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Beenakker, C.W.J.; Schomerus, H.; Titov, M.; Brouwer, P.W.

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Microscopic versus mesoscopic local density of states in one-dimensional localization

H Schomerus,[1] M Titov,[2] P W Brouwer,[3] and C W J Beenakker[2]
[1]*Max Planck Institut fur Physik komplexer Systeme Nothnitzer Strasse 38 01187 Dresden Germany*
[2]*Instituut Lorentz Universiteit Leiden P O Box 9506 2300 RA Leiden The Netherlands*
[3]*Laboratory of Atomic and Solid State Physics Cornell University, Ithaca, New York 14853 2501*
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We calculate the probability distribution of the local density of states ν in a disordered one-dimensional conductor or single mode waveguide, attached at one end to an electron or photon reservoir We show that this distribution does not display a log-normal tail for small ν, but diverges instead $\propto \nu^{-1/2}$ The log normal tail appears if ν is averaged over rapid oscillations on the scale of the wavelength There is no such qualitative distinction between microscopic and mesoscopic densities of states if the levels are broadened by inelastic scattering or absorption, rather than by coupling to a reservoir

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Localization of wave functions by disorder can be seen in the fluctuations of the density of states, provided the system is probed on a sufficiently short length scale [1,2] The local density of states (LDOS) of electrons can be probed using the tunnel resistance of a point contact[3] or the Knight shift in nuclear magnetic resonance,[4] while the LDOS of photons determines the rate of spontaneous emission from an atomic transition [5] In the photonic case one can study the effects of localization independently from those of interactions (For the description of one-dimensional interacting electrons in terms of Luttinger liquids and the interplay of interaction and localization see, e g , Ref 6)

For each length scale δ characteristic for the resolution of the probe, one can introduce a corresponding LDOS ν_δ It is necessary that δ is less than the localization length, in order to be able to see the effects of localization—the hallmark[7] being the appearance of logarithmically normal tails $\propto \exp(-\text{const}\times \ln^2 \nu_\delta)$ in the probability distribution $P(\nu_\delta)$

Much of our present understanding[8] of this problem in a wire geometry builds on the one-dimensional (1D) solution of Altshuler and Prigodin [9] In the simplest case one has a single-mode wire which is closed at one end and attached at the other end to an electron reservoir The optical analogue is a single-mode waveguide that can radiate into free space from one end In 1D the localization length equals twice the mean free path l, which is assumed to be large compared to the wavelength λ One can then distinguish the microscopic LDOS $\nu = \nu_\delta$ for $\delta \ll \lambda$, and the mesoscopic LDOS $\tilde{\nu} = \nu_\delta$ for $\lambda \ll \delta \ll l$ While ν oscillates rapidly on the scale of the wavelength, $\tilde{\nu}$ only contains the slowly varying envelope of these oscillations Altshuler and Prigodin calculated the distribution $P(\tilde{\nu})$ and surmised that $P(\nu)$ would have the same log-normal tails We will demonstrate that this is not the case for the small-ν asymptotics

The calculation of Ref 9 was based on the Berezinskii diagram technique,[10] which reconstructs the probability distribution from its moments (An alternative approach,[11] using the method of supersymmetry, also proceeds via the moments) An altogether different scattering approach has been proposed by Gasparian, Christen, and Buttiker,[12] and more recently by Pustilnik [13] We have pursued this approach and arrive at a relation between ν, $\tilde{\nu}$, and reflection coefficients This allows a direct calculation of the distributions We find that $P(\nu)$ and $P(\tilde{\nu})$ have the same log-normal tail for large densities, but the asymptotics for small ν and $\tilde{\nu}$ is completely different The strong fluctuations of ν on the scale of the wavelength lead to a divergence $P(\nu) \propto \nu^{-1/2}$ for $\nu \to 0$, while the distribution of the envelope vanishes, $P(\tilde{\nu}) \to 0$ for $\tilde{\nu} \to 0$ This qualitative difference between microscopic and mesoscopic LDOS is a feature of an open system Both $P(\nu)$ and $P(\tilde{\nu})$ vanish for small densities if the wire is closed at both ends and the levels are broadened by inelastic scatterers (for electrons) or absorption (for photons)

We consider a 1D wire and relate the microscopic and mesoscopic LDOS at energy E and at a point $x=0$ to the reflection amplitudes r_R, r_L from parts of the wire to the right and to the left of this point The Hamiltonian is $H = -(\hbar^2/2m)\partial^2/\partial x^2 + V(x)$ for noninteracting electrons (For photons of a single polarization we would consider the differential operator of the scalar wave equation) We will put $\hbar = 1$ for convenience of notation We start from the relation between the LDOS and the retarded Green function,

$$\nu = -\pi^{-1}\text{Im}G(0), \tag{1}$$

$$(E + i\eta - H)G(x) = \delta(x), \tag{2}$$

with η a positive infinitesimal We assume weak disorder ($kl \gg 1$, with $k = 2\pi/\lambda$ the wave number), so that we can expand the Green function in scattering states in a small interval around $x=0$,

$$G(x) = c_L(e^{-ikx} + r_L e^{ikx})\,\theta(-x) + c_R(e^{ikx} + r_R e^{-ikx})\,\theta(x) \tag{3}$$

[The function $\theta(x)=1$ for $x>0$ and 0 for $x<0$] The coefficients c_L and c_R are related by the requirement that the Green function be continuous at $x=0, c_L(1+r_L) = c_R(1+r_R)$ Substitution of Eq (3) into Eq (2) gives a second relation between c_L and c_R, from which we deduce

$$G(0) = \frac{(1+r_L)(1+r_R)}{iv(1-r_R r_L)}, \tag{4}$$

with v the velocity Using Eq (1) we arrive at the key relation between the microscopic LDOS and the reflection coefficients,

$$\nu=(\pi v)^{-1}\mathrm{Re}(1+r_L)(1-r_Rr_L)^{-1}(1+r_R) \tag{5}$$

In order to perform the local spatial average that gives the mesoscopic LDOS $\tilde{\nu}$, we use that the reflection coefficients oscillate on the scale of the wavelength If we shift x_0 slightly away from the origin to a point x', one has $r_L \to e^{2ikx'}r_L$ and $r_R \to e^{-2ikx'}r_R$ The product r_Rr_L, however, does not display these oscillations—only this combination should be retained Hence

$$\tilde{\nu}=(\pi v)^{-1}\mathrm{Re}(1+r_Rr_L)(1-r_Rr_L)^{-1} \tag{6}$$

In what follows we will measure ν and $\tilde{\nu}$ in units of $\nu_0=(\pi v)^{-1}$, which is the macroscopic density of states and the ensemble average of $\nu,\tilde{\nu}$

Let us now demonstrate the power of the two simple relations (5) and (6) We take the wire open at the left end and study the density at a distance L from this opening At the right end the wire is assumed to be closed, giving rise to a reflection coefficient $r_R=\exp(i\phi_R)$ with uniformly distributed phase ϕ_R in the interval $(0,2\pi)$ The reflection coefficient $r_L=\sqrt{R}\exp(i\phi_L)$ is parametrized through the uniformly distributed phase ϕ_L and the reflection probability R in the interval (0,1) The assumption of a random scattering phase is justified because we assumed $\lambda \ll l$ [14] The ratio $u=(1+R)(1-R)^{-1}$ has the probability distribution[15]

$$\rho(u)=\frac{e^{-s/4}}{\sqrt{\pi}(2s)^{3/2}}\int_{\operatorname{arccosh} u}^{\infty}dz\frac{ze^{-z^2/4s}}{(\cosh z-u)^{1/2}}, \tag{7}$$

with $s=L/l$ and l the mean free path for backscattering The mesoscopic LDOS (6) can be written in terms of the variables u and $\phi=\phi_L+\phi_R$,

$$\tilde{\nu}=(u-\sqrt{u^2-1}\cos\phi)^{-1} \tag{8}$$

Averaging first over ϕ we find

$$P_{\mathrm{open}}(\tilde{\nu})=\frac{\tilde{\nu}^{-3/2}}{\pi\sqrt{2}}\int_a^{\infty}du\frac{\rho(u)}{\sqrt{u-a}}, \quad a=\tfrac{1}{2}(\tilde{\nu}+\tilde{\nu}^{-1}) \tag{9}$$

The subsequent integration with Eq (7) yields

$$P_{\mathrm{open}}(\tilde{\nu})=\frac{\tilde{\nu}^{-3/2}e^{-s/4}}{2\sqrt{\pi s}}\exp\left(-\frac{1}{4s}\ln^2\tilde{\nu}\right) \tag{10}$$

The distribution function (10) is the celebrated result of Altshuler and Prigodin [9] It displays log-normal tails for both large and small values of $\tilde{\nu}$ Indeed, the two tails are linked by the functional relation[8]

$$P(1/\tilde{\nu})=\tilde{\nu}^3P(\tilde{\nu}) \tag{11}$$

This relation follows directly from Eq (9) and hence requires only a uniformly distributed phase ϕ, regardless of the distribution function $\rho(u)$ of the reflection probability As we will now show, such a relation does not hold, in general, for the microscopic LDOS ν, and the asymptotics of its distribution function for small and large values of ν can be entirely different

The calculation is facilitated by the fact that ν is related to $\tilde{\nu}$ by

$$\nu=2\tilde{\nu}\cos^2(\phi_R/2) \quad \text{if } |r_R|=1 \tag{12}$$

Moreover, $\tilde{\nu}$ is statistically independent of ϕ_R because the latter enters $\tilde{\nu}$ only in combination with ϕ_L, which itself is uniformly distributed The distribution of the microscopic LDOS hence follows directly from Eq (10),

$$P_{\mathrm{open}}(\nu)=\frac{\nu^{-3/2}e^{-s/4}}{\pi\sqrt{2\pi s}}\int_0^1\frac{dt}{\sqrt{1-t}}\exp\left(-\frac{\ln^2(\nu/2t)}{4s}\right), \tag{13}$$

where we substituted $t=\cos^2(\phi_R/2)$ The asymptotic behavior is

$$P_{\mathrm{open}}(\nu)=\frac{\exp(3s/4)}{2^{1/2}\pi}\nu^{-1/2}, \quad \nu\ll e^{-s}, \tag{14a}$$

$$P_{\mathrm{open}}(\nu)=\frac{2^{1/2}\exp(-s/4)}{s^{1/2}\pi^{3/2}}\nu^{-3/2}, \quad e^{-s}\ll\nu\ll e^{s}, \tag{14b}$$

$$P_{\mathrm{open}}(\nu)=\frac{\exp[-s/4-\ln^2(\nu/2)/4s]}{\pi\nu^{3/2}\ln^{1/2}(\nu/2)}, \quad \nu\gg e^{s} \tag{14c}$$

In the second and third region this is similar to the behavior of $P_{\mathrm{open}}(\tilde{\nu})$ in Eq (10) In the region of the smallest densities, however, $P_{\mathrm{open}}(\nu)$ is not log-normal like $P_{\mathrm{open}}(\tilde{\nu})$ but diverges $\propto \nu^{-1/2}$

The different tails arise from two qualitatively different mechanisms that produce small values of ν and $\tilde{\nu}$ For the mesoscopic LDOS this requires remoteness of E from the eigenvalues of wave functions localized within a localization length around x_0 As a consequence, $P(\tilde{\nu})$ is intimately linked to the distribution function of resonance widths [2] Small values of the microscopic LDOS ν are attained at nodes of the wave function which solves the wave equation with open boundary conditions, independent of the energy The nodes are completely determined by the small-scale structure of the wave function, which is a real standing wave $\propto\cos(kx+\alpha)$ with random phase α [8] [We recognize the square of this wave amplitude in Eq (12)] The resulting $\nu^{-1/2}$ divergence of the probability distribution has the same origin as in the Porter-Thomas distribution for chaotic wave functions [16]

The two distributions for the open wire are plotted in Fig 1, together with the result of a numerical simulation in which the Green function inside the wire is calculated recursively [17] The comparison of theory and numerics is free of any adjustable parameter—the velocity was taken from the dispersion

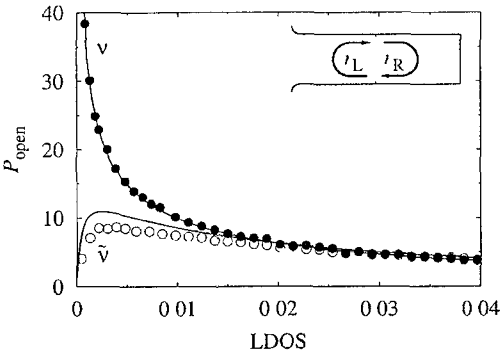

FIG. 1. Distributions of the microscopic local density of states (LDOS) ν and the mesoscopic LDOS $\tilde{\nu}$ for the open wire at a distance $L=2l$ from the opening. [Both are measured in units of their mean $\nu_0=(\pi v)^{-1}$.] Solid curves are given by Eqs. (10) and (13). The data points result from a numerical simulation for a wire of length $10l$ with no adjustable parameter. The inset shows the geometry of the open wire (not to scale).

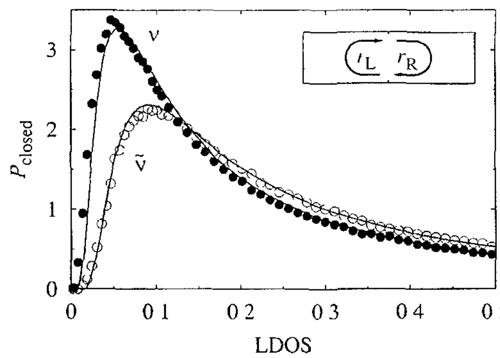

FIG. 2. Same as in Fig. 1 but for the closed wire with dimensionless absorption rate $\omega=1/6$. Solid curves are given by Eqs. (17) and (19). The data points result from a numerical simulation for a wire of length $55l$, with the LDOS computed halfway in the wire.

relation, and the mean free path was obtained from the disorder strength within the Born approximation.

We now show that this qualitative difference between the microscopic and mesoscopic LDOS is absent in a closed wire. If the wire is decoupled from the reservoir we need another source of level broadening to regularize the δ functions in the LDOS. Following Ref. 9, we will retain a finite imaginary part η of the energy, corresponding to spatially uniform absorption (for photons) or inelastic scattering (for electrons), with rate 2η. Equations (5) and (6) still hold provided $\eta \ll E$. The reflection coefficients can be written as $r_{R,L}=\sqrt{R_{R,L}}e^{i\phi_{R,L}}$, where ϕ_R and ϕ_L are uniformly distributed phases if the attenuation length $v/(2\eta) \gg (l\lambda^2)^{1/3}$,[18] and R_R, R_L are independent reflection probabilities. In an infinitely long wire they have the same distribution[19]

$$\rho(R)=\frac{\omega e^{\omega}}{(1-R)^2}\exp[-\omega(1-R)^{-1}], \quad \omega=4\eta l/v \tag{15}$$

After elimination of the phases the distribution of the mesoscopic LDOS takes again the form (9), where u now stands for the combination $u=(1+R_R R_L)(1-R_R R_L)^{-1}$. Equation (15) implies for u the distribution

$$\rho(u)=\omega^2\left(1-\frac{\partial}{\partial\omega}\right)e^{-\omega(u-1)}K_0(\omega\sqrt{u^2-1}) \tag{16}$$

The resulting distribution function of the mesoscopic LDOS is

$$P_{\rm closed}(\tilde{\nu})=\frac{\omega^2\tilde{\nu}^{-3/2}}{\pi\sqrt{2}}\int_a^\infty \frac{du}{\sqrt{u-a}}e^{-\omega(u-1)}[uK_0(\omega\sqrt{u^2-1}) + \sqrt{u^2-1}K_1(\omega\sqrt{u^2-1})], \tag{17}$$

with a defined in Eq. (9). It vanishes for small densities as

$$P_{\rm closed}(\tilde{\nu})=2^{-1/2}\omega\tilde{\nu}^{-2}\exp(-\omega/\tilde{\nu}), \quad \tilde{\nu}\ll\omega \tag{18}$$

This should be compared with the known distribution[9]

$$P_{\rm closed}(\nu)=\left(\frac{2\omega}{\pi}\right)^{1/2}\nu^{-3/2}\exp[\omega-\tfrac{1}{2}\omega(\nu+\nu^{-1})] \tag{19}$$

of the microscopic LDOS. In contrast to the open wire, both distributions vanish for $\nu,\tilde{\nu}\to 0$. This is illustrated in Fig. 2, which compares the analytical predictions to numerical data obtained by diagonalization of a Hamiltonian. The comparison is again free of any adjustable parameter.

We note in passing that the asymptotic behavior (18) differs from the asymptotic behavior

$$P_{\rm closed}(\tilde{\nu})\neq \tfrac{1}{4}(\pi\omega)^{1/2}\tilde{\nu}^{-3/2}\exp(-\pi^2\omega/16\tilde{\nu}), \tag{20}$$

given in Ref. 9 for $\omega\ll 1$. There the distribution function was reconstructed from the leading asymptotics of the moments $\lim_{\omega\to 0}\langle\tilde{\nu}^n\rangle=\omega^{1-n}n!/(2n-1)$. This would be a valid procedure if the distribution depends only on the product $\omega\tilde{\nu}$ in the limit $\omega\to 0$, which it does not. The subleading terms of the moments have to be included for $\tilde{\nu}\lesssim\omega$. Indeed, our distribution function has the same leading asymptotics of the moments, but has a different functional form. This illustrates the potential pitfalls of the restoration procedure which are circumvented by our direct method.

In conclusion, we have given exact results for the distributions of the local densities of states in one-dimensional localization, contrasting the microscopic length scale (below the wavelength) and mesoscopic length scale (between the wavelength and the mean free path). Contrary to expectations in the literature, the log-normal asymptotics at small densities applies only to the mesoscopic LDOS $\tilde{\nu}$, while the distribution of the microscopic LDOS ν diverges $\propto \nu^{-1/2}$ for $\nu\to 0$. This is of physical significance because many of the local probes act on atomic degrees of freedom and hence measure ν rather than $\tilde{\nu}$. The strong length scale dependence of the LDOS disappears if the electrons (or photons) are scattered inelastically (or absorbed) before reaching the reservoir. Both $P(\nu)$ and $P(\tilde{\nu})$ then have an exponential cutoff at small densities.

It is an interesting open problem whether the qualitative distinction between ν and $\tilde{\nu}$ in an open wire carries over to the quasi-one-dimensional geometry with $N>1$ modes An analytic theory could build on the multichannel generalization of Eq (5),

$$\nu = \mathrm{Re\,Tr}\,\hat{M}(1+\hat{r}_L)(1-\hat{r}_R\hat{r}_L)^{-1}(1+\hat{r}_R) \tag{21}$$

Now $\hat{r}_L$ and $\hat{r}_R$ are $N\times N$ reflection matrices and the matrix $\hat{M}_{nm}=2(\pi A)^{-1}(v_n v_m)^{-1/2}\sin(\mathbf{q}_n\ \mathbf{r}_0)\sin(\mathbf{q}_m\ \mathbf{r}_0)$ contains the weights of the N scattering states with transversal momentum $\mathbf{q}_n$ and longitudinal velocity v_n at the transversal position $\mathbf{r}_0$ on the cross section of the wire (area A)

Our approach can be generalized to a number of different situations One example is the LDOS inside a disordered ring penetrated by a magnetic flux [20] Our approach maps this problem onto the problem of reflection and transmission (with amplitude $t_R=t_L\equiv t$ for $\Phi=0$) from the opposite ends of a finite disordered segment The microscopic LDOS is then given by $\nu=(\pi v)^{-1}\mathrm{Re}[(1+r_L)(1+r_R)-t^2](1-2t\cos 2\pi\Phi/\Phi_0+t^2-r_Lr_R)^{-1}$, with the flux quantum $\Phi_0=hc/e$ Another example is the LDOS in a wire coupled to a superconductor at one end [21] The expressions for ν and $\tilde{\nu}$ in terms of the reflection matrices from two independent parts of the wire, derived in this paper, can be directly generalized to include Andreev reflection at the interface

Finally, with our approach one can investigate the relation of wave-function decay to the decay of transmission probabilities These are known to be identical in one dimension Although identity is widely assumed in quasi-one-dimension, it has come under debate recently [22] By cutting the wire at two points instead of one, we can study the correlator $\rho(x,y)=\langle\tilde{\nu}(x)\ln\tilde{\nu}(y)/\tilde{\nu}(x)\rangle$, which selects the localization center at x and then captures the decay of the wave function from x to y [23] In one dimension we now can average over random reflection phases and indeed obtain $\rho(x,y)=\ln T$, where T is the transmission probability from x to y The conditions for a similar relation in quasi-one-dimension are not known

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