit for the exponential integrals the limit (a) is extended.

I have computed numerical values with $T = 100$, collected in the following table; the tabulated quantity is the coefficient of $-\frac{I_{ph}}{2+T}$ in the integrand of the extended limit (a).

ϑ	
90°	0·00
100	0·50
110	2·05
120	3·65
130	3·96
140	2·90
150	1·75
160	0·94
170	0·57
180	0·47

(*b*) *The term factored by* λ.

The integral $\int\limits_0^\infty d\zeta\, e^{-\gamma\frac{(\zeta - w\cos\vartheta)^2}{\sin^2\vartheta}} \left\{\frac{T_s}{T_s(0)}\right\}^{\lambda^2} \psi(T_s)$ is bounded by an upper limit derived from the Schwarz inequality; to this end a factor $e^{-\frac{1}{2}\gamma\zeta^2}$ is taken out of the integrand.

Hence:

$$\int\limits_0^\infty d\zeta\, e^{-\gamma\frac{(\zeta - w\cos\vartheta)^2}{\sin^2\vartheta}} \left\{\frac{T_s}{T_s(0)}\right\}^{\lambda^2} \psi(T_s) \leq \sqrt{\left\{\int\limits_0^\infty d\zeta\, e^{-2\gamma\frac{(\zeta - w\cos\vartheta)^2}{\sin^2\vartheta} + \gamma\zeta^2} \int\limits_0^\infty d\zeta\, e^{-\gamma\zeta^2} \left\{\frac{T_s}{T_s(0)}\right\}^{2\lambda^2} \psi^2(T_s)\right\}}.$$

As $-2\gamma\frac{(\zeta - w\cos\vartheta)^2}{\sin^2\vartheta} + \gamma\zeta^2 = -\frac{\gamma}{\sin^2\vartheta}\left\{\zeta(1+\cos^2\vartheta)^{\frac{1}{2}} - 2\frac{w\cos\vartheta}{(1+\cos^2\vartheta)^{\frac{1}{2}}}\right\}^2 + \gamma w^2\frac{2\cos^2\vartheta}{1+\cos^2\vartheta}$, this inequality can be continued:

$$\leq 2^{-\frac{1}{2}} e^{\gamma w^2\frac{\cos^2\vartheta}{1+\cos^2\vartheta}} \frac{(\sin\vartheta)^{\frac{1}{2}}}{(1+\cos^2\vartheta)^{\frac{1}{4}}} \left(\frac{\pi}{\gamma}\right)^{\frac{1}{2}} T_s(0)^{-\lambda^2-\frac{1}{2}} \sqrt{\int\limits_0^{T_s(0)} T_s^{2\lambda^2}\psi^2(T_s)\,dT_s}$$

$$\leq 2^{-\frac{1}{2}} \left(\frac{\pi}{\gamma}\right)^{\frac{1}{2}} \frac{(\sin\vartheta)^{\frac{1}{2}}}{(1+\cos^2\vartheta)^{\frac{1}{4}}} \left(\frac{\frac{1}{2}T}{\tau}\right)^{\frac{\cos^2\vartheta}{1+\cos^2\vartheta}} T_s(0)^{-\lambda^2-\frac{1}{2}} \sqrt{\int\limits_0^{T_s(0)} T_s^{2\lambda^2}\psi^2(T_s)\,dT_s}$$

Hence the "term factored by λ" that is being considered has the lower limit:

$$(b) - \frac{2^{\frac{1}{2}}}{4}\frac{I_{ph}}{2+T}\int\limits_{\frac{\pi}{2}}^{\pi} d\vartheta \frac{\lambda(\sin\vartheta)^{\frac{1}{2}}}{(1+\cos^2\vartheta)^{\frac{1}{4}}}\left(\frac{T}{2}\right)^{1+\frac{\cos^2\vartheta}{1+\cos^2\vartheta}} T_s(0)^{-\lambda^2-\frac{1}{2}} \int\limits_0^\infty e^{-\tau}\tau^{-\frac{\cos^2\theta}{1+\cos^2\theta}}d\tau \sqrt{\int\limits_0^{T_s(0)} T_s^{2\lambda^2}\psi^2(T_s)\,dT_s};$$

the last integral is equal to:

$$\int\limits_0^{T_s(0)} T_s^{2\lambda^2}\psi^2(T_s)\,dT_s = \frac{1}{2\lambda^2+3} + \frac{9}{4}\left(1+\frac{1}{e^2}\right)^2 \frac{T_s(0)^{2\lambda^2-1}-1}{2\lambda^2-1} \leq \frac{1}{3} + \frac{9}{4}\left(1+\frac{1}{e^2}\right)^2 \frac{T_s(0)^{2\lambda^2-1}-1}{2\lambda^2-1}.$$

The transition of the denominator $2\lambda^2 - 1$ through zero does introduce no singularity. I have computed numerical values with $T = 100$ collected in the following table; the tabulated quantity is the coefficient of $-\frac{I_{ph}}{2+T}$ in the extended lower limit (b); the extension consists in the substitution of $\frac{1}{3}$ for $\frac{1}{2\lambda^2+3}$ in the last integral and the introduction of an upper limit for the exponential integral.

ϑ	
90°	0·00
100	0·12
110	0·50
120	1·22
130	2·16
140	3·04
150	3·57
160	3·60
170	2·91
180	0·00

The integration with respect to ϑ is performed by addition of the tabulated values (with weight $\frac{1}{2}$ if $\vartheta = 90^\circ$ or 180°) and multiplication by the value of the interval.

Collecting the results we have:

$$R_1 \geq -\frac{I_{ph}}{2+T}$$

$$R_2\,(w-;\,\zeta+;\,\cos\vartheta+) \geq -\frac{\pi}{8}\,\frac{I_{ph}}{2+T}$$

$$R_2\,(w-;\,\zeta-) \geq -\frac{T}{2}\,e^{-\frac{T}{2}}\,\frac{I_{ph}}{2+T}$$

$$R_2\,(w-;\,\zeta+;\,\cos\vartheta-)\ (a) \geq -2{\cdot}90\,\frac{I_{ph}}{2+T}$$

$$T=100 \qquad (b) \geq -2{\cdot}99\,\frac{I_{ph}}{2+T}.$$

Hence:

$$T=100 \qquad R\,(w-) \geq -7{\cdot}28\,\frac{I_{ph}}{2+T}.$$

7. The intensity of the radiation emerging in the direction of the positive z-axis is equal to:

$$I_\nu\,(\vartheta=0) = \int_0^T \left(\frac{E_\nu}{x_\nu\rho}\right)_p e^{-\tau}\,d\tau$$

with the value of $\left(\frac{E_\nu}{x_\nu\rho}\right)_p$ as derived in section 4.

Hence it is necessary to evaluate the integral:

$$\int_0^T e^{-\tau}\,d\tau \left\{ |w|\,\gamma^{\frac{1}{2}} \int_0^1 \frac{\eta^2}{(1+\eta^4)^{\frac{1}{4}}}\,e^{-\gamma w^2\eta^4}\,d\eta \right\}.$$

The last integral approximates to zero if w goes to infinity; it approaches zero as $|w|^{-\frac{3}{2}}$. I have computed the value of this integral by numerical integration, the results are contained in the following table. As the interval used was 0·1 and the integral has been obtained by simple summation (first and last term half weight), the accuracy attained is not great.

$\gamma\, w^2$	$(\gamma w^2)^{\frac{3}{4}} \int_0^1 \frac{\eta^2}{(1+\eta^4)^{\frac{1}{4}}}\,e^{-\gamma w^2\eta^4}\,d\eta$
1·0	0·211
2·0	0·266
3·0	0·283
4·0	0·292
5·0	0·296

The integration with respect to τ is performed numerically as shown in the next table.

The value of $T = 100$.

$T = 100$

$e^{-\tau}$	$\gamma^{\frac{1}{2}}\,\lvert w\rvert$	$\lvert w\rvert\,\sqrt{\gamma} \int_0^1 \frac{\eta^2}{(1+\eta^4)^{\frac{1}{4}}}\,e^{-\gamma w^2\eta^4}\,d\eta$
1·0	∞	0·000
0·9	2·17	0·199
0·8	2·01	0·206
0·7	1·90	0·209
0·6	1·82	0·212
0·5	1·74	0·214
0·4	1·67	0·216
0·3	1·59	0·218
0·2	1·51	0·220
0·1	1·41	0·224

The integrand is remarkably constant owing to the fact that only a small interval of w plays a role. Addition of the numbers in the last column and multiplication by the value of the interval in $e^{-\tau}$ gives the value of the integral

$$\int_0^T e^{-\tau}\,d\tau\,|w|\,\gamma^{\frac{1}{2}} \int_0^1 \frac{\eta^2}{(1+\eta^4)^{\frac{1}{4}}}\,e^{-\gamma w^2\eta^4}\,d\eta = 0{\cdot}192.$$

The value of the intensity of the emerging radiation in the direction of the axis of z follows:

$$T=100 \qquad I_\nu\,(\vartheta=0) = \frac{I_{ph}}{2+T}\left\{ 2{\cdot}00 + 0{\cdot}192\,\frac{2^{\frac{5}{4}}}{\pi^{\frac{1}{2}}}\,T \right\} + R\,(w-) + R\,(w+) = 27{\cdot}7\,\frac{I_{ph}}{2+T} + R\,(w-).$$

The value of $R\,(w+)$ is insignificant as the corresponding τ value is larger than 50.

The rest-term satisfies the inequality:

$$T=100 \qquad R\,(w-) \geq -7{\cdot}28\,\frac{I_{ph}}{2+T}$$

as shown in section 6.

8. The preceding computations make it probable that a system of plan-parallel layers scattering monochromatically is far more transparent if in relative motion than if at rest. The numerical example considered yields a value of the intensity of the radiation emerging in a normal direction probably more than 20% of the intensity of the incident radiation, if the optical thickness is 100.

However this percentage depends on the value of the optical thickness. If this value is exceedingly large the radiation travelling outwards is only derived from atoms with very large random velocity component in the z direction; then the extra contribution to the emission coefficient is small as may be seen from the result of section 4.

The foregoing analysis only constitutes a first approximation. It is possible to compute a new value of the emission coefficient starting from the value derived in section 4.

The astrophysical implications seem of some importance.

Firstly the density of the Ca^+ chromosphere as derived from observed intensities in emission or absorption requires a profound modification.

Secondly the possibility exists that the transparency shows a maximum for a certain value of the optical thickness; the value of this last quantity however may depend on the value of the outward drift-velocity by means of the relative number of atoms that are doubly ionised.

A maximum in the transparency thus could arise, that might be connected with the K_2-radiation. However as K_2 has components at the violet *and* at the red side, a modification of the drift-theory would be required that should account for a downward stream of Ca^+ atoms, possibly those atoms that are accelerated downwards after the second ionisation.
