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NECESSARY CONDITIONS FOR THE EXISTENCE OF PERFECT LEE CODES

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Necessary conditions for the existence of perfect Lee codes

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H.W. Lenstra, Jr.

ABSTRACT

Necessary conditions for the existence of perfect Lee codes are obtained.

KEY WORDS & PHRASES: *Perfect code, Lee metric.*

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1. INTRODUCTION

Let  $q, m, e$  be integers, with  $q \geq 2, m \geq 1$  and  $e \geq 0$ . We denote by  $\mathbb{Z}/q\mathbb{Z}$  the ring of integers modulo  $q$ . For  $x \in \mathbb{Z}/q\mathbb{Z}$ , let  $|x| = \min\{|y| \mid y \in \mathbb{Z}, x = (y \bmod q)\}$ .

Let  $X$  denote the  $m$ -fold cartesian product

$$X = (\mathbb{Z}/q\mathbb{Z}) \times \dots \times (\mathbb{Z}/q\mathbb{Z}).$$

This is an abelian group of order  $q^m$ , which we write additively. We endow  $X$  with a metric  $d$  by

$$d((x_i)_{i=1}^m, (y_i)_{i=1}^m) = \sum_{i=1}^m |x_i - y_i|,$$

the so-called *Lee metric*.

A *perfect code of order  $e$*  is a subset  $C$  of  $X$  with the property that for every  $x \in X$  there exists a unique  $c \in C$  for which  $d(x,c) \leq e$ . We are interested in obtaining necessary conditions for the existence of such a code.

Put

$$S_e = \{s \in X \mid d(0,s) \leq e\}.$$

Clearly, a subset  $C \subset X$  is a perfect code of order  $e$  if and only if every  $x \in X$  has a unique decomposition  $x = c + s$ , with  $c \in C$  and  $s \in S_e$ .

By  $G$  we denote the group of group automorphisms of  $X$  which are at the same time isometries. Clearly,  $\#G = 2^m \cdot m!$  for  $q > 2$  and  $\#G = m!$  for  $q=2$ . Notice  $\sigma S_e = S_e$  for every  $\sigma \in G$ .

Let  $\xi_q$  be a fixed primitive  $q$ -th root of unity in  $\mathbb{C}$ . We define a pairing  $\langle, \rangle: X \times X \rightarrow \mathbb{C}$  by

$$\langle (x_i)_{i=1}^m, (y_i)_{i=1}^m \rangle = \xi_q \sum_{1 \leq i \leq m} x_i y_i.$$

We have  $\langle \sigma x, \sigma y \rangle = \langle x, y \rangle$  for all  $\sigma \in G$ ,  $x, y \in X$ .

Let

$$T_e = \{0\} \cup \{x \in X \mid \sum_{s \in S_e} \langle x, s \rangle = 0\} \subset X.$$

For all  $\sigma \in G$  we have  $\sigma T_e = T_e$ . The set  $T_e$  does not depend on the choice of  $\xi_q$ , since all primitive  $q$ -th roots of unity are conjugate over  $\mathbb{Q}$ . For the same reason,  $T_e$  is closed under multiplication by integers which are relatively prime to  $q$ , but we will not use this.

If a group  $H$  acts on a set  $S$ , then the orbit space is denoted by  $S/H$ .

THEOREM 1. *Suppose a perfect code of order  $e$  exists in  $X$ . Then  $\#(T_e/H) \geq \#(S_e/H)$  for all subgroups  $H \subset G$ .*

The case  $H = G$  of this theorem is equivalent to the "Lloyd"-theorem which has been proved by L.A. BASSALYGO [1].

THEOREM 2. *Suppose a perfect code of order  $e$  exists in  $X$ . Then  $\#S_e$  divides  $\# \langle T_e \rangle$ , where  $\langle T_e \rangle$  denotes the subgroup of  $X$  generated by  $T_e$ . More precisely, if*

$$Y_e = \{y \in X \mid \langle t, y \rangle = 1 \text{ for all } t \in T_e\}$$

*then  $Y_e$  is a subgroup of  $X$  of index equal to  $\# \langle T_e \rangle$ , and every perfect code of order  $e$  in  $X$  is periodic modulo  $Y_e$  (i.e.: a union of cosets of  $Y_e$ ).*

Theorem 2 generalizes the "sphere packing bound"  $\#S_e \mid q^m$ , since  $\# \langle T_e \rangle$  obviously divides  $\#X = q^m$ .

THEOREM 3. *Suppose  $q$  is prime, and  $\#S_e = q$ . Then there exists a perfect code  $C \subset X$  of order  $e$  if and only if there exists a subgroup  $C \subset X$  whose underlying set is a perfect code of order  $e$ .*

Section 2 gives some illustrations of theorems 1, 2 and 3, and section 3 contains the proofs. The pleasure of formulating and proving analogues of these theorems for other situations (mixed perfect Lee-codes, for example) is left to the reader.

## 2. EXAMPLES.

We only consider examples which satisfy the sphere packing bound  $\#S_e \mid q^m$ .

(2.1)  $q=5, m=2, e=1$ . It is easily seen that in this case a perfect code exists. We have

$$S_1 = \{(0,0), (\pm 1,0), (0,\pm 1)\} \subset (\mathbb{Z}/5\mathbb{Z}) \times (\mathbb{Z}/5\mathbb{Z}) = X.$$

Let  $x = (a,b) \in X, x \neq (0,0)$ . Then  $x \in T_1$  if and only if  $1 + \xi_5^a + \xi_5^{-a} + \xi_5^b + \xi_5^{-b} = 0$ . Using that  $X^4 + X^3 + X^2 + X + 1$  is the irreducible polynomial of  $\xi_5$  over  $\mathbb{Q}$  one arrives at

$$T_1 = \{(0,0), (\pm 2,\pm 1), (\pm 1,\pm 2)\}.$$

Thus we see  $\#T_1 = 9 > 5 = \#S_1$  and  $\#(T_1/G) = 2 = \#(S_1/G)$ , in accordance with theorem 1.

(2.2)  $q=13, m=2, e=2$ . Also in this case a perfect code exists. One finds that  $T_2$  is the union of the  $G$ -orbits containing

$$(0,0), (1,5), (2,3), (4,6).$$

Hence  $\#(T_2/G) = 4 = \#(S_2/G)$ .

(2.3)  $q=41, m=4, e=2$  or  $q=61, m=5, e=2$ . It has been shown by E. Wattel that no perfect group code exists with these parameters. Since  $\#S_2 = q$  is prime, it follows from theorem 3 that no perfect code at all exists in these cases.

(2.4)  $q=85, m=6, e=2$ . Using the methods of [2] and computer results kindly provided by A.E. Brouwer I checked that  $T_2$  consists of the  $G$ -orbits of

$$\begin{aligned} &(0, 0, 0, 0, 0, 0), \\ &(0, 0, 17, 17, 34, 34), \\ &(0, 17, 17, 17, 17, 34), \\ &(0, 34, 34, 34, 34, 17). \end{aligned}$$

Hence  $\#(T_2/G) = 4 = \#(S_2/G)$  so the necessary condition of Bassalygo's theorem is satisfied (the case  $H=G$  of theorem 1). But by theorem 2 no perfect code exists in this case, since  $\#S_2 = 85$  does not divide  $\#<T_2> = 5^6$ .

(2.5)(Bassalygo)  $q=5, m \geq 2, e=2$ . If a perfect code exists, then theorem 1 (with  $H=G$ ) and the sphere packing bound imply

$$\begin{aligned} \#(T_2/G) &\geq 4, \\ m^2 + (m+1)^2 &= 5^k \quad (\text{for some } k \in \mathbb{Z}). \end{aligned}$$

It can be shown that this leads to a contradiction [1], so no perfect code with these parameters exists.

### 3. PROOFS.

The group ring. Let  $\mathbb{C}[X]$  be the group ring of  $X$  over  $\mathbb{C}$ ; so  $\mathbb{C}[X]$  has, as a  $\mathbb{C}$ -vector space, a basis  $\{e_x \mid x \in X\}$ , and the multiplication is determined by  $e_x \cdot e_y = e_{x+y}$ . For each  $x \in X$  there is a ring homomorphism

$$\begin{aligned} \langle x, - \rangle: \mathbb{C}[X] &\rightarrow \mathbb{C} \\ \langle x, \sum_{y \in X} \lambda_y e_y \rangle &= \sum_{y \in X} \lambda_y \langle x, y \rangle \quad (\lambda_y \in \mathbb{C}) \end{aligned}$$

and it is well known that the map

$$\begin{aligned} \mathbb{C}[X] &\rightarrow \mathbb{C}^X \\ f &\mapsto (\langle x, f \rangle)_{x \in X} \end{aligned}$$

is an isomorphism of  $\mathbb{C}$ -algebras; here  $\mathbb{C}^X$  is the product of  $\#X$  copies of  $\mathbb{C}$ , with addition and multiplication performed componentwise.

For a subset  $D$  of  $X$ , we denote the element  $\sum_{x \in D} e_x$  of  $\mathbb{C}[X]$  by  $\sum D$ .

The group  $G$  acts on  $\mathbb{C}[X]$  in a natural way as a group of algebra automorphisms, by permutation of the basis vectors  $e_x$ . We have  $\langle \sigma x, \sigma f \rangle = \langle x, f \rangle$  for  $x \in X$ ,  $f \in \mathbb{C}[X]$ ,  $\sigma \in G$ .

For a subgroup  $H \subset G$  we define  $\mathbb{C}[X]^H = \{f \in \mathbb{C}[X] \mid \forall \sigma \in H: \sigma f = f\}$ .

Clearly,  $\{\sum \bar{y} \mid \bar{y} \in X/H\}$  is a basis for  $\mathbb{C}[X]^H$ . Let  $f \in \mathbb{C}[X]^H$ . Then for  $x \in X$  and  $\sigma \in H$  we have  $\langle \sigma x, f \rangle = \langle \sigma x, \sigma f \rangle = \langle x, f \rangle$ , so  $\langle x, f \rangle$  only depends on the  $H$ -orbit  $\bar{x}$  of  $x$ . Hence for  $f \in \mathbb{C}[X]^H$ ,  $\bar{x} \in X/H$  we can define  $\langle \bar{x}, f \rangle = \langle x, f \rangle$ , where  $x \in \bar{x}$ . This gives us a ring homomorphism

$$(3.1) \quad \begin{aligned} \mathbb{C}[X]^H &\rightarrow \mathbb{C}^{X/H} \\ f &\mapsto (\langle \bar{x}, f \rangle)_{\bar{x} \in X/H} \end{aligned}$$

which is easily proved to be an isomorphism (e.g.: injectivity follows from injectivity of  $\mathbb{C}[X] \rightarrow \mathbb{C}^X$ , and surjectivity by comparison of dimensions).

Perfect codes. A subset  $C \subset X$  is a perfect code of order  $e$  if and only if the relation

$$(3.2) \quad (\sum S_e) \cdot (\sum C) = \sum X$$

holds in  $\mathbb{C}[X]$ . From this we deduce:

(3.3) LEMMA. Let  $x \in X$ ,  $x \notin T_e$ . Then  $\langle x, \sum C \rangle = 0$  for every perfect code  $C \subset X$  of order  $e$ .

PROOF. Applying the ring homomorphism  $\langle x, - \rangle$  to (3.2) we find

$$\langle x, \sum S_e \rangle \cdot \langle x, \sum C \rangle = \langle x, \sum X \rangle \quad (\text{in } \mathbb{C}).$$

Because of  $x \notin T_e$  we have  $x \neq 0$  so

$$\langle x, \sum X \rangle = \sum_{y \in X} \langle x, y \rangle = 0$$

while further  $x \notin T_e$  implies

$$\langle x, \sum_{S_e} S_e \rangle = \sum_{s \in S_e} \langle x, s \rangle \neq 0.$$

We conclude  $\langle x, \sum C \rangle = 0$ , as required.  $\square$

Let  $H \subset G$  be a subgroup, and for  $f \in \mathbb{C}[X]$  define

$$t_H(f) = \sum_{\sigma \in H} \sigma(f).$$

Clearly,  $t_H$  is a linear map from  $\mathbb{C}[X]$  to  $\mathbb{C}[X]^H$ . Generalizing (3.3) we have:

(3.4) LEMMA. *Let  $\bar{x} \in X/H$ ,  $\bar{x} \notin T_e/H$ . Then  $\langle \bar{x}, t_H(\sum C) \rangle = 0$  for every perfect code  $C \subset X$  of order  $e$ .*

PROOF. For  $x \in \bar{x}$  we have

$$\langle \bar{x}, t_H(\sum C) \rangle = \langle x, \sum_{\sigma \in H} \sigma(\sum C) \rangle = \sum_{\sigma \in H} \langle \sigma^{-1}x, \sum C \rangle$$

and by (3.3) we have  $\langle \sigma^{-1}x, \sum C \rangle = 0$  for each  $\sigma \in H$ .  $\square$

From the isomorphism (3.1) and lemma (3.4) we conclude:

(3.5). The  $\mathbb{C}$ -vector space spanned by  $\{t_H(\sum C) \mid C \subset X \text{ is a perfect code of order } e\}$  has dimension at most  $\#(T_e/H)$ , for every subgroup  $H \subset G$ .

PROOF OF THEOREM 1. Suppose a perfect code  $C \subset X$  of order  $e$  exists. Notice that such a  $C$  has exactly one element in common with  $S_e$ .

For every orbit  $\bar{x} \in S_e/H$ , one can find, by translation, a perfect code  $C_{\bar{x}} \subset X$  of order  $e$  such that the unique element of  $C_{\bar{x}} \cap S_e$  is contained in  $\bar{x}$ . Writing  $t_H(\sum C_{\bar{x}})$  on the basis  $\{\sum \bar{y} \mid \bar{y} \in X/H\}$  of  $\mathbb{C}[X]^H$ :

$$t_H(\sum C_{\bar{x}}) = \sum_{\bar{y} \in X/H} \lambda_{\bar{y}} \cdot (\sum \bar{y}), \quad (\lambda_{\bar{y}} \in \mathbb{C}),$$

we then find

$$\lambda_{\bar{y}} = 0 \quad \text{for } \bar{y} \in S_e/H, \bar{y} \neq \bar{x},$$

$$\lambda_{\bar{x}} > 0$$

(more precisely,  $\lambda_{\bar{x}} = \#H/\#\bar{x}$ ). It follows that  $\{t_H(\sum \bar{C}_{\bar{x}}) \mid \bar{x} \in S_e/H\}$  spans a  $\mathbb{C}$ -vector space of dimension  $\#(S_e/H)$ . Hence (3.5) implies  $\#(S_e/H) \leq \#(T_e/H)$ , as required.  $\square$

PROOF OF THEOREM 2. By the duality theory of finite abelian groups  $Y_e$  is a subgroup of  $X$  of index  $\# \langle T_e \rangle$ . Let

$$V_e = \{f \in \mathbb{C}[X] \mid \langle x, f \rangle = 0 \text{ for all } x \in X, x \notin \langle T_e \rangle\}.$$

We claim

$$V_e = \{\sum_{\bar{t} \in X/Y_e} \lambda_{\bar{t}} \cdot (\sum \bar{t}) \mid \lambda_{\bar{t}} \in \mathbb{C} \text{ for } \bar{t} \in X/Y_e\}.$$

In fact, the inclusion  $\supset$  follows from a direct calculation, and equality follows by comparison of dimensions.

Let  $C \subset X$  be a perfect code of order  $e$ . Then  $\sum C \in V_e$  by lemma (3.3) and the definition of  $V_e$ , so our claim says

$$\sum C = \sum_{\bar{t} \in X/Y_e} \lambda_{\bar{t}} \cdot (\sum \bar{t})$$

for certain complex numbers  $\lambda_{\bar{t}}$ . This exactly means that  $C$  is periodic modulo  $Y_e$ . In particular,  $\#Y_e$  divides  $\#C$ , and since  $\#C \cdot \#S_e = \#X$  it follows that  $\#S_e$  divides  $\#X/\#Y_e = \# \langle T_e \rangle$ .  $\square$

PROOF OF THEOREM 3. We need only prove the "only if"-part. From theorem 1 we see  $\#T_e \geq \#S_e > 1$  so there exists  $x \in T_e$ ,  $x \neq 0$ . Hence

$$(3.6) \quad \sum_{s \in S_e} \langle x, s \rangle = 0$$

for some  $x \in X$ . Thus we have a sum of  $q$   $q$ -th roots of unity which vanishes. Using the irreducibility of the polynomial  $X^{q-1} + \dots + X + 1$  over  $\mathbb{Q}$  (since  $q$  is prime) one easily sees that (3.6) is equivalent to:

$$(3.7) \quad \text{for each } i \in \{0, 1, \dots, q-1\} \text{ there is a unique } s \in S_e \text{ with} \\ \langle x, s \rangle = \xi_q^i.$$

Now let  $C$  be the kernel of the group homomorphism  $X \rightarrow \{\xi_q^i \mid 0 \leq i < q\}$  which sends  $y$  to  $\langle x, y \rangle$ . Then (3.7) is equivalent to:

$$\text{for each } y \in X \text{ there is a unique } s \in S_e \text{ with } y - s \in C.$$

It follows that  $C$  is a perfect code of order  $e$ .  $\square$

More generally, one can prove, using theorems 1 and 2 and the methods of [2]:

COROLLARY. *Suppose  $\#S_e = p$  is prime, and suppose that there exists at most one prime dividing  $q$  which is smaller than  $p$ . Then there exists a perfect code  $C \subset X$  of order  $e$  if and only if there exists a subgroup  $C \subset X$  whose underlying set is a perfect code of order  $e$ . Moreover, every perfect code  $C \subset X$  of order  $e$  is periodic modulo  $pX$ .*

#### REFERENCES.

- [1] BASSALYGO, L.A., *A necessary condition for the existence of perfect codes in the Lee metric*, Mat. Zametki 15 (1974), 313-320 (russian).  
 [2] MANN, H.B., *On linear relations between roots of unity*, Mathematika 12 (1965), 107-117.